ANALYTIC APPROXIMATIONS TO TOPOLOGICAL TRANSFORMATIONS*

BY

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I. INTRODUCTION

A continuous transformation of a two-dimensional region into itself, or into a second such region may be characterized by a pair of continuous functions

$$X = X(x, y), \quad Y = Y(x, y).$$

The continuity of these functions, and the existence of a unique pair of inverse functions are the only restrictions imposed by the topologist. On the other hand, the differential geometer usually demands of the transformations he considers, that the functions defining them be analytic. It is a matter of some interest to investigate the relations connecting these two types of transformation, and in particular to determine whether the general transformations of the first type may not be regarded as in some sense limiting cases of the more special analytic type. This question is here discussed, and we shall prove that *every continuous one-to-one transformation of a two-dimensional region of finite connectivity may be approximated to an arbitrary degree of exactness by an analytic one-to-one transformation.*†

The analogy between the present investigation and the Weierstrass approximation theorem is obvious. The peculiar difficulty of the problem here treated is the need of keeping our approximating transformation one-to-one. This requires a discussion of a combinatorial nature.

II. CONTINUOUS TRANSFORMATIONS

Let the transformation

$$X = X(x, y), \quad Y = Y(x, y)$$

transform the closed region r of the x, y plane into the closed region R of the X, Y plane. Let $X(x, y)$ and $Y(x, y)$ be continuous and single-valued, and let it be possible to write

$$x = x(X, Y), \quad y = y(X, Y),$$

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† This theorem was suggested to the authors by Professor J. W. Alexander.

where these functions are likewise single-valued. We shall then call our transformation of r into R *biunivocal* (or *one-to-one*) and *continuous*. We shall show that the inverse of this transformation is likewise continuous.

The continuity of this inverse transformation from R to r will follow if we show that any sequence of points $P_1, P_2, \cdots$ approaching a limit P is transformed into a sequence $p_1, p_2, \cdots$ approaching a limit p. We form the set p_i, the transform of the set P_i, and notice that, since it is an infinite set in the closed region r, it must contain at least one subsequence with a limit point, say p. That is,

$$\lim_{n=\infty} p_{i_n} = p.$$

From the continuity and biunivocal character of the direct transformation, it follows that

$$\lim_{n=\infty} P_{i_n} = \lim_{n=\infty} P_n = P.$$

Since this argument proves that any subset of the p_i approaching a limit must approach p, we conclude that the set p_i approaches p as a limit.

In order to discuss approximations to transformations, we shall need a measure of the *distance* between two transformations. If S and T are two transformations, with inverses S^{-1} and T^{-1} respectively, then we define the distance between them as the greater of the two numbers

$$\text{maximum } [S(P)-T(P)] \text{ and maximum } [S^{-1}(P)-T^{-1}(P)],$$

where $[A-B]$ denotes the distance from A to B, and the maximum is taken as P ranges over the entire region for which the transformations are defined. To approximate to within a given distance of a transformation T, it is sufficient to approximate for the direct transformation alone. This follows from the fact that if S_n are a series of transformations such that

$$\lim_{n=\infty} \text{maximum } [S_n(P) - T(P)] = 0,$$

then

$$\lim_{n=\infty} \text{maximum } [S_n^{-1}(P) - T^{-1}(P)] = 0.$$

For, we have

$$[S_n^{-1}(P) - T^{-1}(P)] = [T^{-1}TS_n^{-1}(P) - T^{-1}S_nS_n^{-1}(P)],$$

and hence, since S_n is approaching T, and T^{-1} is continuous, our contention follows.

As we shall wish our approximating transformations to extend over the entire regions of definition of the transformation and its inverse, r and R respectively, if we kept to these regions we should have to take the boundary of r into that of R. In general this can not be done by any analytic transformation. We avoid this difficulty by seeking a transformation which shall exist in some region including r, whose inverse exists in some region including R, and which approximates the given transformation in the sense of having its distance from it, as defined above, arbitrarily small, say less than ϵ.

We restrict ourselves to the case where the region r, and hence R, is of finite connectivity, bounded by a number of simple, closed Jordan curves. We embed r and R in two large squares, s and S respectively, so chosen that no point of r is within a distance ϵ of s, and no point of R is within a distance ϵ of S. We then map the region $S-R$ on the region $s-r$ by a continuous transformation, which agrees with the original transformation on the boundary of R and r, and takes the vertices of the first square into those of the second. This is seen to be possible by linear interpolation in the case where r and R are bounded by circles, and the general case reduces to this in virtue of the Jordan-Schoenflies theorem.

If, now, we obtain an analytic biunivocal transformation T_3 which exists in the square s and whose inverse exists in a square inside of S, concentric with it and at distance ϵ from it such that the distance between the transformations T_3 and T_1, the extended continuous transformation, is less than ϵ, we shall have solved our problem for the original regions r and R.

III. The polygonal network

In approximating the extended transformation, T_1, which takes the interior, sides and vertices of a square s into those of a square S, we shall find a simultaneous subdivision of s and S of great service. In this section we shall prove

Lemma 1. *Given a continuous, biunivocal transformation T_1 which takes the interior, sides, and vertices of a square s into those of a square S, we may subdivide s and S into corresponding convex polygons, of which the number meeting at a vertex is one, two, or three according as the vertex is a vertex of the square, a point on a side of the square, or an interior point of the square, respectively, and having the property that any continuous biunivocal transformation T_2 which maps each polygon of s on the corresponding polygon of S is at distance less than $\epsilon/2$ from T_1.*

Let H be selected less than 10ϵ, and $\eta(H)$ be chosen less than H, and so that

$$[P_1 - P_2] < H \text{ if } [p_1 - p_2] < \eta.$$

Let us place on s a square network with mesh $\eta(H)$, and sides parallel to the sides of s, and find its transform under T_1. Let $N_1, N_2, \cdots$ be the intersection points of this transformed network in S, including the points where the network cuts the boundary. Let the least distance between any such point and a point on a side of the net not abutting on it be $3H'$. Surround each of the points $N_1, N_2, \cdots$ in S with a circle with radius H'. No two such circles can intersect. Furthermore, no such circle about N_k can intersect a side of a compartment of the net which does not terminate in N_k.

