ASYMMETRIC DISPLACEMENT OF A VECTOR*

BY

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1. **Introduction.** Levi-Civita's definition of parallel displacement of a vector ‡ has been generalized to non-Riemannian geometries by several writers§ who have replaced the Christoffel symbols of the second kind by a set of quantities Γ^i_{jk} which are symmetric in j and k; and by Schouten|| who has omitted the assumption of symmetry.

The problem of the changes of the connection Γ^i_{jk} which preserve the paths (or geodesics) has been treated by Weyl, by Eisenhart and by Veblen in the symmetric case¶ and by Friedmann and Schouten** in the asymmetric case. Such changes of the connection preserve, in general, only the directions of vectors displaced along themselves. In the present paper are treated changes of connection which preserve the directions of *all* displaced vectors (§5).†† It is readily shown that two distinct symmetric connections cannot give rise to the same displaced directions, so that the connections considered are necessarily asymmetric in general.

In § 7 are given some tensors which are independent of the above change of connection, and a process, like covariant differentiation, for forming tensors of the same nature but of higher rank is indicated.

In the final section we find necessary and sufficient conditions in order that an asymmetric connection may be made symmetric by a change preserving displaced directions.

2. **General linear displacement of a vector.** Consider a vector field ξ^i in an n-dimensional manifold referred to a coördinate system x. The vector

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‡ T. Levi-Civita, *Nozione di parallelismo in una varietà qualunque*, Rendiconti del Circolo Matematico di Palermo, vol. 42 (1917), pp. 173–205.

§ Cf. H. Weyl, *Raum, Zeit, Materie*, 4th edition, p. 100; A. S. Eddington, *The Mathematical Theory of Relativity*, p. 213.

|| J. A. Schouten, *Über die verschiedenen Arten der Übertragung*, Mathematische Zeitschrift, vol. 13 (1922), pp. 56–81; *Der Ricci-Kalkül*, pp. 62–75.

¶ H. Weyl, Göttinger Nachrichten, 1921, p. 99; L. P. Eisenhart, Proceedings of the National Academy of Sciences, vol. 8 (1922), p. 233; O. Veblen, ibid., p. 347.

** Mathematische Zeitschrift, vol. 21 (1924), p. 218.

†† This problem has been touched upon by H. Friesecke in his paper *Vektorübertragung, Richtungsübertragung, Metrik*, Mathematische Annalen, vol. 94 (1925), p. 101.

ξ^i at a point P of coördinates x^i can be thought of as being displaced to a nearby point P' of coördinates x^i+dx^i provided there is given a law specifying the vector at P' which corresponds to the vector ξ^i at P, or, what amounts to the same thing, a law specifying the vector $\delta\xi^i$ at P' which is the difference between the vector of the field at the point P' and the displaced vector. Adopting the latter alternative,* we say that a vector ξ^i suffers a *general linear displacement* when

$$\delta\xi^i = d\xi^i + H^i_{jk}\xi^j dx^k, \tag{2.1}$$

where $d\xi^i$ represent the differentials of the functions ξ^i.

The quantities H^i_{jk}, which will be called the components of the linear connection, are subjected only to the restriction that the quantities given by (2.1) are the components of a vector—a restriction which determines their law of transformation. When the coördinates x are changed by an analytic transformation to $\bar{x}$, we have

$$d\bar{\xi}^i = \frac{\partial\bar{x}^i}{\partial x^\alpha} d\xi^\alpha + \frac{\partial^2\bar{x}^i}{\partial x^j\partial x^k}\xi^j dx^k,$$

$$\bar{H}^i_{jk}\bar{\xi}^j d\bar{x}^k = \bar{H}^i_{\alpha\beta}\frac{\partial\bar{x}^\alpha}{\partial x^j}\frac{\partial\bar{x}^\beta}{\partial x^k}\xi^j dx^k.$$

Hence

$$\delta\bar{\xi}^i = \frac{\partial\bar{x}^i}{\partial x^\alpha} d\xi^\alpha + \left(\frac{\partial^2\bar{x}^i}{\partial x^j\partial x^k} + \bar{H}^i_{\alpha\beta}\frac{\partial\bar{x}^\alpha}{\partial x^j}\frac{\partial\bar{x}^\beta}{\partial x^k}\right)\xi^j dx^k.$$

