A theory of functions of a general variable is due to E. H. Moore.† By a general variable is meant a variable of which the range is a class of elements

\[ \mathcal{Q} = [q] \]

entirely unconditioned. Particular instances of the theory may be obtained by specializing the class \( \mathcal{Q} \). For example, the class \( \mathcal{Q} \) may consist of a finite number of elements, a denumerable infinitude of elements, or a continuous infinitude of elements. The elements themselves may be numbers, real, complex, or hypercomplex; or they may be without numerical character.

A real (single-valued) function \( \mu \) on a general range \( \mathcal{Q} \) is a correspondence between the elements of \( \mathcal{Q} \) and a class of real numbers, such that for every element \( q \) of \( \mathcal{Q} \) there is a definite corresponding real number, notationally \( \mu(q) \).

A property of such a function which is in no way dependent for its definition on any special character which the range \( \mathcal{Q} \) may have in special instances is said to be a property of general reference. For example, a function may be: (a) everywhere zero on \( \mathcal{Q} \), or (b) everywhere positive on \( \mathcal{Q} \), or (c) everywhere negative on \( \mathcal{Q} \), or (d) somewhere positive and nowhere negative on \( \mathcal{Q} \), or (e) somewhere negative and nowhere positive on \( \mathcal{Q} \).

We shall use the following symbolic statements to indicate that a function \( \mu \) has these properties respectively:

(a) \( \mu = 0 (\mathcal{Q}) \),
(b) \( \mu > 0 (\mathcal{Q}) \),
(c) \( \mu < 0 (\mathcal{Q}) \),
(d) \( \mu \geq 0 (\mathcal{Q}) \),
(e) \( \mu \leq 0 (\mathcal{Q}) \).

In the present paper we shall be particularly interested in functions which have the property (d) or the property (e). A function which has either of these properties will be said to be \( M \)-definite. In other words an \( M \)-definite function is one which is not identically zero and does not change sign on \( \mathcal{Q} \).

* Presented to the Society, San Francisco Section, June 12, 1926; received by the editors in July, 1926.
† The theory is called by Professor Moore “General Analysis,” and is developed in his New Haven Mathematical Colloquium Lectures, New Haven, 1910.
Properties of general reference may pertain to a set of real functions

\[ \mu_1, \mu_2, \cdots, \mu_m, \]
each on the general range \( \mathcal{O} \).\(^*\) For example, if there exists a set of real constants \( c_1, c_2, \cdots, c_m \) (not all zero) such that

\[
\sum_{j=1}^{m} c_j \mu_j = 0 \quad (\mathcal{O}),
\]
the set of functions is said to be linearly dependent; otherwise it is linearly independent. Other properties of general reference for a set of functions are obtained by replacing the sign = in (1) by the signs used in (b)-(e). For the special instance in which \( \mathcal{O} \) consists of a finite number of elements, these properties have been considered by the author.\(^\dagger\)

The object of the present paper is to study the condition

\[
\sum_{j=1}^{m} c_j \mu_j \geq 0 \quad (\mathcal{O}),
\]
that is, the condition that a given set of functions on a general range admit an \( M \)-definite linear combination.

The central feature of the theory is a certain integral-valued function of the set of functions \( \{\mu_j\} \) which we shall call the \( M \)-rank of the set. In terms of it may be stated a necessary and sufficient condition that (2) admit a solution \( (c_1, c_2, \cdots, c_m) \) and the maximum number of \( c \)'s that may be zero in such a solution. These results are stated in §4, the earlier sections being preparatory to the definition of \( M \)-rank.

1. Reduction and composition of a general range \( \mathcal{O} \) relative to a function on that range. We consider a class \( \mathcal{O} \) of elements \( q \), notationally

\[ \mathcal{O} = \{ q \}. \]

Let \( \mu \) be any real single-valued function on \( \mathcal{O} \). We shall have occasion to consider three subclasses of \( \mathcal{O} \), relative to \( \mu \), defined as follows:

\[
\mathcal{O}_{p(\mu)} = \{\text{all } q \text{ such that } \mu(q) > 0\} = \{p^{(\mu)}\},
\]
\[
\mathcal{O}_{n(\mu)} = \{\text{all } q \text{ such that } \mu(q) < 0\} = \{n^{(\mu)}\},
\]
\[
\mathcal{O}_{z(\mu)} = \{\text{all } q \text{ such that } \mu(q) = 0\} = \{z^{(\mu)}\}.
\]

\(^*\) It may be noted, however, that a property of such a set of functions can be considered as a property of a single function \( \mu' \) on a composite range \( \mathcal{O}' \), the elements \( q' \) of the range \( \mathcal{O}' \) being bipartite elements of the form \( q' = (q, j) \), the first part \( q \) having the range \( \mathcal{O} \) and the second part \( j \) having the finite range consisting of the numbers 1, 2, \cdots, \( m \).

