ON VOLterra's INTEGRO-FUNCTIONAL EQUATION*

BY

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Although the equation

\[ u(x) = f(x) + s(x)u[\theta_1(x)] + \int_{-x}^{x} K(x, \xi)u[\theta_2(\xi)]d\xi \]

has been discussed by many authors,† nevertheless the question can hardly be considered as completely solved. Hence the following considerations may present points of interest.

1. In the equation (I)

\[ K(x, \xi), \ f(x), \ s(x), \ \theta_1(x), \ \theta_2(x) \]

are given functions which satisfy the following conditions:

(i) The kernel \( K(x, \xi) \) is determined and bounded in the region

\[-X \leq x \leq X, \quad -X \leq \xi \leq X,\]

where \( X \) is a given positive constant. The discontinuities of \( K(x, \xi) \), if there are any, are regularly distributed.† The upper bound of \( |K(x, \xi)| \) is denoted by \( k \).

(ii) The functions \( \theta_1(x), \ \theta_2(x), \ s(x), \ f(x) \) are continuous on the interval

\[-X \leq x \leq X\]

and

\[ |\theta_i(x)| \leq |x| (i = 1, 2); \quad |s(x)| \leq \sigma; \quad |f(x)| \leq f_0 \]

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where \( a \) and \( b_0 \) are given positive constants.

The "solution" of the equation (I) is defined as a function which is integrable in the sense of Lebesgue and remains so if \( x \) is replaced by \( \theta(x) \), and which satisfies the equation. The integrals are to be taken in the sense of Lebesgue.

Instead of the given equation (I) we shall consider the more general equation

\[
(II) \quad u(x) = f(x) + \lambda \left\{ s(x) u[\theta_1(x)] + \int_{-x}^{x} K(x, \xi) u[\theta_2(\xi)] d\xi \right\}
\]

which reduces to (I) for \( \lambda = 1 \). This equation we shall try to satisfy by a power series in \( \lambda \):

\[
(1) \quad u(x) = \sum_{n=0}^{\infty} \lambda^n u_n(x)
\]

where \( u_0(x), u_1(x), \ldots, u_n(x), \ldots \) are to be determined successively by the formulas

\[
(III) \quad u_0(x) = f(x), \quad u_n(x) = s(x) u_{n-1}[\theta_1(x)] + \int_{-x}^{x} K(x, \xi) u_{n-1}[\theta_2(\xi)] d\xi.
\]

The functions (III) are obviously all continuous. If we denote by \( a_n(x) \) the upper bound of \( |u_n(\xi)| \) on the interval \(-x \leq \xi \leq x\), it is readily found that

\[
a_0(x) \leq f_0, \quad a_n(x) \leq \sigma a_{n-1}(x) + 2k \int_0^x a_{n-1}(\xi) d\xi. \quad (*)
\]

Defining the functions

\[ u_0(x), \quad u_1(x), \ldots, \quad U_n(x), \ldots \]

by the relations

\[ U_0(x) = f_0, \quad U_n(x) = \sigma U_{n-1}(x) + 2k \int_0^x U_{n-1}(\xi) d\xi \]

the series

\[
(2) \quad U(x) = \sum_{n=0}^{\infty} \lambda^n U_n(x)
\]

is clearly dominant for the series (1). On the other hand the series (2) is obtained if we formally expand the solution of the equation

\[
U(x) = f_0 + \lambda \left\{ \sigma U(x) + 2k \int_0^x U(\xi) d\xi \right\}
\]

* For the sake of simplicity we consider the case \( x \geq 0 \) only. The case \( x \leq 0 \) can be treated in an entirely analogous way.
in powers of $\lambda$. This solution is uniquely determined and is given by the formula

$$U(x) = \frac{f_0}{1 - \lambda \sigma} e^{\frac{2kx}{1 - \lambda \sigma}},$$

which is expansible in the uniformly convergent power series in $\lambda$ when $|\lambda| < 1/\sigma$. This series must coincide with the series (2) and is uniformly convergent for $\lambda = 1$, provided $\sigma < 1$. In this case the series (1) is also uniformly convergent for $\lambda = 1$ and yields the solution of (I). Thus the existence of a continuous solution of (I) is proved, if in addition to the conditions (i) and (ii) we suppose that

$$(iii) \quad \sigma < 1.$$

It is easy to show now that the solution of (I), as defined above, is unique, under the condition that it be bounded, so that any bounded solution of (I) coincides with (1) (for $\lambda = 1$).