In order that these equations may reduce to

$$\frac{\partial\bar{x}^i}{\partial x^\alpha}\delta\xi^\alpha = \frac{\partial\bar{x}^i}{\partial x^\alpha}d\xi^\alpha + \frac{\partial\bar{x}^i}{\partial x^\alpha}H^\alpha_{jk}\xi^j dx^k,$$

it is necessary and sufficient that

$$H^\alpha_{jk}\frac{\partial\bar{x}^i}{\partial x^\alpha} = \frac{\partial^2\bar{x}^i}{\partial x^j\partial x^k} + \bar{H}^i_{\alpha\beta}\frac{\partial\bar{x}^\alpha}{\partial x^j}\frac{\partial\bar{x}^\beta}{\partial x^k}. \tag{2.2}$$

We write

$$H^i_{jk} = \Gamma^i_{jk} + \Omega^i_{jk}, \tag{2.3}$$

* Although displacement as here defined only applies to a vector belonging to a field, we extend the definition to all vectors by stipulating that the changes in the components be given by $-H^i_{jk}\xi^j dx^k$ as far as terms of the first order.

the Γ's denoting the symmetric part of H^i_{jk},

$$\Gamma^i_{jk} = \tfrac{1}{2}\,(H^i_{jk} + H^i_{kj}), \tag{2.4}$$

and the Ω's the skew-symmetric part,

$$\Omega^i_{jk} = \tfrac{1}{2}\,(H^i_{jk} - H^i_{kj}). \tag{2.5}$$

By adding equations (2.2) to those obtained from them by the interchange of j and k, and dividing by 2, we find the law of transformation for the Γ's is the same as for the H's, namely,

$$\Gamma^\alpha_{jk}\frac{\partial \bar{x}^i}{\partial x^\alpha} = \frac{\partial^2 \bar{x}^i}{\partial x^j \partial x^k} + \bar{\Gamma}^i_{\alpha\beta}\frac{\partial \bar{x}^\alpha}{\partial x^j}\,\frac{\partial \bar{x}^\beta}{\partial x^k}. \tag{2.6}$$

From these equations it is readily proved that

$$\Gamma^\alpha_{\alpha j} = \bar{\Gamma}^\alpha_{\alpha\beta}\frac{\partial \bar{x}^\beta}{\partial x^j} + \frac{\partial \log \Delta}{\partial x^j}, \tag{2.7}$$

where

$$\Delta = \left| \frac{\partial \bar{x}^i}{\partial x^j} \right| .$$

By subtracting equations of the form (2.2), we prove that Ω^i_{jk} are the components of a tensor.*

3. **Fields of vectors generated by displacement and the curvature tensor.** Let us consider a field of vectors generated by the displacement of a vector. The conditions for such a field are $\delta\xi^i = 0$, and can be written as

$$d\xi^i + H^i_{jk}\xi^j dx^k = 0, \tag{3.1}$$

or as

$$\frac{\partial \xi^i}{\partial x^k} + H^i_{jk}\xi^j = 0. \tag{3.2}$$

The conditions of integrability of these equations are

$$Z^i_{jkl}\xi^j = 0, \tag{3.3}$$

where

$$Z^i_{jkl} = \frac{\partial H^i_{jk}}{\partial x^l} - \frac{\partial H^i_{jl}}{\partial x^k} + H^\alpha_{jk}H^i_{\alpha l} - H^\alpha_{jl}H^i_{\alpha k}. \tag{3.4}$$

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* This result is given by Schouten, *Ricci-Kalkül*, p. 67.