From these subclasses we form a certain composite range relative to the function $\mu$,

$$ \mathcal{Q}(\mu) = \mathcal{Q}_p(\mu) \mathcal{Q}_N(\mu) + \mathcal{Q}_s(\mu), $$

consisting of the logical sum of the two classes

$$ \mathcal{Q}_p(\mu) \mathcal{Q}_N(\mu) = [p(\mu) n(\mu)] \text{ and } \mathcal{Q}_s(\mu) = [s(\mu)]. $$

The elements of the new class

$$ \mathcal{Q}(\mu) = [q(\mu)] $$

will therefore be of two kinds: (1) bipartite elements $p(\mu) n(\mu)$ of which the first part $p(\mu)$ ranges over $\mathcal{Q}_p(\mu)$ and the second part $n(\mu)$ ranges over $\mathcal{Q}_N(\mu)$ independently; and (2) unipartite elements $s(\mu)$ ranging over $\mathcal{Q}_s(\mu)$.

The process here indicated may evidently be repeated. If $\sigma$ is a real single-valued function on the new range $\mathcal{Q}(\mu)$, it determines three subclasses of the range, which may be denoted by $\mathcal{Q}_p(\mu \sigma)$, $\mathcal{Q}_N(\mu \sigma)$, and $\mathcal{Q}_s(\mu \sigma)$; and from these may be formed the composite class

$$ \mathcal{Q}(\mu \sigma) = \mathcal{Q}_p(\mu \sigma) \mathcal{Q}_N(\mu \sigma) + \mathcal{Q}_s(\mu \sigma). $$

The process may be repeated indefinitely, provided at each stage a reducing function is available.

It will be noted that if at any stage the reducing function is everywhere positive or everywhere negative, the new composite range will be a null class; while if the reducing function is identically zero, the new composite range will be identical with the old range.

2. Reduced outer multiplication. The reducing function $\mu$ determines with any second function $v$ on the range $\mathcal{Q}$, a real single-valued function on the composite range $\mathcal{Q}(\mu)$ which we will call their reduced outer product, and denote by $((\mu v))$. It is defined as follows:* 

$$ ((\mu v)) = \begin{cases} 
\mu(p) v(n) - v(p) \mu(n) & \text{for } p n \text{ on } \mathcal{Q}_p(\mu) \mathcal{Q}_N(\mu), \\
v(s) & \text{for } s \text{ on } \mathcal{Q}_s(\mu). 
\end{cases} $$

This multiplication is not commutative. Its most obvious property is that $((\mu v)) = 0$ on $\mathcal{Q}(\mu)$. Other properties are developed in the next section.

* The outer product of two functions $f(x)$ and $g(x)$, where $x$ is a real variable on a closed interval, has been defined as $f(x)g(y) - g(x)f(y)$. See Kowalewski's Funktionenräume, Wiener Sitzungsberichte, vol. 120.
3. Reduction of a set of functions. Consider a set of real, single-valued functions
\{\mu_i\} = \{\mu_1, \mu_2, \ldots, \mu_m\},
on a general range \(\Omega\).

Relative to any one of the functions, say \(\mu_k\), we may determine a second set of \(m\) functions
\{\mu_i^{(k)}\} = \{(\mu_k \mu_1), (\mu_k \mu_2), \ldots, (\mu_k \mu_m)\},
each of which is the reduced outer product of the corresponding function in the given set by \(\mu_k\). This new set of functions on the composite range \(\Omega^{(\mu_k)}\) will be called a reduced set, or more explicitly, the reduction of the set \(\{\mu_i\}\) with respect to \(\mu_k\). It has the notable property that its \(k\)th constituent is identically zero. The usefulness of this type of reduction lies in the following lemma:

If the reducing function \(\mu_k\) is not \(M\)-definite, then the set \(\{\mu_i\}\) admits an \(M\)-definite linear combination if and only if the same is true of the reduced set \(\{\mu_i^{(k)}\}\). More explicitly, every set of constants \(c_1, c_2, \ldots, c_m\), satisfying
\[
(2) \quad \sum_{i=1}^{m} c_i \mu_i \geq 0 \quad (\Omega),
\]
will also satisfy the reduced condition
\[
(3) \quad \sum_{i=1}^{m} c_i (\mu_k \mu_i) \geq 0 \quad (\Omega^{(\mu_k)});
\]
and conversely, every solution \(c_1, c_2, \ldots, c_m\) of (3) yields a solution of (2) if the constant \(c_k\), which is arbitrary in a solution of (3), be suitably chosen.