Consider now the side of the transformed net going from N_k to N_j. Of all the points in which this side intersects the circle about N_k, taken in their natural order on the side, there must be a last point P_1. Similarly, of all the points in which the side intersects the circle about N_j, taken in their natural order on the side, there must be a first point P_2. Form such pairs of points on all the sides of the network. Then there must be a minimum distance, 2δ, from the points on the segment of the network $P_1 P_2$ and the points on any other such segment, or the points on the H' circles about vertices other than N_k and N_j. Take $Q_1, Q_2, \cdots, Q_h$ any sequence of points on this segment P_1P_2 such that the distances $P_1 Q_1, Q_1 Q_2, \cdots, Q_h P_2$ are all less than δ. Replace the side $N_k N_j$ by the broken line $N_k P_1 Q_1 \cdots Q_h P_2 N_j$, and similarly replace every other side of a mesh of the net by such a broken line. Then each mesh of the net will go into a simply connected polygonal region, and no point of this region will lie outside of the original mesh, and abutting meshes. Since the distance from a point in s to a point in an abutting mesh is at most $2\sqrt{2}\eta < 3\eta$, if $\bar{T}_2$ is any continuous biunivocal transformation taking each mesh of the net on s into the corresponding polygonal mesh of the modified network just described, we shall have

$$\text{maximum } [T_1(p) - \bar{T}_2(p)] < 3H.$$

Since the inverse transformation $\bar{T}_2^{-1}$ takes a point in S into the same or an abutting mesh into which T_1^{-1} takes this point, we have

$$\text{maximum } [T_1^{-1}(P) - \bar{T}_2^{-1}(P)] < 3\eta < 3H.$$

We must now go from the polygonal network to convex polygons of the kind described in the lemma. We use T_1^{-1} to transform the vertices of the polygonal network back to s. This gives a series of points $n_k\, p_1\, q_1 \cdots q_h\, p_2\, n_j$ on the segment $n_k\, n_j$ dividing it into consecutive segments which we may put into correspondence with the segments of the broken line $N_k\, P_1\, Q_1$

$\cdots Q_h P_2 N_j$. We thus have a correspondence set up between the periphery of a rectangle in s and of a possibly reentrant polygon in S, for each mesh. We wish to show that it is possible to divide the two figures by straight lines into a finite number of similarly placed triangles.

Consider the polygon. We shall say that a point on its boundary is *visible* from another point situated anywhere in its plane, if the two can be joined by a straight line which does not cut the boundary. For any vertex of the polygon, we may find a point inside the polygon, sufficiently close to it so that this vertex and the two adjacent ones are visible from it. If the polygon is not a triangle, we may shift this point so that from it some fourth vertex, as well as as the three consecutive ones we began with, are visible.

Let us now consider one of the rectangles, and its corresponding, possibly reentrant, polygon. Select a vertex of the rectangle, and the corresponding vertex of the polygon. Find a point inside the polygon from which this vertex, the two adjacent ones, and a fourth are visible, and join it to these four by straight lines. Select a point inside the rectangle, so near the chosen vertex that, when the corresponding straight lines are drawn, all the new polygons will be *convex*. Now proceed similarly with one of the vertices of the convex polygons which is not collinear with its adjacent vertices. At each stage we reduce the number of sides of the polygon dealt with, and keep the new polygons convex, so as to allow the process to be repeated. After a finite number of repetitions, no polygon will have more than three sides, and we shall have succeeded in dividing our corresponding meshes into similarly placed triangles.

The sides of these triangles in the two networks will have a lower bound σ. Surround each node of each network with a circle of radius $\sigma/3$. Draw the inscribed polygon meeting the circle in the points where this circle cuts the lines of the network. These polygons will be convex, as will be polygons formed from the triangles by removing the portions inside these polygons. Further, the network of convex polygons will have the property that precisely three of them meet at each vertex inside the square, two at each vertex on a side of the square, and just one abuts on each vertex of the square.

Suppose, now, that T_2 is a continuous, biunivocal transformation which takes each convex polygon in s into its corresponding convex polygon in S. Points in S arising from a given point in s under T_2 and $\bar{T}_2$ respectively come from the same mesh or adjacent meshes under $\bar{T}_2$. Consequently, under T_1 these points come from meshes at worst next but one, or points in s at distance less that $3\sqrt{2}\eta < 5\eta$, and we shall have

$$\text{maximum } [T_1(p) - T_2(p)] < 5H < \epsilon/2.$$

Similarly, the inverse transformations, T_2^{-1} and T_1^{-1}, take a point in S into two points of s at worst in next but one meshes, and we have

$$\text{maximum } [T_1^{-1}(P) - T_2^{-1}(P)] < 5\eta < 5H < \epsilon/2.$$

Thus the network of convex polygons which we have found satisfies all the conditions of Lemma 1, since the distance between T_2 and T_1 is less than $\epsilon/2$.

IV. Differential transformations

We shall now show that a transformation T_2 of the type mentioned in Lemma 1 may be found which is defined by functions having partial derivatives with certain continuity properties. That is, we shall prove

Lemma 2. *Given two squares, s and S, subdivided into corresponding convex polygons, of which the number meeting at a vertex is one, two, or three according as the vertex is a vertex of the square, a point on a side of the square, or an interior point of the square, respectively; there exists a continuous biunivocal transformation T_2: $X = X_2(x, y)$, $Y = Y_2(x, y)$, for which X_2 and Y_2 possess first partial derivatives with respect to x and y continuous in s, and for which the jacobian of X_2, Y_2 with respect to x and y never vanishes in s, which maps the interior and boundary points of each convex polygon in s on those of the corresponding polygon in S.*

In setting up the differentiable transformation T_2, we shall frequently have occasion to change our system of coördinates. To carry over the properties of the transformation from one of these systems to another, we prove a general result once for all.

If a point transformation of one region on another is given parametrically, the parametric curves $U = U(X, Y)$, $V = V(X, Y)$, $u = u(x, y)$, $v = v(x, y)$ being such that (1) *the (U, V)-(X, Y) relation is one-to-one, so that it may be solved in the form $X = X(U, V)$, $Y = Y(U, V)$, and similarly the (u, v)-(x, y) relation may be solved in the form $x = x(u, v)$, $y = y(u, v)$;* (2) *the functions $U(X, Y)$ and $V(X, Y)$ possess continuous first partial derivatives with respect to X and Y, and similarly the functions $u(x, y)$ and $v(x, y)$ possess continuous first partial derivatives with respect to x and y;* (3) *the jacobian of U, V with respect to X, Y never vanishes, and similarly that of u, v with respect to x, y never vanishes; then a one-to-one relation between U, V and u, v: $U = U(u, v)$, $V = V(u, v)$, for which U and V possess continuous first partial derivatives with respect to u and v, and for which the jacobian of U, V with respect to u, v never vanishes, implies a one-to-one relation between X, Y and x, y: $X = X(x, y)$, $Y = Y(x, y)$, with similar properties at interior points of the regions, and conversely.*

For, by hypothesis (1), X and Y are single-valued functions of U and V. Also, by hypotheses (2) and (3), these functions possess continuous first partial derivatives with respect to U and V.* But, we are also assuming a differentiable one-to-one relation of U, V to u, v, and by hypotheses (1) and (2), there is a differentiable one-to-one relation between u, v and x, y. Since a differentiable, uniform function of a pair of similar functions yields a similar function, we see that X and Y are uniform, differentiable functions of x and y. The property of the jacobian follows from the fact that

$$\frac{\partial(X,Y)}{\partial(x,y)} = \frac{\partial(X,Y)}{\partial(U,V)}\,\frac{\partial(U,V)}{\partial(u,v)}\,\frac{\partial(u,v)}{\partial(x,y)},$$

and

$$1 = \frac{\partial(X,Y)}{\partial(U,V)}\,\frac{\partial(U,V)}{\partial(X,Y)}.$$

The second relation shows that the first factor above can not vanish, being the reciprocal of a finite quantity, and the second and third factors do not vanish by our hypothesis.