In order that there exist a vector at each point of space which can be obtained by displacement of a vector which has arbitrary initial components at an arbitrary starting point, it is necessary and sufficient that equations (3.2) be completely integrable. The conditions of integrability (3.3) of these equations will be satisfied identically if, and only if,

$$Z^i_{jkl} = 0.$$

From the way in which these functions Z^i_{jkl} arise it is evident that they are the components of a tensor. By substitution from (2.3) we have

$$Z^i_{jkl} = B^i_{jkl} + C^i_{jkl}, \tag{3.5}$$

where $B^i_{jk,l}$ is the ordinary curvature tensor for the Γ's, namely,

$$B^i_{jkl} = \frac{\partial \Gamma^i_{jk}}{\partial x^l} - \frac{\partial \Gamma^i_{jl}}{\partial x^k} + \Gamma^\alpha_{jk}\Gamma^i_{\alpha l} - \Gamma^\alpha_{jl}\Gamma^i_{\alpha k}, \tag{3.6}$$

and where

$$C^i_{jkl} = \Omega^i_{jk,l} - \Omega^i_{jl,k} + \Omega^\alpha_{jk}\Omega^i_{\alpha l} - \Omega^\alpha_{jl}\Omega^i_{\alpha k}, \tag{3.7}$$

$\Omega^i_{jk,l}$ being the covariant derivative of Ω^i_{jk} formed with respect to the Γ's:

$$\Omega^i_{jk,l} = \frac{\partial \Omega^i_{jk}}{\partial x^l} + \Omega^\alpha_{jk}\Gamma^i_{\alpha l} - \Omega^i_{\alpha k}\Gamma^\alpha_{jl} - \Omega^i_{j\alpha}\Gamma^\alpha_{kl}. \tag{3.8}$$

4. **The paths or geodesics.** If we define a path (geodesic) as a curve whose tangent vector at any point is obtained by displacement of the tangent vector at a nearby point, the differential equations of the paths are

$$\frac{d^2x^i}{ds^2} + \Gamma^i_{jk}\frac{dx^j}{ds}\frac{dx^k}{ds} = 0. \tag{4.1}$$

Only the symmetric part of the coefficients H^i_{jk} enters into these equations.*

By choosing for Ω^i_{jk} an arbitrary tensor, skew-symmetric in j and k, we can associate with any geometry of paths† a displacement of the type (2.1).

5. **Changes of linear connection which preserve displaced directions.** The vector arising from ξ^i by displacement to the point x^i+dx^i has for its components, as far as terms of the first order,

$$\xi^i - H^i_{jk}\xi^j dx^k. \tag{5.1}$$

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* Schouten, *Ricci-Kalkül*, p. 76.

† Cf. O. Veblen and T. Y. Thomas, *The geometry of paths*, these Transactions, vol. 25 (1923), p. 551–580, for a general account of the geometry of paths.

If we consider another set of H's, say

$$H'^{i}_{jk} = H^{i}_{jk} - a^{i}_{jk}, \tag{5.2}$$

then the new vector arising by displacement has for its components

$$\xi^{i} - H^{i}_{jk}\xi^{j}dx^{k} + a^{i}_{jk}\xi^{j}dx^{k}. \tag{5.3}$$

In order that the directions of the vectors (5.1) and (5.3) may be the same, it is necessary and sufficient that their difference be in the direction of either, or that

$$a^{i}_{jk}\xi^{j}dx^{k} = \lambda\xi^{i}, \tag{5.4}$$

to within terms of the first order. The conditions can also be written

$$a^{i}_{jk}\xi^{j}dx^{k} = \lambda\delta^{i}_{j}\xi^{j}.$$

Since these equations must hold for arbitrary ξ^{j}, we have

$$a^{i}_{jk}dx^{k} = \lambda\delta^{i}_{j}.$$

If we determine λ from this last relation by contraction, substitute back and equate the coefficients of dx^{k} to zero, we find

$$a^{i}_{jk} = \frac{\delta^{i}_{j}}{n} a^{\alpha}_{\alpha k}. \tag{5.5}$$

It is seen from (2.2) and (5.2) that a^{i}_{jk} is a tensor. Hence $a^{\alpha}_{\alpha k}$ is a vector, which will be denoted by $2n\varphi_{k}$. In terms of this vector, equations (5.5) become*

$$a^{i}_{jk} = 2\delta^{i}_{j}\varphi_{k},$$

and from (2.4) and (2.5) the changes in the Γ's and Ω's are, therefore, respectively

$$\Gamma^{i}_{jk} - \Gamma'^{i}_{jk} = \delta^{i}_{j}\varphi_{k} + \delta^{i}_{k}\varphi_{j}, \tag{5.6}$$

$$\Omega^{i}_{jk} - \Omega'^{i}_{jk} = \delta^{i}_{j}\varphi_{k} - \delta^{i}_{k}\varphi_{j}. \tag{5.7}$$

It is to be noted that the skew-symmetric part of the change vanishes only if the symmetric part does and vice-versa.

