SIMPLER PROOFS OF WARING'S THEOREM ON CUBES, WITH VARIOUS GENERALIZATIONS*

BY

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1. **Introduction.** In 1770 Waring conjectured that every positive integer is a sum of nine integral cubes $\geqq 0$. The first proof was given by Wieferich;† but owing to a numerical error he failed to treat a wide range of numbers corresponding to $\nu = 4$. Bachmann‡ indicated a long method to fill the gap, but himself made certain errors. The latter were incorporated in the unsuccessful attempt by Lejneek.§ The gap was first filled by Kempner.‖

All of these writers make use of three tables. The computation of each of the last two tables is considerably longer than the first. The third table as given by Wieferich and reproduced by Bachmann contains six errors, corrected by Kamke (cf. Kempner, Mathematische Annalen, loc. cit., p. 399). It is shown here that the last two tables may be completely avoided. The resulting simple proof of Waring's theorem in §§2, 3 is based on the customary prime 5. The second simple proof in §4 is based on the prime 11. By §5, we may also use the prime 17.

However, the main object of the paper is to prove generalizations of two types. Let C_n denote the sum of the cubes of n undetermined integers $\geqq 0$. Waring's theorem states that C_9 represents all positive integers. It is proved in §§ 4, 5 that tx^3+C_8 represents all positive integers if $1 \leqq t \leqq 23$, $t \neq 20$, but not if $t > 23$. To complete the discussion for $t = 20$ would require the extension of von Sterneck's table from 40,000 to 61,500.

It is proved in §6 that $tx^3+2y^3+C_7$ represents all positive integers if $1 \leqq t \leqq 34$, $t \neq 10, 15, 20, 25, 30$. Also that $tx^3+3y^3+C_7$ represents all if $1 \leqq t \leqq 9$, $t \neq 5$. Various similar theorems are highly probable in view of Lemma 8. More interesting empirical theorems on cubes were announced by the writer in the American Mathematical Monthly for April, 1927, and on biquadrates in the Bulletin of the American Mathematical Society, May-June, 1927.

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† Mathematische Annalen, vol. 66 (1909), pp. 99–101.

‡ *Niedere Zahlentheorie*, vol. 2, 1910, pp. 477–8.

§ Mathematische Annalen, vol. 70 (1911), pp. 454–6.

‖ *Über das Waringsche Problem und einige Verallgemeinerungen*, Dissertation, Göttingen, 1912. Extract in Mathematische Annalen, vol. 72 (1912), pp. 387–399.

If N is prime to 6, it is shown in §7 that every integer k is represented by $6x^2+6y^2+6z^2+Nw^3$, and that we may take $w\geqq 0$ if $k\geqq 23^3N$. In §8 is discussed the representation of all large integers by ly^3+C_7 when $l\leqq 5$.

The tables and computations in §§ 2–4, 6 and the first part of §5 were kindly checked with great care by Lincoln La Paz.

2. **Three lemmas needed for Waring's theorem.** We prove the following lemmas.

LEMMA 1. *If p is a prime $\equiv 2 \pmod 3$ and if l is an integer not divisible by p, every integer not divisible by p is congruent modulo p^n to a product of a cube by l.*

From the positive integers $\leqq p^n$ we omit the p^{n-1} multiples of p and obtain $\phi=(p-1)p^{n-1}$ numbers $a_1, \cdots, a_\phi$. Each la_i^3 is not divisible by p and hence is congruent to one of the a's modulo p^n. We shall prove that no two of the la_i^3 are congruent. It will then follow that $la_1^3, \cdots, la_\phi^3$ are congruent to $a_1, \cdots, a_\phi$ in some order. Since every integer not divisible by p is congruent to a certain a_j, it will therefore be congruent to a certain la_i^3.

If possible, let $la_i^3\equiv la_k^3 \pmod{p^n}$. Since $a_i\equiv a_kx \pmod{p^n}$ determines an integer x, we have $x^3\equiv 1$. By Euler's theorem, $x^\phi\equiv 1 \pmod{p^n}$. Since ϕ is not divisible by 3, $\phi=3q+r$, $r=1$ or 2. Hence $x^r\equiv 1$, $x\equiv 1$, $a_i\equiv a_k \pmod{p^n}$, contrary to hypothesis.

LEMMA 2. *Let P and e be given integers $\geqq 0$, such that P is of the form $5+48l$. Then every integer $\geqq P^e\cdot 22^3$ can be represented by $P^e\gamma^3+6(x^2+y^2+z^2)$, where γ, x, y, z are integers and $\gamma\geqq 0$.*

It is known that every positive integer not of the form $4^r(8s+7)$ is a sum of three integral squares. Hence this is true of positive integers congruent modulo 16 to one of the following:

$$1,2,3,4,5,6,8,9,10,11,13,14. \tag{1}$$

If n is any integer, we shall prove that

$$n \equiv P^e\gamma^3 + 6\mu \pmod{96} \tag{2}$$

has integral solutions γ, μ such that $0\leqq\gamma\leqq 22$, and such that μ is one of the numbers (1). Then there is an integer q for which

$$n = P^e\gamma^3 + 6\mu + 96q = P^e\gamma^3 + 6m, \qquad m = \mu + 16q.$$

When $n\geqq P^e\cdot 22^3$, then $n\geqq P^e\gamma^3$, $m\geqq 0$, whence m is a sum of three integral squares. Thus Lemma 2 will follow if we show that (2) has solutions of the specified type.

We shall first treat the case $e=0$:

$$n \equiv {}^3\gamma + 6\mu \qquad \text{(mod 96)}. \tag{3}$$

The method for (3) is such that, by multiplying it by $P, P^2, \cdots$, we can deduce at once the solvability of (2). With this end in view, we omit 3 and 11 from (1) and obtain the numbers

$$1,2,4,5,6,8,9,10,13,14, \tag{4}$$

whose products by 5 (and hence by P) are congruent modulo 16 to the same numbers (4) rearranged.

At the top of the following table we list certain values of γ and below them the least residues modulo 96 of their cubes. The body of the table shows the residue modulo 96 of $\gamma^3+6\mu$ for certain values (4) of μ.