The converse is proved in a similar manner.

The principal types of coördinate systems we shall use are three. Those of the first type will be rectangular cartesian coördinates, with various orientations, and choice of origin. These clearly satisfy our three hypotheses, with relation to a fundamental coördinate system, say that with origin at one corner of the square s or S, and axes along two of its sides.

The second type are polar coördinates, with the pole at an arbitrary place. It is evident that these satisfy all our conditions, for any region not including the pole, when the angular coördinate t is reduced modulo 2π, and all the functions involving this coördinate are assumed to be of period 2π with respect to it.

The third type are modified polar coördinates. We begin with a closed curve which surrounds the pole, or origin, with a continuously turning tangent which never coincides with a radius vector. We use a series of curves, similar to this and similarly placed with reference to the pole, as our u-curves, taking the scale such that the length cut off by these curves on any fixed radius vector increases uniformly with u. The v curves are our radii vectores. Analytically, if $r=f(t)$ is the equation of the closed curve in polar coördinates, our parametric curves are related to polar coördinates by the equations

$$u=a+b\,r\,f(t), \quad v=t.$$

* Cf. Osgood, *Funktionentheorie*, Berlin, 1920, p. 69.

We may now show that, in any region not including the pole, these parametric curves satisfy our three hypotheses. In such a region, a pair of values x, y yields a single pair of values r, t and hence a single pair of values u, v. Conversely, a pair u, v fixes t by the second relation and then r by the first and thence x, y. Thus the first hypothesis is satisfied. The partial derivatives with respect to r and t are $\partial u/\partial r = bf(t)$, $\partial u/\partial t = br f'(t)$, $\partial v/\partial r = 0$, $\partial v/\partial t = 1$. These are all continuous, since our assumption about the tangent to the closed curve involves the existence of $f'(t)$, finite and continuous. From the relation of polar to cartesian coördinates, we see that, in a region excluding the pole, the second hypothesis is satisfied. For the third hypothesis, we have

$$\frac{\partial(u,v)}{\partial(x,y)} = \frac{\partial(u,v)}{\partial(r,t)}\,\frac{\partial(r,t)}{\partial(x,y)} = \frac{bf(t)}{r},$$

which does not vanish since $f(t)$ does not vanish.

We next proceed to the explicit construction of the differentiable transformation T_2. Our method is first to set up affine transformations applicable to suitable neighborhoods of the vertices of the polygonal network, then to interpolate between these along the sides of the polygons, and finally to interpolate into their interiors.

Since only one polygon abuts on the vertices of the squares s and S, at these points we may use the identity, or a similarity transformation. We use the latter:

$$X = kx, \quad Y = ky,$$

taking k positive and so small that the adjacent vertices of the network in s are mapped by this transformation on the adjacent sides of the network in S. At vertices of the network on a side of the square, we have to transform this side, and a line meeting it, into the corresponding side and a line meeting it. Taking the vertices in question as our origin, and the sides of the squares as the y axes, so oriented that, with the standard relation of the axes, the positive x axes point inside the squares, we have to take the lines $x=0$, $y=mx$ into $X=0$, $Y=MX$ respectively, where m and M may be positive or negative, by a transformation with positive jacobian. We may use

$$X = kx, \quad Y = ky + k(M - m)x,$$

where we again take k positive and so small that the three adjacent vertices of the network in s are each mapped on the interior of the corresponding adjacent side in S. At the interior vertices of the network, we have to transform three lines forming three convex sectors into three similar lines. Taking

the vertices in question as our origin, and one of the lines as the positive x axis, we have to transform the lines $y=0$, $y=mx$, $y=-nx$ into $Y=0$, $Y=MX$, $Y=-NX$ respectively, where m, n, M and N are positive, by a transformation with positive jacobian. We may use

$$X = kmn(M+N)x + k(mN - nM)y,$$

$$Y = kMN(m+n)y,$$

where we take k, as before, positive and so small that the three adjacent vertices of the network in s are each mapped on the interior of the corresponding adjacent side in S.

We nowhave affine transformations defined at each vertex of our network. We surround each vertex in s by a small triangle, with sides perpendicular to the sides of the network meeting there, and define T_2 inside and on the boundary of this triangle as the affine transformation just referred to. This gives a corresponding set of triangles in S. We must now interpolate between these transformations along the sides of the polygon. Let $p_1\,p_2$ be a side in s, and $P_1\,P_2$ be the corresponding side in S. If we take p_1 as origin, and $p_1\,p_2$ as the positive x axis, and P_1 as origin and $P_1\,P_2$ as the positive X axis, the affine transformation at p_1 will have as its equations

$$X = a_1x + b_1y, \quad Y = c_1y,$$

where a_1 is positive from our choice of direction along the x axis, and c_1 is positive since the jacobian remains positive for all axes with the standard orientation. In terms of the same coördinates, the affine transformation at p_2 will be

$$X = a_2x + b_2y + d_2, \quad Y = c_2y.$$

Here a_2 and c_2 are positive as before. We see that $d_2 = P_1\,P_2 - a_2(p_1\,p_2)$ is positive, since our condition on adjacent vertices makes the image of p_1 under this transformation go into a point between P_1 and P_2, i.e. $(d_2, 0)$, the image of $(0, 0)$, is a point with positive abscissa. We also assume that $a_2 \geqq a_1$, which is permissible, since, if it were not the case, we could reverse the rôles of p_1 and p_2 which would interchange these quantities.

If $x=x_1$ and $x=x_2$ are the equations of the sides of the small triangles which cut $p_1\,p_2$, our problem is to find a differentiable, biunivocal transformation which agrees up to the first partial derivatives with the transformation at p_1 for $x=x_1$ and with that at p_2 for $x=x_2$. We write

$$
\begin{aligned}
X = a_1 \frac{(x-x_2)^2}{(x_1-x_2)^2}\left\{x_1 + \frac{(x_1+x_2)(x-x_1)}{x_2-x_1}\right\} \\
+ a_2 \frac{(x-x_1)^2}{(x_2-x_1)^2}\left\{x_2 + \frac{(x_1+x_2)(x-x_2)}{x_1-x_2}\right\} \\
+ b_1 y \frac{(x-x_2)^2}{(x_1-x_2)^2}\left\{1 + \frac{2(x-x_1)}{x_2-x_1}\right\} \\
+ b_2 y \frac{(x-x_1)^2}{(x_2-x_1)^2}\left\{1 + \frac{2(x-x_2)}{x_1-x_2}\right\} \\
+ d_2 \frac{(x-x_1)^2}{(x_2-x_1)^2}\left\{1 + \frac{2(x-x_2)}{x_1-x_2}\right\},
\end{aligned}
$$

$$
\begin{aligned}
Y = c_1 y \frac{(x-x_2)^2}{(x_1-x_2)^2}\left\{1 + \frac{2(x-x_1)}{x_2-x_1}\right\} \\
+ c_2 y \frac{(x-x_1)^2}{(x_2-x_1)^2}\left\{1 + \frac{2(x-x_2)}{x_1-x_2}\right\}.
\end{aligned}
$$