Conversely, if the connection is changed in accordance with (5.6) and (5.7), φ_{i} being an arbitrary vector, conditions (5.4) are fulfilled.

In the associated geometry of paths, there is brought about a change of the affine connection Γ^{i}_{jk} which preserves the paths. Moreover, every

* Friesecke, loc. cit., p. 106, obtains these conditions.

change preserving the paths is of the form (5.6).* Hence a study of displacements of the form (2.1) is one approach to a *projective* geometry of paths.

From (5.6) and (5.7) follow

$$(5.8)\qquad \varphi_j = \frac{1}{n+1}(\Gamma^{\alpha}_{\alpha j} - \Gamma'^{\alpha}_{\alpha j}),$$

$$(5.9)\qquad \varphi_j = \frac{1}{n-1}(\Omega'^{\alpha}_{j\alpha} - \Omega^{\alpha}_{j\alpha}).$$

From these equations, (5.6), and (5.7) we find that the following expressions are independent of φ_j and are therefore invariant under the change of connection being considered:†

$$(5.10)\qquad \Pi^{i}_{jk} = \Gamma^{i}_{jk} - \frac{\delta^{i}_{j}}{n+1}\Gamma^{\alpha}_{\alpha k} - \frac{\delta^{i}_{k}}{n+1}\Gamma^{\alpha}_{\alpha j}\ ;$$

$$(5.11)\qquad \mathfrak{L}^{i}_{jk} = \Omega^{i}_{jk} - \frac{\delta^{i}_{j}}{n-1}\Omega^{\alpha}_{\alpha k} - \frac{\delta^{i}_{k}}{n-1}\Omega^{\alpha}_{j\alpha}\ ;$$

$$(5.12)\qquad \Sigma^{i}_{jk} = \Gamma^{i}_{jk} + \frac{\delta^{i}_{j}}{n-1}\Omega^{\alpha}_{k\alpha} + \frac{\delta^{i}_{k}}{n-1}\Omega^{\alpha}_{j\alpha}\ ;$$

$$(5.13)\qquad \frac{1}{n+1}\Sigma^{\alpha}_{\alpha k} = \frac{\Gamma^{\alpha}_{\alpha k}}{n+1} + \frac{\Omega^{\alpha}_{k\alpha}}{n-1}\ ;$$

$$(5.14)\qquad L^{i}_{jk} = H^{i}_{jk} - \frac{\delta^{i}_{j}}{n}H^{\alpha}_{\alpha k}.$$

We note then that from these definitions, it follows that

$$\Pi^{\alpha}_{\alpha k} = \mathfrak{L}^{\alpha}_{\alpha k} = L^{\alpha}_{\alpha k} = 0.$$

6. **Normal affine connection.** The quantities Π^{i}_{jk} defined by (5.10) are the components of the projective connection.‡ From (5.10) it follows that

$$(6.1)\qquad \Pi^{i}_{jk} = \Gamma^{i}_{jk}$$

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* Cf. H. Weyl, Göttinger Nachrichten, 1921, p. 99. Also L. P. Eisenhart, Proceedings of the National Academy of Sciences, vol. 8 (1922), p. 233, and O. Veblen, ibid., p. 347.

† We shall use German characters to denote tensors which are independent of φ.