0	1	2	3	4	5	6	7	8	9	10	11	13	14	15	17	18	22
0	1	8	27	64	29	24	55	32	57	40	83	85	56	15	17	72	88
6	7	14	33	70	35												
12	13	20	39	76	41		67		69		95	1		27	29		
24	25	32	51	88	53					64						0	
30	31	38	57	94	59												
36	37	44	63	4	65		91		93		23						
48	49	56	75	16	77	72		80					8				40
54	55	62	81	22	83												
60	61	68	87	28	89		19		21		47						
78	79	86	9	46	11												
84	85	92	15	52	17		43		45		71	73		3	5		

In the body of the table occur $0, 1, \cdots, 95$ with the exception* of

$$2,10,18,26,34,42,50,58,66,74,82,90. \tag{5}$$

The latter give all the positive integers <96 of the form $2+8r$.

But 3 and 11 are also available values of μ. For

$$\gamma = 0,2,4,6,8,10, \tag{6}$$

the residues modulo 96 of $\gamma^3+6 \cdot 3$ and $\gamma^3+6 \cdot 11$ are together found to be the numbers (5). This can be proved without computation as follows. In (6), $\gamma=2g$, $g=0, 1, 2, 3, 4, 5$. Thus $\gamma^3+18=2+8(g^3+2)$, $\gamma^3+66=2+8(g^3+8)$. Hence it remains only to show that the values of g^3+2 and

* However large a γ we take, we cannot reach an exceptional number (5). For $\gamma^3+6\mu\equiv 2+8r$ (mod 96) implies that γ is even and hence $6\mu\equiv 2$ (mod 8), $\mu\equiv 3$ (mod 4), $\mu\equiv 3, 7, 11, 15$ (mod 16). But none of these four occur in (4).

g^3+8 are together congruent to 0, 1, $\cdots$, 11 modulo 12. But $g^3 \equiv g \pmod 6$. Hence g^3+2 takes six values incongruent modulo 6 and therefore also modulo 12. Likewise for g^3+8. But $g^3+2 \equiv G^3+8 \pmod{12}$ would imply $g \equiv G \pmod 6$, $g = G$, a contradiction.

Hence for every integer n, (3) has integral solutions γ, μ, $0 \leqq \gamma \leqq 22$, μ in (1).

In $5(2+8r) = 2+8\rho$, $\rho = 1+5r$ ranges with r over a complete set of residues modulo 12. In other words, the products of the numbers (5) by 5 are congruent modulo 96 to the same numbers (5) rearranged. The same is true of their products by $P = 5+48l$, since $2k \cdot P \equiv 2k \cdot 5 \pmod{96}$. Evidently the products of 0, 1, $\cdots$, 95 by P are congruent modulo 96 to 0, 1, $\cdots$, 95 rearranged. Hence the products of the numbers in the above table by P are congruent to the same numbers modulo 96. Those numbers are therefore the residues modulo 96 of $P(\gamma^3+6\mu)$ for $0 \leqq \gamma \leqq 22$ and for μ in (4). We saw that the products $P\mu$ are congruent modulo 16 to the same numbers (4) rearranged. Hence the residues modulo 96 of $P\gamma^3+6\nu$ for $0 \leqq \gamma \leqq 22$ and for ν in (4) are the numbers in the table and hence are the numbers 0, 1, $\cdots$, 95 other than (5).

To complete the proof of the statement concerning (2) when $e=1$, it remains to show that, by choice of γ in (6) and for $t = 18$ or 66, $P\gamma^3+t$ is congruent modulo 96 to any assigned number in (5). Since the last was proved for γ^3+t, we need only show that γ^3 and $P\gamma^3$ take the same values modulo 96 when γ takes the values (6). Then $\gamma = 2g$, $g = 0, 1, 2, 3, 4, 5$. Thus $g^3 \equiv 0, 1, 8, 3, 4, 5$; $5g^3 \equiv 0, 5, 4, 3, 8, 1 \pmod{12}$, respectively. Hence γ^3 and $5\gamma^3$ take the same values modulo $8 \cdot 12$. But the products of 5 and $P = 5+48l$ by the same even number γ^3 are congruent modulo 96.

The insertion of the factor P may be repeated e times. This proves the statement concerning (2).

LEMMA 3. *Given the positive numbers s and t and a number B for which $0 \leqq B \leqq s$, $t \leqq 9^3 s$, we can find an integer $i \geqq 0$ such that*

$$B \leqq s - ti^3 < B + 3(ts^2)^{1/3}. \tag{7}$$

Denote the last member of (7) by L. If $s < L$, take $i = 0$. Next, let $s \geqq L$ and determine a real number r so that $s - tr^3 = B$. Then

$$tr^3 = s - B \geqq L - B = 3(ts^2)^{1/3} \geqq t/27, \quad 3r \geqq 1.$$

We may write $r = i + f$, where $0 \leqq f < 1$, and i is an integer $\geqq 0$. Since $i \leqq r$, $B = s - tr^3 \leqq s - ti^3$, as desired in (7). Next,

$$s - ti^3 - B = s - t(r-f)^3 - s + tr^3 = tw,$$

where

$$w = r^3 - (r - f)^3 = 3r^2f - f^2(3r - f) < 3r^2f < 3r^2,$$

since $3r \geqq 1, f < 1$. Since $B \geqq 0$,

$$tr^3 \leqq s, \quad r^2 \leqq (s^2/t^2)^{1/3}, \quad s - ti^3 - B < 3tr^2 \leqq 3(ts^2)^{1/3}.$$

3. **Proof of Waring's theorem.** We first prove that every integer s exceeding $9 \cdot 5^{12}$ is a sum of nine integral cubes $\geqq 0$. For this proof we take $C=9, p=5, t=1$ in our formulas. Since $s > Cp^{3\cdot 4}$ there exists an integer $n \geqq 4$ such that

(8) $$Cp^{3n} < s \leqq Cp^{3(n+1)}.$$

Write

(9) $$k = 3(tC^2)^{1/3}p^{2n+2}.$$

Hence

(10) $$3(ts^2)^{1/3} \leqq k.$$

We separate two cases. First, let $Cp^{3n}+2k \leqq s$. Then Cp^{3n} and $Cp^{3n}+k$ are both $\leqq s$. Taking them in turn for B in Lemma 3, and using (10), we conclude that there exist integers I and J, each $\geqq 0$, such that

$$Cp^{3n} \leqq s - tI^3 < Cp^{3n} + k,$$
$$Cp^{3n} + k \leqq s - tJ^3 < Cp^{3n} + 2k.$$

Hence there are two distinct integral values I and J of i which satisfy

(11) $$Cp^{3n} \leqq s - ti^3 < Cp^{3n} + 2k, \quad i \geqq 0.$$

Second, let $Cp^{3n}+2k > s$. Then (11) holds for $i=0$ and (when $t=1$) for $i=1$, since the integer Cp^{3n} is less than s and hence is $\leqq s \dot{-} 1$.