By direct calculation, we find that, for $x=x_1$, these equations give

$$
\begin{aligned}
X &= a_1x_1 + b_1y, & \partial X/\partial x &= a_1, & \partial X/\partial y &= b_1, \\
Y &= c_1y, & \partial Y/\partial x &= 0, & \partial Y/\partial y &= c_1.
\end{aligned}
$$

For $x=x_2$, they give

$$
\begin{aligned}
X &= a_2x_2 + b_2y + d_2, & \partial X/\partial x &= a_2, & \partial X/\partial y &= b_2, \\
Y &= c_2y, & \partial Y/\partial x &= 0, & \partial Y/\partial y &= c_2.
\end{aligned}
$$

Thus this transformation joins to those at p_1 and p_2 as desired. We also note that, for $y=0$, we have

$$
\begin{aligned}
\partial X/\partial x = a_1 \frac{(x_2-x)}{(x_2-x_1)^3}\left\{(x_2-x_1)^2 - 3(x_1+x_2)(x-x_1)\right\} \\
+ a_2 \frac{(x-x_1)}{(x_2-x_1)^3}\left\{(x_2-x_1)^2 + 3(x_1+x_2)(x_2-x)\right\} \\
+ d_2 \frac{6(x-x_1)(x_2-x)}{(x_2-x_1)^3}.
\end{aligned}
$$

For $0<x_1<x<x_2$, the coefficients of a_2 and d_2 are positive. Since the sum of the coefficients of a_1 and a_2 is unity, and we have arranged matters so that $a_2 \geqq a_1 > 0$, and $d_2 > 0$, we see that the sum of the three terms is positive.

Hence, along the line $y=0$, X is a monotonic function of x, and the mapping is one-to-one. Again, for $y=0$, we have

$$\partial Y/\partial x = 0,$$

$$\partial Y/\partial y = c_1 \frac{(x-x_2)^2}{(x_1-x_2)^2}\left\{1+\frac{2(x-x_1)}{x_2-x_1}\right\} + c_2\frac{(x-x_1)^2}{(x_2-x_1)^2}\left\{1+\frac{2(x-x_2)}{x_1-x_2}\right\}.$$

For $0<x_1<x<x_2$, the coefficients of c_1 and c_2 are positive, and since c_1 and c_2 are positive, $\partial Y/\partial y$ is positive. Hence the jacobian of X, Y with respect to x, y for the cubic transformation under discussion is positive along the line $y=0$. Thus the transformation is one-to-one im kleinen, and, since it is one-to-one along the line $y=0$, and continuous, we may find a strip including this line in which it is one-to-one im grossen. Furthermore, since the transformation has continuous derivatives, if this strip is taken sufficiently small, the jacobian of the transformation will be positive throughout the strip.

We have now to interpolate into the interior of our polygons. In each polygon of the figure s, we draw a differentiable closed curve, close to the boundary of the polygon. We may form it of straight lines parallel to the sides between the x_1 and x_2 used above in the strips, and circular arcs in the triangular sectors. If it is taken sufficiently close to the polygon, it will be visible from some point c inside the polygon, i.e. such that each radius vector drawn from c to the curve cuts it in a single point, at an angle not zero. Thus it may be used as the closed curve of a system of modified polar coördinates u, v of the kind previously described, with the pole at c. Consider the transforms of these u curves in S, under the cubic and affine transformations. In a sufficiently narrow strip, these curves will be visible from some point C inside the polygon. In this strip we shall take as parametric curves the curves $U=$const., the transforms of $u=$const., and the curves $T=$const., the radii vectores drawn from C, T being the angle made with a fixed vector. To show that these are admissible parameters, we note that the one-to-one relation of U, T to X, Y follows from the fact that the U curves are visible from C, so that a pair of values U, T gives a single point X, Y. On the other hand, a pair X, Y clearly gives one value of T, and a single U resulting from the u obtained from x, y. The U and T curves clearly are differentiable. For the condition on the jacobian, we have

$$\frac{\partial(U,T)}{\partial(X,Y)} = \frac{\partial(U,T)}{\partial(U,V)}\,\frac{\partial(U,V)}{\partial(x,y)}\,\frac{\partial(x,y)}{\partial(X,Y)},$$

where the curves $V=$const. are the transforms of $v=$const. under the affine and cubic transformations. From the definition of the U, V curves, the

jacobian of U, V with respect to x, y is the same as that of u, v with respect to x, y and hence can not vanish, since u, v were admissible parameters. The last jacobian does not vanish, since it relates to the cubic and affine transformations. For the first factor we have

$$\frac{\partial(U,T)}{\partial(U,V)} = \frac{\partial T}{\partial V}.$$

If this vanished, we should either have $\partial T/\partial U = 0$ at the same time, in which case since U and V are admissible parameters, we would be at a singular point on the curve (or in its representation), or we should have $\partial T/\partial U \neq 0$, and the T and U curves would be tangent. The first case is inadmissible, since the angular coördinate represents the radii without singularities in any admissible system of coördinates, and the second is excluded since the U curves we are using are all visible from C. Thus the U, T parameters satisfy our three conditions, and form an admissible system.

We choose the scales so that the curves $u=0$ and $u=1$ are respectively the inner and outer boundaries of a strip entirely inside the region where our affine and cubic transformations are valid, while their transforms $U=0$ and $U=1$ are the boundaries of a strip inside which the U, T coördinates are admissible. In these coördinates, the affine and cubic transformations will have the equations

$$u = U, \quad t = F(T,U),$$

where we replace v by t, its equal. We note that

$$\frac{\partial F}{\partial T} = \frac{\partial(u,t)}{\partial(U,T)} > 0.$$

In the strip between $u=1$ and the polygon, we define T_2 as the affine and cubic transformation, which is that written above near $u=1$. In the strip from $u=1$ to $u=0$, we use a derived transformation, such that it agrees with this transformation up to the first derivatives at $u=1$, and takes the curves $t=$const. into curves *tangent to the curves* $T=$const. at $u=0$. The derived transformation is

$$u = U, \quad t = F(T, 2U^2 - U^3) = F_1(T,U).$$

This is one-to-one for values of u, or U, between 0 and 1. For, since $u=U$, $t=F(T, U)$ is biunivocal for these values, for any value of u or U in this range, $t=F(T, U)$ gives a one-to-one relation of t to T. But $2U^2-U^3$ takes on the values between 0 and 1 when U does, and accordingly for fixed u or U, $t=F_1(T, U)$ gives a one-to-one relation of t to T. The tangency prop-

erty required follows from the fact that $(\partial F_1/\partial U)_{U=0}=0$, while $(\partial F_1/\partial T)_{U=0} = (\partial F/\partial T)_{U=0}>0$. For $u=U=1$, we easily see that the derived transformation and its derivatives agree with the earlier transformation. Its jacobian is $\partial F_1/\partial T=\partial F/\partial T>0$.