‡ These quantities were first employed and named by T. Y. Thomas in the Proceedings of the National Academy of Sciences, vol. 11 (1925), pp. 199–203.

if, and only if, $\Gamma^{\alpha}_{\alpha i}=0$. From (2.7) it is seen that this will be an invariant property only under transformations of constant jacobian. An affine connection for which (6.1) is true will be called normal* for the given coördinate system and those arising from it by transformations with constant jacobian.†

If *in the given coördinate system* we choose the components φ_i equal to $\Gamma^{\alpha}_{\alpha i}/(n+1)$, then from (5.8) it follows that

$$\Gamma'^{\alpha}_{\alpha i} = 0, \tag{6.2}$$

and conversely. Hence we have proved

THEOREM 1. *For any coördinate system there is a unique normal affine connection.*

T. Y. Thomas‡ states that the components of the projective connection "constitute a normalized affine connection." In the light of the above discussion this statement must be interpreted as follows: for each coördinate system there exists an affine connection whose components are equal to the corresponding components of the projective connection in the given coördinate system. The same equality of components also holds in coördinate systems arising from the given one by transformations of constant jacobian. In coördinate systems connected with the given one by transformations of variable jacobian, however, a *different* affine connection will be normalized and will have its components equal to those of the projective connection.

It also follows that the skew-symmetric tensor

$$S_{ij} = B^{\alpha}_{\alpha ij} = \frac{\partial \Gamma^{\alpha}_{\alpha i}}{\partial x^j} - \frac{\partial \Gamma^{\alpha}_{\alpha j}}{\partial x^i} \tag{6.3}$$

vanishes for a normal affine connection in a properly chosen coördinate system, and therefore in any coördinate system.§ Conversely, we can show

* E. Cartan, adopting a viewpoint entirely different from that of the present paper, includes $\Gamma^{\alpha}_{\alpha i} = 0$ in the definition of his normal projective connection. He proves a result equivalent to Theorem 1. Cf. his paper *Sur les variétés à connexion projective*, Bulletin de la Société Mathématique de France, vol. 52 (1924), p. 211.

† T. Y. Thomas (loc. cit.) places $\Delta = 1$ as a device to make the Π's have a law of transformation like that of the Γ's. He calls transformations of coördinates of this type *equi-transformations*. For a determination in finite form of the equations of all such transformations, cf. E. Goursat, Bulletin des Sciences Mathématiques, vol. 41 (1917), p. 211.

‡ Loc. cit., p. 200.

§ That it is always possible to choose the affine connection so that the skew-symmetric tensor vanishes was first established by L. P. Eisenhart, Proceedings of the National Academy of Sciences, vol. 8 (1922), p. 233.

that if the skew-symmetric tensor vanishes, then it is possible to choose the coördinate system so that $\bar{\Gamma}^{\alpha}_{\alpha i}=0$. We have by hypothesis

$$\frac{\partial \Gamma^{\alpha}_{\alpha i}}{\partial x^{j}} = \frac{\partial \Gamma^{\alpha}_{\alpha j}}{\partial x^{i}}.$$

Hence the equations

$$\frac{\partial \log \Delta}{\partial x^{i}} = \Gamma^{\alpha}_{\alpha i}$$

are completely integrable, and it is possible to determine a Δ satisfying them and assuming an initial value different from 0. Suppose that such a solution is

$$\Delta = f(x^1, x^2, \cdots, x^n). \tag{6.4}$$

Choose the variables $(\bar{x}^2, \bar{x}^3, \cdots, \bar{x}^n)$ as arbitrary analytic functions of $(x^1, x^2, \cdots, x^n)$ subject only to the restriction that the minor of $\partial\bar{x}^1/\partial x^1$ in Δ shall initially be different from zero. Then (6.4) can be solved for this derivative and Cauchy's theorem applied to show the existence of functions $\bar{x}^1$ satisfying (6.4). In the coördinate system so obtained it follows from (2.7) that $\bar{\Gamma}^{\alpha}_{\alpha i}=0$.

THEOREM 2. *A necessary and sufficient condition for the existence of coördinate systems for which a given affine connection is normal is the vanishing of the skew-symmetric tensor.*

Veblen* has pointed out that for affine connections of the type under consideration there is a definition of volume:

$$V = \int \gamma dx^1 dx^2 \cdots dx^n, \qquad \frac{\partial \log \gamma}{\partial x^{i}} = \Gamma^{\alpha}_{\alpha i}.$$

In the coördinate systems referred to in Theorem 2 this definition takes the form

$$V = \gamma \int dx^1 dx^2 \cdots dx^n,$$

γ being a constant. In a Riemann geometry, the skew-symmetric tensor is zero. Hence coördinate systems exist for which the affine connection given by the Christoffel symbols is normal. Since in this case

$$\Gamma^{\alpha}_{\alpha i} = \frac{\partial \log \sqrt{g}}{\partial x^{i}},$$

* O. Veblen, Proceedings of the National Academy of Sciences, vol. 9 (1923), p. 3. Cf. also L. P. Eisenhart, ibid., p. 4.

where g is the determinant of the fundamental tensor g_{ij}, they are the coördinate systems for which g is constant.