Hence in both cases there exist two distinct integers and hence two consecutive integers $j-1$ and j, which are both values of i satisfying (11).

At least one of the integers $s-t(j-1)^3$ and $s-tj^3$ is not divisible by 5. For, their difference is the product of t by $3j^2-3j+1$. The double of the latter is congruent to $(j+2)^2-2$, modulo 5. But 2 is not congruent to a square.

Hence there exists an integer $a \geqq 0$ such that (11) holds when $i=a$, and such that $s-ta^3$ is not divisible by $p=5$. By Lemma 1, there exist integers b and M such that

(12) $$s - ta^3 = b^3 + p^nM, \quad 0 < b < p^n.$$

When $n \geqq 4$, we have

$$(13) \qquad Cp^{3n} + 2k \leqq 12p^{3n}.$$

For, if we insert the value (9) of k, divide all terms by p^{3n}, and note that $1/p^{n-2} \leqq 1/p^2$, we see that (13) holds if

$$(14) \qquad C + \frac{6}{p^2}(tC^2)^{1/3} \leqq 12.$$

When $C=9$, $p=5$, this holds if

$$t \leqq \frac{25^3}{8 \cdot 9^2} = 24.1.$$

By (11) with $i=a$, (12) and (13), we get

$$Cp^{3n} \leqq b^3 + p^n M < 12p^{3n}, \quad (C-1)p^{3n} < Cp^{3n} - b^3.$$

Hence

$$(C-1)p^{2n} < M < 12p^{2n}.$$

Write $M = N + 6p^{2n}$. Thus

$$(15) \qquad (C-7)p^{2n} < N < 6p^{2n},$$

$$(16) \qquad s = ta^3 + b^3 + p^n(N + 6p^{2n}).$$

We seek integers c and m, each $\geqq 0$, such that

$$(17) \qquad p^n N = c^3 + p^n \cdot 6m, \quad m = d_1^2 + d_2^2 + d_3^2,$$

for integers d_i. Then will

$$(18) \qquad s = ta^3 + b^3 + c^3 + p^n(6p^{2n} + 6m).$$

Writing A for p^n, we then have

$$(19) \qquad s = ta^3 + b^3 + c^3 + \sum_{i=1}^{3} [(A + d_i)^3 + (A - d_i)^3].$$

These cubes are all $\geqq 0$. For, if $d_i^2 > A^2$, then $m > A^2 = p^{2n}$, and, by (17), $p^n N > 6p^n p^{2n}$, contrary to (15). Hence s is a sum of nine integral cubes $\geqq 0$.

It remains to select c and m. Choose an integer e so that

$$(20) \qquad e = 0,1,2, \quad e + n \equiv 0 \pmod{3}.$$

The condition in Lemma 2 is $N \geqq 5^e \cdot 22^3$. By (15), this will be satisfied if $(C-7)p^{2n} \geqq 5^e \cdot 22^3$. When $n \geqq 4$, the minimum value of $2n - e$ is 6. Hence it suffices to take

$$(21)\qquad (C-7)5^6 \geqq 22^3,\quad C-7 \geqq \left(\frac{22}{25}\right)^3 = (0.88)^3 = 0.681472.$$

Thus if $C \geqq 7.682$, Lemma 2 shows the existence of integers γ and m, each $\geqq 0$, such that $N = 5^e\gamma^3 + 6m$, where m is a sum of three integral squares. By (20), $5^{e+n}\gamma^3$ is the cube of an integer $c \geqq 0$. Thus (17) holds when $p=5$.

This completes the proof that every integer s exceeding $9\cdot 5^{12}$ is a sum of nine integral cubes $\geqq 0$. The same is true when $s<40{,}000$ by the table of von Sterneck,* which shows also that if $8042<s<40{,}000$, s is a sum of six integral cubes $\geqq 0$. To utilize the latter result, let $10^4 \leqq s \leqq 9\cdot 5^{12}$. By Lemma 3 with $B=10^4$, there exists an integer $u\geqq 0$ satisfying

$$(22)\qquad 10^4 \leqq \sigma < 10^4 + 3(ts^2)^{1/3},\qquad \sigma = s - tu^3.$$

We have $s<5^{14}$. For $t<5^2$, the radical is $<5^{10}$. Also, $10^4 = 5^4 2^4 < 5^9$. Hence $\sigma < 16\cdot 5^9 < 4^3\cdot 5^9$.

Apply Lemma 3 with $t=1$, $B=10^4$, and s replaced by σ. Thus there exists an integer $v\geqq 0$ satisfying

$$(23)\qquad 10^4 \leqq \tau < 10^4 + 3\sigma^{2/3},\qquad \tau = \sigma - v^3.$$

The radical is $<4^2\cdot 5^6$. Also, $10^4<4^2\cdot 5^6$. Hence $\tau<4^3\cdot 5^6$. As before, there exists an integer $w\geqq 0$ satisfying

$$(24)\qquad 10^4 \leqq \tau - w^3 < 10^4 + 3\tau^{2/3} = 4\cdot 10^4 = 40{,}000.$$

Since $\tau - w^3$ is therefore a sum of six cubes, while $s = tu^3+v^3+\tau$, s is a sum of nine integral cubes $\geqq 0$. This completes the proof of Waring's theorem.

4. **The first generalizations.** Let C_n denote the sum of the cubes of n undetermined integers $\geqq 0$. Let t be an integer $\geqq 0$.

LEMMA 4. *The form $f_t = tx^3 + C_8$ represents all positive integers $\leqq 40{,}000$ if and only if $0<t\leqq 23$.*

If $t>23$ or if $t=0$, C_8 and hence f_t fail to represent 23. Next, let $0<t\leqq 23$. By von Sterneck's table, every positive integer $\leqq 40{,}000$, except 23 and 239, is a sum of eight integral cubes $\geqq 0$. It remains only to show that f_t represents 23 and 239. Take $x=1$. Since

$$0 \leqq 23 - t < 23,\quad 23 < 239 - t < 239,$$

both $23-t$ and $239-t$ are represented by C_8.