In continuing our interpolation, it will be convenient to use polar coördinates. As our lemma on systems of coördinates only applies to interior points, we must extend our transformation beyond the inner boundary, $u=U=0$, to make this curve an interior curve. We do this by putting

$$u = U, \quad t = F_1(T,0),$$

in a narrow strip inside the curves $u=U=0$. This joins with our preceding transformation along these curves, continuity of the derivatives being retained, since $(\partial F_1/\partial U)_{U=0}=0$. Its jacobian is $(\partial F_1/\partial T)_{U=0}>0$.

We next draw two circles of radius a about the points c and C as centers respectively, taking a so small that these circles lie entirely inside the curves $u=0$ and $U=0$. To extend our transformation so that it maps the region between $u=0$ and the first circle on to that between $U=0$ and the second, we introduce polar coördinates t, r and T, R. Let the equations of $u=0$ and $U=0$ in these coördinates be

$$r = w(t) = v(T), \quad R = V(T),$$

where, as in the following equations, all the functions of t and T are of period 2π. Along these curves we have

$$\partial t/\partial R = 0, \quad \partial t/\partial T = (\partial F_1/\partial T)_{U=0}, \quad \partial r/\partial R = E(T), \quad \partial r/\partial T = G(T).$$

Since $dr/dT=(\partial r/\partial R)\ (dR/dT)+\partial r/\partial T$, we have $v'(T)=E(T)V'(T)+G(T)$. We put

$$t = F_1(T,0), \quad r = G(R,T),$$

where $G(R, T)$ is periodic in T, has $\partial G/\partial R>0$, and satisfies the further conditions

$$G(a,T) = a, \quad (\partial G(R,T)/\partial R)_{R=a} = 1,$$
$$G(V,T) = v, \quad (\partial G(R,T)/\partial R)_{R=V} = E(T).$$

The determination of $G(R, T)$ for a fixed value of T reduces to the problem of finding a function of R, with positive derivative, which, with its derivative takes two positive assigned values at each end of an interval. The function $E(T)$ is positive, since

$$\left(\frac{\partial(r,t)}{\partial(R,T)}\right)_{U=0} = E(T)\left(\frac{\partial F_1}{\partial T}\right)_{U=0} > 0, \text{ and } \left(\frac{\partial F_1}{\partial T}\right)_{U=0} > 0.$$

We define the auxiliary function $H(R)$ as follows:

$$\begin{aligned} H(R) &= 1 + (R - a)(M - 1)/k, & a \leqq R \leqq a + k, \\ H(R) &= M, & a + k \leqq R \leqq V - k, \\ H(R) &= E + (V - R)(M - E)/k, & V - k \leqq R \leqq V. \end{aligned}$$

Here we take

$$M = \frac{v - a - \frac{1}{2}(1 + E)k}{V - a - k},$$

and k a positive number so small that

$$k < \tfrac{1}{2}(V-a), \quad k < \frac{2(v-a)}{1+E}.$$

This makes $H(R)$ a continuous, broken-line function, with positive ordinate throughout, and such that the area under it is $v-a$. Also, $H(a)=1$ and $H(V)=E$. Consequently if we put

$$G(R,T) = a + \int_a^R H(R)dR,$$

it will satisfy all the conditions required of it above. We may take k an absolute constant, since the right members of the inequalities it satisfies are positive continuous periodic functions of T, and hence have positive minima.

We note that the transformation given above maps our ring shaped regions on one another in a one-to-one manner, since the relation $t=F_1(T,0)$ is clearly one-to-one, and if either t or T is fixed, the second relation $r=G(R, T)$ is one-to-one, since $\partial G/\partial R>0$. On the outer boundary it agrees with our former transformation, since $\partial r/\partial R = E(T)$ there, and $t=F_1(T, 0)$, $r=G(V, T)=v(T)$. From these we deduce

$$dr/dT = (\partial r/\partial R)(dR/dT) + \partial r/\partial T, \text{ or } v'(T) = E(T)V'(T) + \partial r/\partial T,$$

which, combined with the earlier relation $v'(T)=E(T)V'(T)+G(T)$, shows that $\partial r/\partial T=G(T)$. The transformation has a positive jacobian, since this equals $(\partial F_1/\partial T)_{U=0}(\partial G/\partial R)$.

In the ring between $R=a$ and $R=a/2$, corresponding to $r=a$, $r=a/2$, we put

$$\begin{aligned} r &= R, \\ t &= 4T(R - a)^2(4R - a)/a^3 + F_1(T,0)(2R - a)^2(5a - 4R)/a^3. \end{aligned}$$

For $R=a$, this gives us $r=a$, $\partial r/\partial R=1$, $\partial r/\partial T=0$, $t=F_1(T, 0)$, $\partial t/\partial T=(\partial F_1/\partial T)_{U=0}$, $\partial t/\partial R=0$. These values agree with those of the last trans-

formation at this circle, all except the value of $\partial r/\partial R$ following directly, and that from the relation

$$dr/dT = (\partial r/\partial R)(dR/dT) + \partial r/\partial T, \quad \text{or} \quad 0 = \partial r/\partial T.$$

For $R=a/2$, the equations give $r=a/2$, $\partial r/\partial R=1$, $\partial r/\partial T=0$, $t=T$, $\partial t/\partial T=1$, $\partial t/\partial R=0$. Thus it joins on to the identity along this circle. The transformation is one-to-one, since, for fixed r or R, the t-to-T relation is one-to-one. For, we have

$$\frac{\partial t}{\partial T} = \frac{4(R-a)^2(4R-a)}{a^3} + \left(\frac{\partial F_1}{\partial T}\right)_{U=0} \frac{(2R-a)^2(5a-4R)}{a^3},$$

which is clearly positive for $a/2<R<a$. As this is the value of the jacobian, we see that our transformation has a positive jacobian.

Inside the circles $r=a/2$, $R=a/2$, we use the identity transformation

$$r = R, \quad t = T,$$

to complete the mapping. As the identity has the same form in all coördinates, the singularity of the coördinates at the pole is of no importance.

There is now a transformation for each polygon, given in polar coördinates with pole in the polygon, and defined in a region including the polygon in its interior. Thus, by our lemma on systems of coördinates, it may be expressed in terms of the particular set of cartesian coördinates used for T_2. The totality of these yields a differentiable transformation T_2 of the kind required.