It is likewise seen from (5.9) that by choosing φ_i properly we can make

$$\Omega^{\alpha}_{\alpha j} = 0$$

for the displacement, so that

$$\mathfrak{L}^{i}_{jk} = \Omega^{i}_{jk}. \tag{6.5}$$

These are invariant conditions, but in general they can not be realized simultaneously with (6.1).

A proper choice of φ_i will in a similar manner make

$$L^{i}_{jk} = H^{i}_{jk} \tag{6.6}$$

for the given coördinate system.

7. Tensors independent of the change of the linear connection. By computation from (3.6) or by reference to Veblen and T. Y. Thomas* we find

$$B'^{i}_{jkl} = B^{i}_{jkl} - \delta^{i}_{j}(\Phi_{kl} - \Phi_{lk}) - \delta^{i}_{k}\Phi_{jl} + \delta^{i}_{l}\Phi_{jk}, \tag{7.1}$$

where

$$\Phi_{kl} = \varphi_{k,l} + \varphi_k\varphi_l,$$

and $\varphi_{k,l}$ is the covariant derivative of φ_k formed with respect to the Γ's. By contraction, the following expressions are found:

$$\Phi_{jk} = \frac{R'_{jk} - R_{jk}}{n-1} + \frac{S'_{jk} - S_{jk}}{n^2-1}, \tag{7.2}$$

$$\Phi_{jk} - \Phi_{kj} = \frac{S_{jk} - S'_{jk}}{n+1}, \tag{7.3}$$

where $R_{jk} = B^{\alpha}_{jk\alpha}$ is the Ricci tensor and S_{ij} is given by (6.3). Substitution from (7.2) in (7.1) and separation of the accented and unaccented terms show that the quantities

$$\begin{aligned}\mathfrak{W}^{i}_{jkl} = B^{i}_{jkl} - \frac{\delta^{i}_{j}}{n+1}S_{kl} + \frac{1}{n-1}(\delta^{i}_{k}R_{jl} - \delta^{i}_{l}R_{jk}) \\ + \frac{1}{n^2-1}(\delta^{i}_{k}S_{jl} - \delta^{i}_{l}S_{jk})\end{aligned} \tag{7.4}$$

* Loc. cit., p. 559; also L. P. Eisenhart, Annals of Mathematics, ser. 2, vol. 24 (1923), p. 377, where there are differences of sign due to different definitions of φ_i and B^{i}_{jkl}.

are the components of a tensor which is independent of the vector φ. It is the projective curvature tensor discovered by Weyl.* It can also be obtained by expressing integrability conditions of the equations of transformation of the components of the projective connection (5.10).† When it is obtained in the latter way, it is expressed in terms of the Π's. That it can be so expressed, follows directly from the observations leading to (6.1). In fact, we can prove

THEOREM 3. *Any projective tensor formed from the components of the affine connection can be expressed in terms of the components of the projective connection.*

By a *projective tensor* is meant a tensor independent of the vector φ. To prove the theorem, we compute the values of the components of the tensor for a normal affine connection. Equations (6.1) show that each Γ can be replaced by the corresponding Π. Since the values of the components of the tensor are by hypothesis independent of the affine connection for which they are computed, the theorem is proved.