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* Akademie der Wissenschaften, Wien, Sitzungsberichte, vol. 112, IIa (1903), pp. 1627–1666. Dahse's table to 12,000 was published by Jacobi, Journal für Mathematik, vol. 42 (1851), p. 41; Werke, vol. 6, p. 323.

THEOREM I. *If* $1 \leqq t \leqq 23$, $t \neq 20$, *every positive integer is represented by* $f_t = tx^3 + C_8$.

We proceed as in §3 with $p=5$ or $p=11$ according as t is not divisible by 5 or 11, and with $n \geqq 4$ or $n \geqq 3$, respectively. We shall find limits within which C may be chosen. But we refrain from making a definite choice for C initially, since we may need to decrease C slightly to meet the difficulty arising below (11) when $t>1$. Then (11) does not hold for $i=1$ if

$$Cp^{3n} > s - t. \tag{25}$$

In the latter case, we employ a new constant C'. Then

$$C'p^{3n+3} = Cp^{3n} \cdot p^3C'/C > (s-t)p^3C'/C$$

will be $\geqq s$ if

$$C' \geqq \frac{C}{p^3} \cdot \frac{s}{s-t},$$

and hence if $C' > \frac{1}{2}C$. Thus if C' lies between $\frac{1}{2}C$ and C, (8) will remain true after C is replaced by C'. By (25), $Cp^{3n} = s - t + P$, $P>0$. By (8), $P < t \leqq 23$. Write $q = P/p^{3n}$. Since $n \geqq 4$ or $\geqq 3$, according as $p=5$ or 11, q is very small. We take $C' = C - q$. Then C' lies between $\frac{1}{2}C$ and C, and $C'p^{3n} = s - t$. Hence after taking C' as a new C, we have (8) and the desired two integral solutions i of (11) in all cases.

For $p=5$, $t \leqq 23$, (14) holds when $C \leqq 9.03$. Reduction to $C=9$ permits us to avoid the difficulty mentioned before. The entire proof in §3 now holds if $p=5$ and if t is not divisible by 5.

LEMMA 5. *Let* $P = 11 + 48l$ *and* e *be given integers* $\geqq 0$. *Every integer* $\geqq P^e \cdot 23^3$ *is represented by* $P^e\gamma^3 + 6(x^2+y^2+z^2)$, $\gamma \geqq 0$.

We now omit 4, 5, and 13 from the available numbers (1) and have

$$1, 2, 3, 6, 8, 9, 10, 11, 14, \tag{26}$$

whose products by P are congruent modulo 16 to the same numbers (26) rearranged.

The following table shows the residues modulo 96 of $\gamma^3 + 6\mu$ for $\gamma = 0, 1, \cdots, 23$ and for μ in (26). It was computed as in §2, with also $19^3 \equiv 43$, $21^3 \equiv 45$, $23^3 \equiv 71 \pmod{96}$.

$\gamma=0$	1	2	3	4	5	6	7	8	9	10	11
6	7	14	33	70	35	30		38		46	
12	13	20	39	76	41				69		
18	19	26	45	82	47	42	73	50		58	5
36	37	44	63	4	65		91				23
48	49	56	75	16	77	72		80	9	88	
54	55	62	81	22	83	78		86		94	
60	61	68	87	28	89				21		
66	67	74	93	34	95	90	25	2	27	10	53
84	85	92	15	52	17		43				71

6μ	$\gamma=13$	14	15	17	18	19	21	22	23
12	1			29			57		
36			51			79			11
48		8			24			40	
84			3			31			59

In the body of the table occur $0, 1, \cdots, 95$ with the exception of

(27) $$0, 32, 64.$$

Since $32P\equiv 64$, $64P\equiv 32 \pmod{96}$, and since the products of $0, 1, \cdots, 95$ by P are evidently congruent modulo 96 to the same numbers rearranged, the same is true of the numbers in the table. Using the omitted value 4 of μ, we get

(28) $$18^3+24\equiv 0,\quad 2^3+24\equiv 32,\quad 10^3+24\equiv 64 \qquad (\text{mod } 96).$$

Since $4(P^2-1)\equiv 4(11^2-1)\equiv 0 \pmod{96}$, $4P^{2k}\equiv 4$, and multiplication of (28) by P^{2k} yields

(29) $$18^3P^{2k}+24\equiv 0,\quad 2^3P^{2k}+24\equiv 32,\quad 10^3P^{2k}+24\equiv 64 \quad (\text{mod } 96).$$

This completes the proof of Lemma 5 where e is even.

Since $P+1$ is divisible by 12, the product of an even cube by P is congruent to its negative, modulo 96. Hence

(30) $$6^3P\equiv -24,\quad 22^3P\equiv 8,\quad 14^3P\equiv 40 \qquad (\text{mod } 96).$$

As before, multiplication of (30) by P^{2k} yields

(31) $$6^3P^e+24\equiv 0,\quad 22^3P^e+24\equiv 32,\quad 14^3P^e+24\equiv 64 \quad (\text{mod } 96),$$

where $e=2k+1$. Thus Lemma 5 follows when e is odd.

Let $p=11$, $n\geqq 3$, $t\leqq 15$, $t\neq 11$. Then (13) holds if

(32) $$C+\frac{6}{11}(tC^2)^{1/3}\leqq 12 \qquad \text{for } t=15.$$

When $C=7.05$, the left member is 11.9960. By (15), the condition in Lemma 5 is satisfied if $(C-7)11^{2n}\geqq 11^{e}\cdot 23^{3}$. By (20), the minimum of $2n-e$ for $n\geqq 3$ is 6. Hence it suffices to take

$$(33)\qquad (C-7)11^{6}\geqq 23^{3},\qquad C-7\geqq 0.006868.$$

Hence all the conditions on C are satisfied if $C=7.01$, and the reduction from 7.05 avoids the difficulty arising when (25) holds. Next,

$$4(3j^{2}-3j+1)\equiv (j+5)^{2}+1 \pmod{11},$$

while -1 is not congruent to a square. For $C=7.01$, the proof in §3 now shows that every integer s exceeding $C\cdot 11^{9}$ is represented by f_t. It remains to prove this also when $10^{4}\leqq s\leqq C\cdot 11^{9}$. Consider (22) and (23). Now

$$(34)\qquad (tC^{2})^{1/3}=9.033,\quad 10^{4}<(0.01)11^{6},\quad \sigma<(27.11)11^{6},$$

$$\sigma^{2/3}<(9.0245)11^{4},\quad 10^{4}<(0.69)11^{4},\quad \tau<28\cdot 11^{4}<7^{3}\cdot 11^{3}.$$

In place of (24), we now have $10^{4}\leqq\tau-w^{3}<28{,}000$. This proves Theorem I when $t\leqq 15$, $t\neq 11$.