We note first that if \(\mu_k = 0(\Omega)\), the proposition is true though trivial, since in that case the two sets \(\{\mu_i\}\) and \(\{\mu_i^{(k)}\}\) are identically the same. We may then assume in the proof that the function \(\mu_k\) changes sign on \(\Omega\).

Suppose that the condition (2) has a solution \(\bar{c}_1, \bar{c}_2, \ldots, \bar{c}_m\).
Taking account of the three subclasses of the range \(\Omega\) relative to the function \(\mu_k\), we obtain from this hypothesis the three statements
\[
(4) \quad \sum_{i=1}^{m} \bar{c}_i \mu_i(\phi) \geq 0 \quad (\phi \text{ on } \Omega_{P^{(\mu_k)}}),
\]
\[
(5) \quad \sum_{i=1}^{m} \bar{c}_i \mu_i(n) \geq 0 \quad (n \text{ on } \Omega_{N^{(\mu_k)}}),
\]
(6) \[ \sum_{j=1}^{m} \varepsilon_j \mu_j(x) \geq 0 \quad (x \text{ on } \mathcal{A}^{(p)}) , \]

with the understanding that the sign \( \geq \) has the significance of \( \geq' \) in at least one of the three statements.

Multiplying (4) by \( -\mu_k(n) \) and (5) by \( \mu_k(p) \) and adding the results, we have

\[ \sum_{j=1}^{m} \varepsilon_j [\mu_k(p)\mu_j(n) - \mu_j(p)\mu_k(n)] \geq 0 \quad (\text{on } \mathcal{A}^{(p)}\mathcal{A}^{(n)}) . \]

This together with (6) may be written

(7) \[ \sum_{j=1}^{m} \varepsilon_j (\mu_k \mu_j) \geq 0 \quad (\mathcal{A}^{(n)}) , \]

which proves the first part of the proposition.

Suppose conversely that the condition (3) has a solution \( \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_m \), as expressed by (7).

We note first that the constant \( \varepsilon_k \) is arbitrary, since its coefficient is identically zero. We may therefore omit the term corresponding to \( j = k \) from the summation, indicating the omission by an apostrophe', and write (7) in the two statements

(8) \[ \sum_{j=1}^{m} \varepsilon_j [\mu_k(p)\mu_j(n) - \mu_j(p)\mu_k(n)] \geq 0 \quad (\text{on } \mathcal{A}^{(p)}\mathcal{A}^{(n)}) , \]

(9) \[ \sum_{j=1}^{m} \varepsilon_j \mu_j(x) \geq 0 \quad (x \text{ on } \mathcal{A}^{(n)}) , \]

one of the signs \( \geq \) having the significance of \( \geq' \).

Since \( -\mu_k(p)\mu_k(n) \) is positive, we may obtain from (8) an equivalent statement

(10) \[ \sum_{j=1}^{m} \varepsilon_j \frac{\mu_j(p)}{\mu_k(p)} \geq \sum_{j=1}^{m} \varepsilon_j \frac{\mu_j(n)}{\mu_k(n)} \quad (\mathcal{A}^{(p)}{\mathcal{A}^{(n)}}). \]

Now the values on the left side of (10) must have a greatest lower bound, and those on the right a least upper bound, which bounds may or may not coincide. In any case we may choose the arbitrary \( \varepsilon_k \) so that

\[ \sum_{j=1}^{m} \varepsilon_j \frac{\mu_j(p)}{\mu_k(p)} \geq - \varepsilon_k \geq \sum_{j=1}^{m} \varepsilon_j \frac{\mu_j(n)}{\mu_k(n)} . \]
And from this double relation we obtain

\[ \sum_{j=1}^{\infty} e_{j} \mu_{j}(p) + e_{n} \mu_{n}(p) \geq 0, \quad \sum_{j=1}^{\infty} e_{j} \mu_{j}(m) + e_{n} \mu_{n}(m) \geq 0, \]

which together with (9) may be written

\[ \sum_{j=1}^{\infty} e_{j} \mu_{j} \geq 0 \quad (\mathcal{D}). \]

This completes the proof of the lemma.