In order to prove this, it is sufficient to show that any bounded solution of the homogeneous equation

$$v(x) = s(x)v[\theta_1(x)] + \int_{-\infty}^{x} K(x, \xi)v[\theta_2(\xi)]d\xi$$

is identically zero. Let us denote by $\bar{v}(x)$ the upper bound of $|v(\xi)|$ on the interval $-x \leq \xi \leq x$. Then

$$\bar{v}(x) \leq \sigma \bar{v}(x) + 2k \int_{-\infty}^{x} \bar{v}(\xi)d\xi.$$

If $v_0$ is the upper bound of $\bar{v}(x)$ on the whole interval $(-X, X)$, we obtain successively

$$\bar{v}(x) \leq v_0 \frac{2kx}{1 - \sigma}; \quad \bar{v}(x) \leq v_0 \left(\frac{2kx}{1 - \sigma}\right)^2 \frac{1}{2!}; \quad \cdot \cdot \cdot \quad \bar{v}(x) \leq v_0 \left(\frac{2kx}{1 - \sigma}\right)^n \frac{1}{n!}; \quad \cdot \cdot \cdot$$

whence

$$\bar{v}(x) \leq v_0 \left(\frac{2kX}{1 - \sigma}\right)^n$$

for all $n$, however large. This is possible only if $\bar{v}(x) = 0$, that is, $v(x) = 0$.

The same results hold true if we drop the condition of the continuity of the functions $s(x), \theta_1(x), f(x)$, and suppose only that
(iv) The functions \( s(x) \), \( \theta_t(x) \), \( f(x) \) are measurable and bounded and all the terms of the sequence (III) are measurable and remain measurable if \( x \) is replaced by \( \theta_t(x) \).

Under the conditions (i)-(iv) the bounded solution of (I) is unique and is given by the formulas above.

2. Consider now the case \( \sigma \geq 1 \), under the hypotheses

\[ |\theta_t(x)| \leq \alpha |x| ; \quad \alpha \sigma < 1. \]

As we shall see below, the equation (I), if \( \sigma \geq 1 \), may possess infinitely many continuous solutions. Therefore in this section we confine our discussion to the continuous solutions of (I) for which the ratio \( (u(x) - u(0))/x \) remains bounded as \( x \to 0 \).

If \( s(0) \neq 1 \), the equation (I) determines uniquely the initial value \( u(0) \):

\[ u(0) = \frac{f(0)}{1 - s(0)}. \]

If \( s(0) = 1 \), it is necessary for the existence of a solution that \( f(0) = 0 \), and if this condition is satisfied, then \( u(0) \) may be chosen arbitrarily. We introduce now the further restriction:

(vi) The ratios

\[ \frac{f(x) - f(0)}{x}, \quad \frac{s(x) - s(0)}{x} \]

remain bounded as \( x \to 0 \), and

\[ f(0) = 0 \quad \text{if} \quad s(0) = 1. \]