5. The case $t=20$. First, take $p=11$. The proof fails if $n=3$, since (32) requires $C<7$. Hence $n\geqq 4$, and (14) holds if $C\leqq 11.3$. Thus every s exceeding $C\cdot 11^{12}$ is represented by f_{20}, where $C^{2}=50$. But if $s<C\cdot 11^{12}$, we obtain by (22)–(24) the condition $\tau-w^{3}<152{,}794$, which is far beyond the limit of von Sterneck's table.

A better result is obtained by taking $p=17$, $n\geqq 3$. Lemma 2 holds also when $P=17+48l$. For, by multiplying (3) by this P, we see that every integer is congruent modulo 96 to $P\gamma^{3}+6P\mu$. But $6P\equiv 6 \pmod{96}$. If $3j^{2}-3j+1\equiv 0 \pmod{17}$, multiplication by 6 gives $(j+8)^{2}\equiv 7$, whereas 7 is a quadratic non-residue of 17. The two conditions on C are both satisfied if $C^{2}=50$. Then (22)-(24) yield $\tau-w^{3}<65{,}500$. A still lower limit will be found by employing

LEMMA 6. *Given the positive numbers s and t, and a number B for which $0\leqq B\leqq s$, $t\leqq 9^{3}s$, we can find an integer $i\geqq 0$ such that*

$$B\leqq s-ti^{3}<B+t(3r^{2}-3r+1),\quad r^{3}=(s-B)/t.$$

The proof consists in the following modification of the last part of the proof of Lemma 3. The condition for $w<3r^{2}-3r+1$ is

$$(1-f)[3r^{2}-3r(1+f)+1+f+f^{2}]>0.$$

Since $0\leqq f<1$, this is evidently satisfied when $r\geqq 1+f$. In the contrary case, $i=0$, $r=f$, and the quantity in brackets is $(1-f)^{2}>0$.

By (21) with 5 replaced by 17, $C-7 \geqq 0.0004411$. This and condition (13) are both satisfied if $C=7.00045$. It remains to treat integers $s<C\cdot 17^9$. By Lemma 4 we may take $s>40,000$. By Lemma 6 with $t=20$, $B=8043$, there exists an integer $u \geqq 0$ such that

$$8043 \leqq \sigma < 8043 + 20(3r^2 - 3r + 1), \quad \sigma = s - 20u^3,$$

where $r^3=s/20$ slightly exceeds the initial r^3. Then

$$\log r = 3.5393787, \quad r^2 = 11{,}988{,}290, \quad r = 3462, \quad \sigma - 8043 < 719{,}089{,}700.$$

Apply Lemma 6 with $t=1$, $B=8043$, s replaced by σ. Hence there exists an integer $v \geqq 0$ such that

$$8043 \leqq \tau < 8043 + 3R^2 - 3R + 1, \quad \tau = \sigma - v^3, \quad R^3 = \sigma - 8043,$$

$$\log R = 2.9522610, \quad R^2 = 802{,}642.2, \quad R = 895.9, \quad \tau - 8043 < 2{,}405{,}240.$$

By Lemma 6 with $t=1$, there exists an integer $w \geqq 0$ such that

$$8043 \leqq \tau - w^3 < 8043 + 3\rho^2 - 3\rho + 1, \quad \rho^3 = \tau - 8043,$$

$$\log \rho = 2.1270528, \quad \rho^2 = 17{,}951.7, \quad \rho = 134, \quad \tau - w^3 < 61{,}497.$$

Hence Theorem I would hold also for $t=20$ provided an extension of von Sterneck's table would show that every integer between 8043 and 61,497 is a sum of six cubes.

To prove Waring's theorem by means of $p=17$, $n \geqq 3$, and the same C, we find by three applications of Lemma 3 with $t=1$, $B=8043$, that $\tau - w^3 < 42{,}846.7$. This limit is reduced by using Lemma 6.

6. **The second generalizations.** We employ two lemmas.

LEMMA 7. *$F_l = ly^3 + C_7$ represents all positive integers $\leqq 40{,}000$ if and only if $l=2$-6, 9-15. F_7 represents all $\leqq 40{,}000$ except* 22. *F_8 represents all except* 23, 239, *and* 428.

By the tables of Dahse and von Sterneck, C_7 represents every positive integer $\leqq 40{,}000$ except

(35) 15, 22, 23, 50, 114, 167, 175, 186, 212, 231, 238, 239, 303, 364, 420, 428, 454.

Thus $F_0 \neq 15$. Also $F_1 = C_8 \neq 23$. If $l>15$, evidently $F_l \neq 15$. Hence let $2 \leqq l \leqq 15$. The successive differences of the numbers (35) are

(36) 7, 1, 27, 64, 53, 8, 11, 26, 19, 7, 1, 64, 61, 56, 8, 26.

Hence every positive difference of two numbers (35), not necessarily consecutive, is 1, 7, 8, 11, or is >15.

First, let $l \neq 7, 8, 11$. If n and m $(n > m)$ are any two numbers (35), then $n - m \neq l$. Since $n - l$ is therefore not one of the numbers (35), it is represented by C_7. Hence n is represented by F_l with $y = 1$.

A like result holds also if $l = 11$. By (36) the only pair of numbers (35) with the difference 11 is the pair 186, 175. But $186 - 11 \cdot 2^3 = 98$ is not in (35) and hence is represented by C_7. Hence $F_{11} = 186$ for $y = 2$.

For $l = 7$, it remains to consider $n = 22$ and 238, which alone exceed predecessors by 7, as seen from (36). But $238 - 7 \cdot 2^3 = 182$ is not in (35) and hence is represented by C_7.