The process of reduction may be repeated. As reduction of the set \( \{\mu_{j}\} \) with respect to \( \mu_{k} \) yields the set \( \{\mu_{j}^{(k)}\} \), so reduction of this latter set with respect to one of its constituents \( \mu_{j}^{(k)} \) yields a set which we shall denote by \( \{\mu_{j}^{(k)}\} \).

In general, we define the set

(11) \( \{\mu_{i}^{(k_1 k_2 \cdots k_s)}\} \)

as the reduction of the set

(12) \( \{\mu_{i}^{(k_1 k_2 \cdots k_{s-1})}\} \)

with respect to the function \( \mu_{k_1 k_2 \cdots k_{s-1}}^{(k)} \).

The set (11) will be called an \( s \)th reduction of the set \( \{\mu_{j}\} \). Clearly there are many \( s \)th reductions, depending on the choice of the sequence \( k_1, k_2, \ldots, k_s \). If the integers in this sequence are distinct, \( s \) constituent functions of the \( s \)th reduction are identically zero.

4. The \( M \)-rank of a set of functions. We recall that a function \( \mu \) is said to be \( M \)-definite if either of the conditions

\[ \mu \geq 0 \quad (\mathcal{D}) \text{ or } \mu \leq 0 \quad (\mathcal{D}) \]

is satisfied.

A set of \( m \) functions \( \{\mu_{j}\} \) is said to be of \( M \)-rank \( r \) if at least one of its \((m-r)\)th reductions contains an \( M \)-definite constituent function while no one of its \((m-r-1)\)th reductions contains such a constituent function. If the given set contains an \( M \)-definite function the set is of \( M \)-rank \( m \). If neither it nor any of its reductions contains such a function it is of \( M \)-rank zero.

Theorem. A necessary and sufficient condition that the set of \( m \) functions admit an \( M \)-definite linear combination is that its \( M \)-rank be greater than zero.
If the $M$-rank is $r (0 < r < m)$, then there is a subset of $m - r + 1$ of the functions which admits an $M$-definite linear combination, but there is no subset of $m - r$ functions for which this is true.

First, if the given set $\{\mu_i\}$ admits an $M$-definite linear combination, its $M$-rank is greater than zero. For otherwise the $(m - 1)$th reduction $\{\mu_{1, 2, \ldots, m-1}\}$ would contain no $M$-definite function, while the corresponding reduced condition

$$\sum_{j=1}^{m} c_j \mu_j^{1, 2, \ldots, m-1} \geq 0 \quad (Q^{(1, 2, \ldots, m-1)})$$

must admit a solution $c_1, c_2, \ldots, c_m$ by the lemma of §3. These two requirements are incompatible since all functions of the $(m - 1)$th reduction are zero except one.

Conversely, suppose the $M$-rank of the given set is $r (>0)$. Then there is an $(m - r)$th reduction of the set which contains an $M$-definite function. Suppose, for simplicity of notation, it is $\{\mu_{1, 2, \ldots, m-r}\}$, and suppose the $(m-r+1)$th function of this set is $M$-definite. Then the condition

$$\sum_{j=1}^{m} c_j \mu_j^{1, 2, \ldots, m-r} \geq 0 \quad (Q^{(1, 2, \ldots, m-r)})$$

admits a solution $c_1, c_2, \ldots, c_m$, in which

- $c_j = 0$ for $j > m-r+1$,
- $c_j$ is arbitrary for $j < m-r+1$,
- $c_{m-r+1} = +1$ or $-1$ according as $\mu_{m-r+1}$ is positive or negative.

Hence by repeated application of the lemma of §3, we find that the condition

$$\sum_{j=1}^{m} c_j \mu_j \geq 0 \quad (Q)$$

admits a solution in which $c_j = 0 (j > m-r+1)$, suitable values being assigned to $c_j (j < m-r+1)$. The given set therefore admits an $M$-definite linear combination; indeed a subset of $m-r+1$ of them has this property.

It remains to be proved that when the $M$-rank is $r$, no subset of $m-r$ of the given functions admits an $M$-definite linear combination. Suppose for definiteness that the subset

$$\mu_1, \mu_2, \ldots, \mu_{m-r}$$

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did this property. Then by the first proposition of the theorem (already established) the $M$-rank of this subset must be greater than zero, call it $r'$. That means that some $(m-r-r')$th reduction of the subset (12) would contain an $M$-definite constituent function. The corresponding $(m-r-r')$th reduction of the original set of $m$ functions would contain the same $M$-definite function, and hence the $M$-rank of the set would be $r+r'$, contrary to our assumption that it was $r$.

This completes the proof of the theorem.

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