V. Analytic transformations

Weierstrass* has given an example of an analytic function approximating to any degree of accuracy a given continuous function of two variables, $f(x, y)$, in a finite region R. This approximating function is

$$f_k(x,y) = \frac{1}{k^2\pi} \int_R \int f(X,Y) e^{-((X-x)/k)^2-((Y-y)/k)^2} dX dY,$$

when k is sufficiently large. It is easy to show that if $f(x, y)$ possesses continuous first partial derivatives over any closed region D interior to R, we have

$$\lim_{k=\infty} f_k(x,y) = f(x,y)\ ;\ \lim_{k=\infty} (\partial f_k/\partial x) = \partial f/\partial x\ ;\ \lim_{k=\infty} (\partial f_k/\partial y) = \partial f/\partial y$$

uniformly over D.

* Weierstrass, Berliner Sitzungsberichte, 1885.

This result enables us to approximate to any biunivocal, differentiable transformation by an analytic transformation, the approximation applying to the derivatives, but the analytic transformation thus obtained is not necessarily one-to-one. For the case in which the differentiable transformation has a non-vanishing jacobian, we may show that, when the approximation is sufficiently close, it is biunivocal, and shall prove the following lemma:

LEMMA 3. *If*

$$X = X(x,y),\ Y = Y(x,y)$$

is a continuous, biunivocal transformation, holding in the closed convex region s, for which X and Y possess first partial derivatives with respect to x and y, continuous in s, and for which the jacobian of X and Y with respect to x and y never vanishes in s, and

$$X = X_a(x,y),\ Y = Y_a(x,y)$$

are a pair of differentiable functions, which approximate X and Y, and whose first partial derivatives approximate those of X and Y, uniformly in s, then, if the approximation is sufficiently close, these equations represent a biunivocal transformation for the interior of s.

We note from the form of the approximating functions, that they take each point x, y into a point X_a, Y_a, but may take two points x, y into the same point X_a, Y_a. We must show this to be impossible when the approximation is sufficiently good. We shall divide the proof into two parts, first proving the lemma for the case in which the given functions X, Y are linear functions, and then for the general case.

1. Let

$$X = ax + by + e,\ Y = cx + dy + f$$

be the given transformation. To predict the necessary degree of approximation, we make some preliminary calculations. Consider the radius drawn out from any fixed point p_1 to a variable point p, which moves out along this radius in s. If the line makes an angle A with the X axis, we shall have

$$X - X_1 = a(x - x_1) + b(y - y_1) = r(a \cos A + b \sin A),$$
$$Y - Y_1 = c(x - x_1) + d(y - y_1) = r(c \cos A + d \sin A).$$

For the distance in the transformed figure, S, we have

$$R^2 = (X - X_1)^2 + (Y - Y_1)^2 = r^2[(a^2 + c^2) \cos^2 A + 2(ab + cd) \cos A \sin A + (b^2 + d^2) \sin^2 A].$$

As A varies, the ratio R^2/r^2 varies between the maximum and minimum values

$$\tfrac{1}{2}(a^2+b^2+c^2+d^2) \pm \tfrac{1}{2}\sqrt{(a^2+c^2-b^2-d^2)^2+4(ab+cd)^2}.$$

The product of these is $(ad-bc)^2$, which is the square of the jacobian. This might have been predicted since the square root of the product of the maximum and minimum values, and the jacobian both measure the ratio of the area of a circle in s to that of its corresponding ellipse in S. As the jacobian does not vanish, neither the maximum nor minimum value is zero. Let the value of the minimum be q^2. Then, for the affine transformation, the minimum value of R/r is $q>0$.

Now consider the image of the variable radius $p_1\,p$ under the approximating transformation. Let the transformed radius be $P_1\,P$, where $P_1=(X_1, Y_1)$ and $P=(X, Y)$. We have for its length

$$R = \sqrt{(X-X_1)^2+(Y-Y_1)^2}.$$

We calculate its derivative with respect to r, and find

$$\frac{dR}{dr} = \frac{[(X-X_1)(\partial X/\partial x)+(Y-Y_1)(\partial Y/\partial x)]\cos A}{\sqrt{(X-X_1)^2+(Y-Y_1)^2}} + \frac{[(X-X_1)(\partial X/\partial y)+(Y-Y_1)(\partial Y/\partial y)]\sin A}{\sqrt{(X-X_1)^2+(Y-Y_1)^2}}.$$

We may write this in a form involving derivatives only, by noting that $(X-X_1)$ and $(Y-Y_1)$ enter, essentially, through their ratio, only. But, since the image of the straight line $p_1\,p$ is a differentiable curve, having $P_1\,P$ as a chord, there is some intermediate point on this curve for which the tangent is parallel to the chord. If this point is $\overline{P}$ we shall have

$$\frac{Y-Y_1}{X-X_1} = \frac{d\overline{Y}}{d\overline{X}} = \frac{(\partial\overline{Y}/\partial x)\cos A + (\partial\overline{Y}/\partial y)\sin A}{(\partial\overline{X}/\partial x)\cos A + (\partial\overline{X}/\partial y)\sin A}.$$

We notice that both terms of this ratio can not be zero unless the jacobian of the transformation is zero at $\overline{P}$. We may now write for the derivative

$$\frac{dR}{dr} = \frac{(X_x\cos A+X_y\sin A)(\overline{X}_x\cos A+\overline{X}_y\sin A)}{\sqrt{(\overline{X}_x\cos A+\overline{X}_y\sin A)^2+(\overline{Y}_x\cos A+\overline{Y}_y\sin A)^2}} + \frac{(Y_x\cos A+Y_y\sin A)(\overline{Y}_x\cos A+\overline{Y}_y\sin A)}{\sqrt{(\overline{X}_x\cos A+\overline{X}_y\sin A)^2+(\overline{Y}_x\cos A+\overline{Y}_y\sin A)^2}}$$

where the subscripts indicate partial derivatives. If in this expression we replace the partial derivatives X_x, X_y, Y_x, Y_y, as well as $\overline{X}_x$, $\overline{X}_y$, $\overline{Y}_x$, $\overline{Y}_y$,

by a, b, c, d, their values for the affine transformation, it reduces to R/r for the affine transformation, or

$$\sqrt{(a^2 + c^2)\cos^2 A + 2(ab + cd)\cos A \sin A + (b^2 + d^2)\sin^2 A} \geqq q > 0.$$

As the expression is a continuous function of these partial derivatives for values in some neighborhood of a, b, c, d, a u_A may be found such that, if these partial derivatives are all within u_A of their values for the affine transformation, the expression will be $>q/2$, for a given A. But, as the function is continuous in A, for values in some neighborhood of this given value, it will be $>q/4$. By the Heine-Borel theorem, a finite number of such neighborhoods may be found which include all values of A, and hence a number u may be found less than any of the corresponding u_A. If the approximation has its partial derivatives within u of those of the affine transformation, for any A, we shall have $dR/dr > q/4 > 0$.