Finally, for $l = 8$, (36) shows that only $n = 23$, 175, 239, and 428 exceed smaller numbers in (35) by 8. Since F_8 is a sum of eight cubes, it does not represent 23 or 239. Next, $175 - 8 \cdot 2^3 = 111$ is not in (35). But $428 - 8 = 420$, $428 - 8 \cdot 2^3 = 364$, and $428 - 8 \cdot 3^3 = 212$ are all in (35), while $428 < 8 \cdot 4^3$. Hence $F_8 \neq 428$.

LEMMA 8. *$F_{k,l} = kx^3 + ly^3 + C_7$ represents all positive integers $\leqq 40{,}000$ when $l = 2$–6, 9–15, and k is arbitrary; when $l = 7$ if and only if $1 \leqq k \leqq 22$; when $l = 8$ if and only if $1 \leqq k \leqq 23$; but not if both k and l exceed 15.*

In the final case, $F \neq 15$. The first case follows from Lemma 7. Next, let $l = 7$. If $k = 1$, F is $7y^3 + C_8$, which represents all integers $\leqq 40{,}000$ by Lemma 4. If $k > 22$, $F = 22$ requires $x = 0$, whereas $7y^3 + C_7 \neq 22$ by Lemma 7. It remains to consider the case $l = 7$, $1 < k \leqq 22$. By Lemma 7, we have only to verify that $F = 22$ has integral solutions. When $k = 7$, take $x = y = 1$, since C_7 represents 8. When $k \neq 7$, take $x = 1$, $y = 0$, since C_7 represents $22 - k$, which is $\geqq 0$, < 22, and $\neq 15$.

Finally, let $l = 8$. If $k = 1$, apply Lemma 4. If $k > 23$, $F = 23$ implies $x = 0$, whereas $8y^3 + C_7 \neq 23$ by Lemma 7. Hence let $1 < k \leqq 23$. By Lemma 7, we have only to verify that F represents 23, 239, 428. If $k \neq 8$, take $x = 1$, $y = 0$; then $F = k + C_7$ represents 23, since C_7 represents $23 - k \neq 15, 22, 23$; $F = 239$, since $239 - k$ is not 231 and is in the interval from 216 to 237 and hence is represented by C_7; $F = 428$, since $428 - k$ is not 420 and is in the interval from 405 to 426 and hence is represented by C_7. If $k = 8$, take $x = y = 1$ and note that C_7 represents 7, 223, and 413.

THEOREM II. *$tx^3 + ly^3 + C_7$ represents all positive integers if $l = 2$, $1 \leqq t \leqq 34$, $t \neq 10, 15, 20, 25, 30$, and if $l = 3$, $1 \leqq t \leqq 9$, $t \neq 5$.*

Let neither t nor l be divisible by the prime $p \equiv 2 \pmod 3$. By §§ 3, 4, there exists an integer $a \geqq 0$ such that (11) holds when $i = a$ and such that $s - ta^3$ is not divisible by p. By Lemma 1, there exist integers b and M such that

$$s - ta^3 = lb^3 + p^nM, \qquad 0 < b < p^n. \tag{37}$$

We shall presently choose C and t so that (13) is satisfied. Using also (11) with $i=a$, we have

$$Cp^{3n} \leqq lb^3 + p^nM < 12p^{3n}, \quad (C-l)p^{3n} < Cp^{3n} - lb^3.$$

Hence

$$(C-l)p^{2n} < M < 12p^{2n}.$$

Write $M = N + 6p^{2n}$. Then

$$(C-l-6)p^{2n} < N < 6p^{2n}, \quad s = ta^3 + lb^3 + p^n(N + 6p^{2n}). \tag{38}$$

(I) Let $p=5$. As in (21), the condition $N \geqq 5^6 \cdot 22^3$ in Lemma 2 is satisfied if $C-l-6 \geqq 0.68148$. Since $t \geqq 1$, (13) fails if $n=3$. Hence $n \geqq 4$.

First, let $l=2$ and take $C=8.68148$. Condition (14) gives $t \leqq 35.076$. Hence 34 is the maximum t. It remains to consider integers s satisfying $10^4 \leqq s \leqq C \cdot 5^{12}$. Since $t < 5^3$, $C^2 < 5^3$, the radical in (22) is $< 5^{10}$, and $\sigma < 16 \cdot 5^9$. By Lemma 3 with $B = 10^4$, $t=2$, and s replaced by σ, there exists an integer $v \geqq 0$ such that

$$10^4 \leqq \tau < 10^4 + 3(2\sigma^2)^{1/3}, \qquad \tau = \sigma - 2v^3. \tag{39}$$

Since $2\sigma^2 < 4^6 \cdot 5^{18}$, we have (24). This proves Theorem II for $l=2$, $t \leqq 34$, t prime to 5.

Second, let $l=3$ and take $C=9.68148$. By (14), $t \leqq 9.6186$. Hence $t \leqq 9$. Since $tC^2 < 10^3$, (22) gives $\sigma < 31 \cdot 5^8 < 7 \cdot 5^9$. By Lemma 3 with $B=10^4$, $t=3$, and s replaced by σ,

$$10^4 \leqq \tau < 10^4 + 3(3\sigma^2)^{1/3}, \qquad \tau = \sigma - 3v^3.$$

Since $3\sigma^2 < 4^6 \cdot 5^{18}$, (24) holds. This proves Theorem II for $l=3$, $t \leqq 9$, $t \neq 5$.

Finally, if $l \geqq 4$, then $C \geqq 10.682$, and (14) fails if $t \geqq 2$. But if $t=1$, we have the form treated in §4.

(II) Let $p=11$. Whether $n \geqq 3$ or $n \geqq 4$, the condition in Lemma 5 is satisfied if $C-l-6 \geqq 0.006868$, as in (33).

First, let $n \geqq 3$. If $l \geqq 3$ and $t \geqq 5$, (32) fails. Hence let $l=2$, $C=8.006868$. Then $C^2=64.11$ and (32) requires that $t \leqq 6$. But (32) holds if $t=5$ since $(5C^2)^{1/3} < 6.844$. The only new case is $t=5$. It remains to consider integers s satisfying $10^4 \leqq s \leqq C \cdot 11^9$. We employ (22), (39), and (24):

$$10^4 < (0.006)11^6, \quad \sigma < (20.538)11^6, \quad (2\sigma^2)^{1/3} < (9.45)11^4,$$

$$\tau < 30 \cdot 11^4 = 330 \cdot 11^3 < 7^3 \cdot 11^3, \quad \tau^{2/3} < 6000, \quad \tau - w^3 < 28{,}000.$$

This proves Theorem II for $t=5$, $l=2$.