Treating (3.7) in an entirely analogous manner, we get

$$C'^{i}_{jkl} = C^{i}_{jkl} - \delta^{i}_{j}(\Phi_{kl} - \Phi_{lk}) + \delta^{i}_{k}\Phi_{jl} - \delta^{i}_{l}\Phi_{jk},$$

$$\Phi_{jk} - \Phi_{kj} = \frac{D_{jk} - D'_{jk}}{n-1},$$

$$\Phi_{jk} = \frac{C_{jk} - C'_{jk}}{n-1} + \frac{D_{jk} - D'_{jk}}{(n-1)^2},$$

$$C_{jk} = C^{\alpha}_{jk\alpha}, \quad D_{jk} = C^{\alpha}_{\alpha jk}.$$

It then is found that the following are the components of tensors independent of φ:

$$\mathfrak{X}^{i}_{jkl} = C^{i}_{jkl} - \frac{\delta^{i}_{j} D_{kl}}{n-1} + \frac{1}{n-1}(\delta^{i}_{k}C_{jl} - \delta^{i}_{l}C_{jk}) + \frac{1}{(n-1)^2}(\delta^{i}_{k}D_{jl} - \delta^{i}_{l}D_{jk}), \tag{7.5}$$

* H. Weyl, Göttinger Nachrichten, 1921, p. 99.

† Cf. a paper by the writer, Proceedings of the National Academy of Sciences, vol. 11 (1925), p. 207.

$$\mathfrak{X}^{\alpha}_{\alpha jk} = \frac{(3-n)D_{jk}}{(n-1)^2} + \frac{C_{jk} - C_{kj}}{n-1}, \tag{7.6}$$

$$\mathfrak{Y}_{jk} = \frac{D_{jk}}{n-1} - \frac{S_{jk}}{n+1}. \tag{7.7}$$

The tensor given by (7.7) can also be obtained by expressing integrability conditions of the equations of transformation of the quantities (5.13).

We find also

$$\mathfrak{W}^{\alpha}_{\alpha kl} = \mathfrak{W}^{\alpha}_{kl\alpha} = \mathfrak{X}^{\alpha}_{kl\alpha} = 0,$$

and that the Weyl tensor has the property of cyclic symmetry, namely,

$$\mathfrak{W}^{i}_{jkl} + \mathfrak{W}^{i}_{klj} + \mathfrak{W}^{i}_{ljk} = 0,$$

whereas the tensor $\mathfrak{X}^{i}_{jkl}$ does not.

We note also that the tensor

$$\mathfrak{Z}^{i}_{jkl} = Z^{i}_{jkl} - \frac{\delta^{i}_{j}}{n} Z^{\alpha}_{\alpha kl}$$

arises by treating (3.4) in the same way that we did (3.6) and (3.7). From a consideration of equations (6.6) we see that this tensor can be expressed in terms of the quantities (5.14), and a theorem similar to Theorem 3 can be stated for projective tensors formed from the H's.

In addition to the above tensors which are independent of φ, we have also the projective invariants given by formulas (5.10) to (5.14). Of these, the quantities Σ^{i}_{jk} present special interest. They constitute an affine (symmetric) connection which is uniquely determined by the linear connection and which is the same for all linear connections yielding the same displaced directions. Since it is readily proved that the Σ's have the same law of transformation as the Γ's, they form a basis for covariant differentiation which is independent of φ. Thus from the projective tensors given in this section we can form infinite sequences of tensors of the same character. To obtain formulas for covariant differentiation we need only replace the Γ's by Σ's in the ordinary formulas.*

8. **Semi-symmetric displacements.** We next inquire under what conditions it is possible to choose φ so that $\Omega^{i}_{jk}=0$. A necessary condition is obviously the vanishing of the tensor $\mathfrak{L}^{i}_{jk}$, that is,

$$\Omega^{i}_{jk} = \frac{\delta^{i}_{j}}{n-1}\Omega^{\alpha}_{\alpha k} + \frac{\delta^{i}_{k}}{n-1}\Omega^{\alpha}_{j\alpha}. \tag{8.1}$$

* Cf. O. Veblen and T. Y. Thomas, loc. cit., p. 571.

This condition is also sufficient; for, as remarked above, the vector φ can be chosen so as to reduce $\mathfrak{L}^i_{jk}$ to Ω^i_{jk}. We can write (8.1) in the form

$$\Omega^i_{jk} = \delta^i_j\psi_k - \delta^i_k\psi_j, \tag{8.2}$$

where ψ_j is a vector. This gives the type of asymmetric displacement studied by Friedmann and Schouten and called by them a semi-symmetric displacement (halb-symmetrische Übertragung).* We can therefore state the following theorems.