We now assert that a transformation approximating the affine transformation to this degree of accuracy is necessarily one-to-one in s. For, if it were not, it would take two points in s, p_2 and p_3, into the same point P_2. Since s is convex, $p_2\, p_3$ lies entirely in s. Let p_1 be a point on the extended line $p_3\, p_2$, and P_1 its transform. Let a variable point p, at distance r from p_1, move along $p_2\, p_3$, and let the distance of its transform, P, from P_1 be R. As r changes from $p_1\, p_2$ to $p_1 p_3$, R changes from $P_1\, P_2$ back to $P_1 P_2$. Hence, at some point between p_2 and p_3, we must have $dR/dr = 0$. But this is impossible, since $dR/dr > q/4 > 0$.

This proves the lemma for the affine case. We notice that here the limit on the approximation involves only the derivatives.

2. Now consider a general differentiable transformation, with nonvanishing jacobian in a closed region s. At every point p in this region, we form an approximating linear transformation, having the same derivatives as the given transformation has at this point. By the first part of this proof, we may find a u_p for this linear transformation such that any transformation whose derivatives approximate its derivatives to within u_p will be one-to-one. Since the given transformation has continuous derivatives, we may surround each point by a circle of radius v_p such that, in this circle, the partial derivatives never differ from those at the center by more than $u_p/2$. Surround each point p by a circle of radius $v_p/2$. By the Heine-Borel theorem, we may find a finite number of such circles such that each point of s is inside some one of them. Let u be the minimum u_p for any of these points, and v the minimum v_p. We see that, if any point in s is taken as the center of a circle of radius $v/2$, the partial derivatives of the given transformation will not

differ by more than $u/2$ from some linear transformation for which u may be taken as the u of part one.

Since the given transformation is continuous in s, its inverse is, and from the nature of s we may find a V such that, if the images of two points are at distance less than V, the points themselves will be at distance less than $v/4$. We now assert that any approximating transformation which takes a point in s into a point at distance less than V from its transform under the given transformation, and whose derivatives are within $u/2$ of those of the given transformation, is necessarily one-to-one. For, suppose it took two points p_1 and p_2 into the same point P'. Let the transforms of these points under the given transformation be P_1 and P_2. From the nature of our approximation, P_1P' and P_2P' are both less than V. Hence P_1P_2 is less than $2V$, and p_1p_2 is less than $v/2$. Draw a circle with center p_1 and radius $v/2$. If this circle does not lie entirely in s, the part of it contained in s is convex, since s is convex, and hence this part may be used as the region of part one. But, in this circle, the derivatives of the given transformation do not differ by more than $u/2$ from those of some linear transformation for which this is the u of part one. But, since the derivatives of the approximating transformation are within $u/2$ of those of the given transformation, they are within u of those of this linear transformation. Thus the approximating transformation is one-to-one in this circle, and can not take p_1 and p_2 into the same point. Thus our lemma is proved.

We are now in a position to prove

THEOREM I. *Given a pair of continuous functions*

$$X = X(x,y), \quad Y = Y(x,y),$$

defining a continuous, biunivocal transformation of some closed, two-dimensional region r of finite connectivity of the x, y plane, into a closed region R of the X, Y plane, and a positive constant ϵ, there exists a pair of analytic functions

$$X = X_a(x,y), \quad Y = Y_a(x,y),$$

defining a biunivocal transformation of some closed region of the x, y plane including r, into a closed region of the X, Y plane including R, whose distance from the given transformation is less than ϵ.

For, from the discussion of §2 we may embed the regions r and R in two squares s and S, and find a continuous transformation T_1 of s on S, which agrees with the given transformation in r, and takes the vertices of s into those of S. Then we dissect the squares into convex polygons, by Lemma 1, such that any continuous biunivocal transformation T_2 which maps

each polygon of s into the corresponding polygon of S is at distance less that $\epsilon/2$ from T_1. By Lemma 2, we then find a pair of functions

$$X = X_2(x,y), \quad Y = Y_2(x,y),$$

with continuous first partial derivatives, whose jacobian never vanishes in s, and which has the properties demanded of T_2. Consequently its distance from T_1 is less than $\epsilon/2$. We next apply Lemma 3 to this transformation to find a V and a $u/2$ such that, if a pair of differentiable functions are given which approximate X and Y to within $V/\sqrt{2}$, and whose derivatives approximate those of X and Y to within $u/2$, they define a one-to-one transformation in s. Also we find a W such that the transforms of any two points in S whose distance is less than W, under the inverse of T_2, are at distance less than $\epsilon/2$. Finally, we use the method of Weierstrass recalled at the beginning of this section to find a pair of analytic functions

$$X = X_3(x,y), \quad Y = Y_3(x,y),$$

which approximate $X_2(x, y)$ and $Y_2(x, y)$ to within $\epsilon/2\sqrt{2}$, $V/\sqrt{2}$, $W/\sqrt{2}$, and whose derivatives approximate those of these functions to within $u/2$. These equations define the analytic transformation whose existence the theorem asserts. For, by Lemma 3, it is one-to-one in s. From the choice of the approximation, and of W, we have

$$[T_3(p) - T_2(p)] < \epsilon/2,$$
$$[T_3^{-1}(P) - T_2^{-1}(P)] = [T_2^{-1}T_2T_3^{-1}(P) - T_2^{-1}T_3T_3^{-1}(P)] < \epsilon/2.$$

Thus the distance of the transformation T_3 to T_2 is less than $\epsilon/2$, and since that of T_2 to T_1 is less than $\epsilon/2$, the distance from T_3 to T_1 is less than ϵ, in s. Since no point of R is within ϵ of the boundary of S, T_3 maps s on a region including R, and hence T_3^{-1} is defined in a region including R. As the distance from T_3 to T_1 is less than ϵ wherever both are defined, we see that T_3 satisfies all the conditions demanded of it in the theorem.

COROLLARY. *The functions $X_3(x, y)$ and $Y_3(x, y)$ of the theorem may be taken as a pair of polynomials.*

For we may approximate to any analytic function, as well as to its first derivatives, by a polynomial.

VI. TRANSFORMATIONS ON A SPHERE

Our argument may be extended to transformations of a sphere into itself with but slight modification. A natural definition of an analytic transformation on a sphere is *a transformation which is expressed by analytic functions*

in terms of some standard set of coördinates, such as the polar coördinates ϕ and θ. To avoid trouble at the poles, we require that the functions be analytic and single-valued in ϕ and θ for *every* possible choice of the axis. In our discussion it will be more convenient to use an equivalent form of this definition, and note that any transformation of a three-dimensional region, not necessarily simply connected, including the points of the sphere as interior points, analytic in terms of cartesian coördinates, which leaves the sphere invariant, determines a transformation on the sphere which is analytic in the above defined sense, and conversely any transformation analytic in the earlier sense may be extended to give an analytic space transformation of the kind here used. With this definition, we may formulate

THEOREM II. *Given a set of three continuous functions*

$$X = X(x,y,z),\ Y = Y(x,y,z),\ Z = Z(x,y,z),$$

which leave the unit sphere invariant,

$$X^2 + Y^2 + Z^2 = 1 \text{ if } x^2 + y^2 + z^2 = 1,$$

and define a continuous biunivocal transformation on this sphere, and a positive number ϵ, *there exists a set of three analytic functions*

$$X = X_a(x,y,z),\ Y = Y_a(x,y,z),\ Z = Z_a(x,y,z),$$

which leave the unit sphere invariant,

$$X_a^2 + Y_a^2 + Z_a^2 = 1 \text{ if } x^2 + y^2 + z^2 = 1,$$

and define a biunivocal transformation of the unit sphere into itself whose distance from the given transformation is less than ϵ.