Second, let $n \geqq 4$, $l=2$. Using the same C, we find that (14) holds when $t \leqq 8145$. But the proof of Theorem II fails for the first new case $t=10$ when $s<C \cdot 11^{12}$. We employ (22), (39), and (24) with the refinement of replacing 10^4 by 8042. We obtain

$$\sigma < (25.8682)11^8, \quad \tau < (73.576)11^5, \quad \tau - w^3 < 163{,}969,$$

where the final number is beyond the limit 40,000 of von Sterneck's table.

7. **Generalization of Lemmas 2 and 5.** These lemmas can be generalized as follows.

THEOREM III. *If N is a positive integer divisible by neither* 2 *nor* 3, *every integer** $\geqq 23^3 N$ *is represented by* $N\gamma^3+6(x^2+y^2+z^2)$, *where* γ, x, y, z *are integers and* $\gamma \geqq 0$.

As in the proof of Lemma 2 this will follow from

LEMMA 9. *Every integer n is congruent modulo* 96 *to* $N\gamma^3+6\mu$ *for* $0 \leqq \gamma \leqq 23$, *with μ in the set* (1).

Proof was given in §5 when $N \equiv 17 \pmod{48}$. It is true by the proof in Lemma 2 when $N \equiv 5, 5^2, 5^3 \equiv 29, 5^4 \equiv 1 \pmod{48}$.

If $N=41+48l$, $N \equiv 5^2 \pmod{16}$. We saw that the products of the numbers (4) by 5 are congruent modulo 16 to the same numbers (4) rearranged. Hence the same is true of their products by N. Also $3N \equiv 3 \cdot 9 \equiv 11$, $11N \equiv 3 \pmod{16}$. Hence the products of all the numbers (1) by N are congruent modulo 16 to the same numbers (1) rearranged. Multiplication of (3) by N proves Lemma 9.

For $N=37+48l$, we proceed as in the last part of the proof of Lemma 2. In $37(2+8r)=2+8\rho$, $\rho=9+37r$ ranges with r over a complete set of residues modulo 12. Finally, $37g^3 \equiv g^3 \pmod{12}$. The lemma follows also for $N \equiv 37^3 \equiv 13 \pmod{48}$.

By Lemma 5, the lemma holds when $N \equiv 11$ or $11^3 \equiv 35 \pmod{48}$.

Let $N=19+48l$. The products of the numbers (26) by 3 and hence by N are congruent modulo 16 to the same numbers rearranged. Since $N \equiv 1 \pmod 3$, $32N \equiv 32$, $64N \equiv 64 \pmod{96}$. Since $N+5$ is divisible by 12, the product of an even cube by N is congruent to its product by -5 modulo 96. Hence

$$N \cdot 6^3 \equiv -5 \cdot 24 \equiv -24, \quad N \cdot 22^3 \equiv (-5)(-8)=40,$$
$$N \cdot 14^3 \equiv (-5)(-40) \equiv 8 \qquad \pmod{96}.$$

* Except for $N \equiv 11, 19, 35, 43 \pmod{48}$, we may replace 23 by 22. But when $N=1$, $S=9832$ is between 21^3 and 22^3 and is not represented by $\gamma^3+6(x^2+y^2+z^2)$. For, that requires $\gamma^3 \equiv S \equiv 4$, $\gamma \equiv 4 \pmod 6$. But no one of $(1/6)(S-4^3)=4 \cdot 407$, $(1/6)(S-10^3)=16 \cdot 92$, $(1/6)(S-16^3)=4 \cdot 239$ is a sum of three squares.

Adding 24 to each, we get 0, 64, 32, respectively. The lemma follows also for $N\equiv 19^3\equiv 43 \pmod{48}$.

Let $N=23+48l$. Then $N\equiv 7 \pmod{16}$. Omitting 1, 4, 9 from (1), we get

$$2,3,5,6,8,10,11,13,14, \tag{40}$$

whose products by 7 are congruent modulo 16 to the same numbers permuted. For μ in (40), the residues modulo 96 of $\gamma^3+6\mu$ are shown in the following table having the values of γ at the top:

0	1	2	3	4	5	6	7	8	9	10	11	13	14	15	17	18	22
12	13	20	39	76	41				69			1			29		
18	19	26	45	82	47	42	73	50		58	5			33			
30	31	38	57	94	59	54		62		70							
36	37	44	63	4	65		91				23			51			
48	49	56	75	16	77	72	7	80		88	35		8			24	40
60	61	68	87	28	89				21								
66	67	74	93	34	95	90	25	2	27	10	53	55		81	83		
78	79	86	9	46	11	6		14		22							
84	85	92	15	52	17		43				71			3			

In the body of the table occur $0, 1, \cdots, 95$ with the exception of 0, 32, 64. But $32N\equiv 64$, $64N\equiv 32 \pmod{96}$. We proceed as in the proof of Lemma 5. Since $N+1$ is divisible by 12, the product of an even cube by N is congruent to its negative modulo 96. Hence

$$6^3N\equiv -24,\quad 22^3N\equiv 8,\quad 14^3N\equiv 40 \qquad \pmod{96}.$$

Adding 24, we get 0, 32, 64, respectively. The same proof holds for $N=47+48l$.

For $N\equiv 7$ or $31 \pmod{48}$, the preceding proof is to be modified as for $N=19+48l$.

This completes the proof of Theorem III.