THEOREM 4. *A necessary and sufficient condition that an asymmetric displacement be semi-symmetric is the vanishing of the tensor* $\mathfrak{L}^i_{jk}$.

THEOREM 5. *A semi-symmetric displacement can always be replaced by a symmetric one with preservation of displaced directions.*

It is seen from (5.12) that for this displacement, reduced to the symmetric form,

$$\Sigma^i_{jk} = \Gamma^i_{jk}.$$

We can also arrive at the semi-symmetric displacement in another manner. Let the components of the linear connection in a coördinate system y be denoted by $\bar{H}^i_{jk}$, and consider a vector with components η^i in that system. The vector η^i and that obtained from it by displacement, namely

$$\eta^i - \bar{H}^i_{jk}\eta^j dy^k,$$

will have proportional components in the y coördinate system, if, and only if,

$$\bar{H}^i_{jk} = \frac{\delta^i_j}{n}\bar{H}^\alpha_{\alpha k} \tag{8.3}$$

at the given point. The proof of the foregoing statement is entirely analogous to the derivation of equations (5.5). Denoting by p_k the expressions

$$\frac{1}{n}\bar{H}^\alpha_{\alpha\beta}\frac{\partial y^\beta}{\partial x^k}$$

we find from (2.2)

$$H^i_{jk} = \frac{\partial^2 y^\alpha}{\partial x^j \partial x^k}\frac{\partial x^i}{\partial y^\alpha} + \delta^i_j p_k,$$

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* Loc. cit. Cf. also J. A. Schouten, Proceedings, Koninklijke Akademie van Wetenschappen te Amsterdam, vol. 26 (1923), p. 850, where certain applications to physics are indicated. In the latter connection, cf. H. Eyraud, Comptes Rendus, January, 1925, pp. 127–129.

whence by interchange of j and k and subtraction,

$$2\Omega^{i}_{jk} = \delta^{i}_{j} p_k - \delta^{i}_{k} p_j. \tag{8.4}$$

Contraction gives

$$p_k = \frac{2}{n-1} \Omega^{\alpha}_{\alpha k},$$

and substitution in (8.4) shows that (8.1) are satisfied. Hence the displacement is semi-symmetric. Conversely, if (8.1) are fulfilled and we define a system of coördinates with origin at the point $x^i = x^i_0$ by the equations

$$x^i = x^i_0 + y^i - \tfrac{1}{2} (\Sigma^{i}_{\alpha\beta})_0 y^\alpha y^\beta, \tag{8.5}$$

then by means of equations (2.2) written in the form

$$\bar{H}^{i}_{jk} = \frac{\partial^2 x^\alpha}{\partial y^j \partial y^k} \frac{\partial y^i}{\partial x^\alpha} + H^{\alpha}_{\beta\gamma} \frac{\partial y^i}{\partial x} \frac{\partial x^\beta}{\partial y^j} \frac{\partial x^\gamma}{\partial y^k},$$

and the relations

$$\left(\frac{\partial x^i}{\partial y^j}\right)_0 = \delta^{i}_{j}, \quad \left(\frac{\partial^2 x^i}{\partial y^j \partial y^k}\right)_0 = -(\Sigma^{i}_{jk})_0,$$

which are consequences of (8.5), we find that

$$(\bar{H}^{i}_{jk})_0 = \frac{2\delta^{i}_{j}(\Omega^{\alpha}_{\alpha k})_0}{n-1}.$$

Hence (8.3) are satisfied, and it is seen that the quantities (5.14) vanish at the origin in the y coördinate system.

THEOREM 6. *A necessary and sufficient condition in order that a displacement be semi-symmetric is that there exist for each point of space a coördinate system in terms of which any vector at the given point and that arising from it by displacement to a nearby point have proportional components.**

It is to be noted that the coördinate system given by (8.5) is independent of the vector φ and is the geodesic coördinate system for the associated symmetric connection.

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* Weyl, in *Raum, Zeit, Materie*, p. 100, arrives at the affine connection by postulating the existence of a coördinate system in terms of which original and displaced vectors have equal components.

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