We divide the sphere by two networks of convex polygons, whose sides are great circles, the number of polygons meeting at each vertex being three, which have the property that any continuous biunivocal transformation T_2 of the sphere into itself, which maps each polygon of the first network on the corresponding polygon of the second network, is at distance less than $\epsilon/2$ from the given transformation, T_1. This is shown to be possible by the method used to prove Lemma I. To set up a differentiable transformation T_2, we begin with the neighborhood of a vertex and its transform, gnomonically project these neighborhoods on the tangent planes at the vertices in question, and set up affine transformations in these planes as in the proof of Lemma 2. They project back into differentiable transformations on the sphere, which will yield affine transformations when projected from the center on any tangent plane. To interpolate along a side, we work in the

tangent plane at its midpoint, and in that at the midpoint of the corresponding side, and form the cubic transformations of Lemma 2. These project into differentiable transformations on the sphere. For the interpolation into the interior of a polygon, we work in the tangent plane at some interior point of it, and that at some interior point of its transform, and proceed as in Lemma 2. We thus obtain a differentiable transformation of the sphere into itself which is one-to-one, possesses a non-vanishing jacobian with respect to any system of ϕ, θ coördinates, and approximating T_1 to within $\epsilon/2$. We readily extend this to a region between two spheres concentric with the unit sphere, and enclosing it, by adding to the ϕ, θ transformation the equation $R=r$. This transformation, when expressed in cartesian coördinates, gives three differentiable functions

$$X = X_2(x,y,z),\ Y = Y_2(x,y,z),\ Z = Z_2(x,y,z),$$

such that $X_1^2+Y_1^2+Z_1^2=1$ if $x^2+y^2+z^2=1$.

We shall now use the considerations of Lemma 3 to show that any differentiable transformation in three variables which approximates this one, both as to coördinates and their derivatives, sufficiently closely, will, when the transforms of points on the unit sphere under it are projected gnomonically on the sphere, yield a one-to-one transformation of the sphere. At every point p of the sphere we draw the tangent plane, and project the points in some neighborhood of p on this tangent plane. Similarly we project their transforms on the tangent plane at the transformed point P. This gives a differentiable transformation of a part of one of these planes on the other. We form an approximating linear transformation, having the same derivatives as this transformation at p. By part 1 of Lemma 3, we may find a u_p for this linear transformation such that any transformation of the plane whose derivatives approximate its derivatives to within u_p will be one-to-one. We surround each point p by a circle of radius v_p such that, in the circle which is the projection of this on the tangent plane, the partial derivatives of the plane transformation never differ from those at the center by more than $u_p/2$. We then surround each point p on the sphere by a circle of radius $v_p/2$. We pick out a finite number of such circles with centers at p_i such that each point on the sphere is inside some one of them. If, now, u is the minimum u_p and v the minimum v_p for the corresponding points, if any point q on the sphere is taken as the center of a circle of radius $v/2$, there is a point p_i on the sphere within a distance $v/2$ of q, such that if the points inside the circle at q are projected on the tangent plane to the sphere at p, and their transforms are projected on the tangent plane at P_i, the transform of p_i, a plane differentiable transformation will arise with the property that any

other differentiable transformation whose partial derivatives differ from those for it by less than $u/2$ will be one-to-one.

We now find a V for the transformation on the sphere such that, if the images of two points are at distance less than V the points themselves will be at distance less than $v/4$. Now consider any approximating differentiable transformation, T_3. Project the points in a circle of radius $v/2$ about q on the tangent plane at the nearest p_i, and their transforms under T_3 on the tangent plane at P_i, the transform of p_i under the given transformation. A plane differentiable transformation will arise whose partial derivatives are seen to be continuous functions of the partial derivatives of T_3, and certain quantities connected with the given transformation and its derivatives at p_i. Consequently we may find a u' such that, if the derivatives of T_3 are within u' of those of the given transformation, the derivatives of the plane transformation just found will be within $u/2$ of the plane transformation previously found for p_i. Thus the plane transformation will be one-to-one inside the circle of radius $v/2$. If T_3 in addition approximates the given transformation to within V in distance, it will project on the sphere into a transformation one-to-one throughout. For, if it took two points p_1 and p_2 into the same point P', whose transforms under the given transformation were P_1 and P_2, we should have P_1P' and P_2P' less than V, and hence P_1P_2 less than $2V$. Thus p_1p_2 would be less than $v/2$, and a circle with center at p_1 and radius $v/2$ could be drawn in which the projected transformation on the tangent plane at the nearest p_i would be one-to-one, contradicting the assumption that p_1 and p_2 gave a single point P'.

We now find, by the Weierstrass method quoted in Section 5, applied to functions of three variables, a set of three analytic functions $X_3(x, y, z)$, $Y_3(x, y, z)$, $Z_3(x, y, z)$ approximating the functions defining the differentiable transformation, $X_2(x, y, z)$, $Y_2(x, y, z)$, $Z_2(x, y, z)$, to within $\epsilon/4\sqrt{3}$, $V/\sqrt{3}$, $W/\sqrt{3}$, and whose derivatives approximate those of these functions to within u', where W is chosen such that two points of the sphere whose distance is less than W, under the inverse of the differentiable transformation, go into two points whose distance is less than $\epsilon/4$. We write, finally,

$$X = \frac{X_3(x,y,z)}{\sqrt{X_3^2 + Y_3^2 + Z_3^2}},$$

$$Y = \frac{Y_3(x,y,z)}{\sqrt{X_3^2 + Y_3^2 + Z_3^2}},$$

$$Z = \frac{Z_3(x,y,z)}{\sqrt{X_3^2 + Y_3^2 + Z_3^2}},$$

as the approximating analytic transformation. It clearly takes the sphere into itself, is one-to-one, and is at distance less than ϵ from the given transformation T_1, since it is at distance less than $\epsilon/2$ from T_2.

COROLLARY. *Given a continuous, biunivocal transformation of any simply connected closed surface into itself, a positive number ϵ and any set of curves on the surface deformable into meridians and parallels, we may find an analytic transformation, the analyticity being given in terms of the given curves, whose distance from the given transformation, measured in the same terms, is less than ϵ.*

For we have merely to deform the closed surface into a sphere, and the curves into meridians and parallels, and apply Theorem II.

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