Introducing in (I) a new dependent variable \( z(x) \) given by

\[ z(x) = \frac{u(x) - u(0)}{x} ; \quad u(x) = u(0) + xz(x), \]

we obtain

\[ z(x) = \varphi(x) + r(x) [\theta_t(x)] + \int_{-\infty}^{\infty} L(x, \xi) z[\theta_s(\xi)] d\xi, \]

where

\[ \varphi(x) = \begin{cases} \frac{u(0)}{x} \int_{-\infty}^{\infty} K(x, \xi) d\xi + \frac{f(x) - f(0) + s(x)f(0) - s(f(x))}{x[1 - s(0)]}, & \text{if } s(0) \neq 1 ; \\ \frac{u(0)}{x} \int_{-\infty}^{\infty} K(x, \xi) d\xi + \frac{s(x) - 1}{x} u(0), & \text{if } s(0) = 1 ; \end{cases} \]

\[ r(x) = s(x) \theta_t(x) x^{-1} ; \quad L(x, \xi) = K(x, \xi) \theta_s(\xi) x^{-1}. \]

* This implies, of course, \( \alpha < 1 \).
The functions \( \varphi(x), r(x), L(x, \xi) \) replace the functions \( f(x), s(x), K(x, \xi) \) of §1. By virtue of (v) and (vi) they are bounded and

\[ |r(x)| \leq \alpha \sigma < 1. \]

It is obvious that the conditions (i)–(iv) are satisfied for the equation (4) and therefore the equation (4) admits of a unique bounded solution, \( z(x) \), which is given by the method above.

Thus, under the conditions (i)–(iii), (v), (vi), the equation (1) possesses a continuous solution for which the ratio \( (u(x) - u(0))/x \) remains bounded as \( x \to 0 \). If \( s(0) \neq 1 \), this solution is unique. If \( s(0) = 1 \), the initial value \( u(0) \) can be chosen arbitrarily, and when \( u(0) \) is prescribed the solution is uniquely determined.

3. A series of examples in which the theorem of uniqueness for bounded solutions fails, was given by C. Popovici.* The example which follows is simpler and apparently more general.† We consider a particular case of the homogeneous equation (1), namely

\[ u(x) = \sigma u(ax) ; \quad 0 < \alpha < 1 ; \quad x > 0. \]

The most general solution of (5) can be found explicitly. Setting

\[ y = \log x; \quad \mu = \log \sigma; \quad \nu = \log \alpha; \quad v(\log x) = \log u(x), \]

we get

\[ v(y) = \mu + v(y + \nu), \]

which gives

\[ v(y) = -\frac{\mu}{\nu} y + \omega(y), \]

where \( \omega(y) \) is an arbitrary periodic function of \( y \), of period \( \nu \). Hence the general solution of (5) is

\[ u(x) = x^{-\mu/\nu} e^{\omega(\log x)} \]

Let \( \omega(x) \) be a continuous function which is not constant. Then \( e^{\omega(\log x)} \) has positive lower and upper bounds on the interval \((0, \infty)\) and tends to no limit as \( x \to 0 \). The corresponding solution \( u(x) \), which is given by (6), is not bounded as \( x \to 0 \), because \( -\mu/\nu < 0 \). This was to be expected, for it was proved above that the bounded solution of (5) is identically zero.

* Loc. cit.
† This example was indicated by A. Friedmann.
Suppose now that \( \sigma \geq 1 \), but \( \alpha \sigma < 1 \). Since \( -\mu/\nu > 0 \) the formula (6) gives us infinitely many solutions which are bounded and even continuous if we set \( u(0) = u(0+) = 0 \).

For all these solutions, however, the ratio \( (u(x) - u(0))/x \) is not bounded. Suppose finally \( \alpha \sigma \geq 1 \). In this case all the solutions determined by (6) are bounded and continuous (if we set \( u(0) = 0 \)) and for all these solutions the ratio \( (u(x) - u(0))/x \) remains bounded. The particular example of the equation

\[
\frac{u(x)}{u(x) - u(0)} = \frac{\sigma}{\sigma - 1}
\]

which admits of infinitely many linear solutions

\[
u(x) = Cx, \quad C \text{ constant},
\]

shows that there is no question about the theorem of uniqueness in this case.

Analogous results may be obtained for the more general equation

\[
u(x) = \sigma u(\alpha x) + \int_0^x K(x, \xi) u(\xi) d\xi.
\]