If $0<n<23^3N$ in Lemma 9, write $\Gamma=\gamma-96$. Then

$$n\equiv N\Gamma^3+6\mu \pmod{96},\quad n\geqq N\Gamma^3. \tag{41}$$

If n is negative, write $\Gamma=\gamma-96w$, and choose a positive integer w so that $n\geqq N\Gamma^3$. If $n>23^3N$, take $\Gamma=\gamma$. In every case, (41) holds. As in the proof of Lemma 2, this implies

THEOREM IV. *If N is any integer prime to 6, every integer is represented by $N\Gamma^3+6(x^2+y^2+z^2)$, where the integer Γ may be negative.*

8. Representation of all large numbers. We prove the following theorem.

THEOREM V. *For $l=1, 2, 3, 4,$ or 5, $F_l=ly^3+C_7$ represents all sufficiently large integers.**

Let r be the real ninth root of $12/(6.9+l)$. Then $r>1$. The number of primes $\equiv 2 \pmod 3$ which exceed x and are $\leqq rx$ is known to increase indefinitely with x. Choose as x the first radical in (42). Hence for all sufficiently large integers n, there exist at least ten primes p such that

$$\left(\frac{n}{12}\right)^{1/9} < p \leqq \left(\frac{n}{6.9+l}\right)^{1/9}, \quad p \equiv 2 \pmod 3. \tag{42}$$

The product of the ten primes exceeds $(n/12)^{10/9}$ and hence exceeds n if $n>12^{10}$. Hence not all ten are divisors of n. Henceforth, let p be a prime $>l$ not dividing n and satisfying (42). By Lemma 1 there exist integers δ and M satisfying

$$n \equiv l\delta^3 \pmod{p^3}, \quad n - l\delta^3 = p^3M, \quad 0 < \delta < p^3.$$

By (42), $(6.9+l)p^9 \leqq n < 12p^9$. Hence

$$(6.9+l)p^9 - lp^9 \leqq n - l\delta^3 = p^3M, \quad p^3M < n < 12p^9.$$

Cancellation of factors p^3 gives

$$6.9p^6 < M < 12p^6.$$

Write $M=N+6p^6$. Then $0.9p^6<N<6p^6$. Let $p\geqq 11$. Then $N>22^8$. By Lemma 2 with $e=0$, N can be represented by $\gamma^3+6(d_1^2+d_2^2+d_3^2)$ with $\gamma\geqq 0$. If any $|d_i|\geqq p^3$, then $N\geqq 6p^6$, contrary to the above. Hence in

$$n = l\delta^3 + p^3M = l\delta^3 + 6p^9 + p^3\gamma^3 + 6p^3(d_1^2 + d_2^2 + d_3^2)$$
$$= l\delta^3 + (p\gamma)^3 + \sum_{i=1}^{3} [(p^3 + d_i)^3 + (p^3 - d_i)^3],$$

each cube is $\geqq 0$. This proves Theorem V.

The following second proof applies to numbers exceeding a much smaller limit. For n sufficiently large, there exist seven primes P satisfying

$$(n/12)^{1/6} < P \leqq (n/C)^{1/6}, \quad P \equiv 2 \pmod 3, \quad C < 12. \tag{43}$$

The earlier discussion applies when p^3 is replaced by P^2 and gives

$$n = l\delta^3 + P^2M, \quad M = N + 6P^4, \quad (C - l - 6)P^4 < N < 6P^4.$$

* For $l=1$, the case of 8 cubes, see Landau, Mathematische Annalen, vol. 66 (1909), pp. 102–5; *Verteilung der Primzahlen*, vol. 1, 1909, pp. 555–9. For $l=2$, Dickson, Bulletin of the American Mathematical Society, vol. 33 (1927), p. 299.

Thus $N \geqq 23^3P$ if

$$l + 6 + \left(\frac{23}{P}\right)^3 \leqq C < 12, \tag{44}$$

which holds if $C = l + 6.9$. Then by Theorem III, N is represented by $P\gamma^3 + 6(d_1^2 + d_2^2 + d_3^2)$ with $\gamma \geqq 0$. Hence

$$n = l\delta^3 + (P\gamma)^3 + \sum_{i=1}^{3} [(P^2 + d_i)^3 + (P^2 - d_i)^3],$$

where each cube is $\geqq 0$, since each $|d_i| < P^2$.

We may now readily verify that all integers of a wide range are sums of eight cubes. For $P > 1150$, (44) is satisfied if $C = 7.00001$. Take $n = Cm^6$. Then (43) gives

$$rm < P \leqq m, \quad r = (C/12)^{1/6}, \quad \log r = \bar{1}.9609862.$$

Start with $m = 1500$. Then $rm = 1371.1$. The ten primes $\equiv 2 \pmod 3$ between 1371 and 1500 are

$$1373, 1409, 1427, 1433, 1439, 1451, 1481, 1487, 1493, 1499.$$

Equating the fourth to rm', we get $m' = 1567.7$. Hence the last seven primes serve for every m from 1500 to m'. Repeating with m' in place of m, we get as further P's 1511, 1523, 1553, 1559. Hence 1487, 1493, 1499, and these four serve for every m from m' to 1626.7. We advance similarly to 1705.5, 1751.4, and $M = 1771.2$. But the four primes between M and the seventh prime 1733 serving for the third interval are all $\equiv 1 \pmod 3$. We may employ 41^2 and the last six of the seven primes, since their product by 41 exceeds the n corresponding to M, since (43) holds when $P = 41^2$, and since Lemma 1 holds when p is replaced by any product P of primes each $\equiv 2 \pmod 3$. Hence we advance from M to $1637/r = 1790.9$, and thence to 1823.7 (again using 41^2), 1856.5, $\mu = 1869.6$. Lacking new primes $\equiv 2 \pmod 3$, we use $P = 11 \cdot 167$ and note that the product of 11 and the last six of the seven primes exceeds the n corresponding to μ. We therefore advance to 1882.7. The next 13 steps proceed to 3307.1 by means of primes only, the number of available new primes being 2, 1, 5, 7, 7, 6, 4, 9, 8, 7, 11, 12, 10 respectively.

We may also proceed from 1500 to smaller values of m. Without new device, we reach 1163. For the next step we have available only five primes 1091, 1097, 1103, 1109, 1151, and $P = 5 \cdot 227, 11 \cdot 101, 23 \cdot 47$. The advance to $1061/r = 1160.7$ requires the verification that the integers n in the interval which are divisible by the five primes and one of the factors of each of the three P's are actually sums of 8 cubes. With occasionally a like verification, we may advance in 26 steps to 821. The next step would involve serious additional verifications, since there are available only 761, 773, 797, 809, $11 \cdot 71$, $17 \cdot 47$ as values of P.

The n corresponding to the final $m=821$ is $10^{17}(21.436)$. Employing technical theory of primes, Baer* proved that every integer $>23 \cdot 10^{14}$ is a sum of eight cubes. The interest of our work lies in its very elementary character.

By two applications of Lemma 6 with $t=1$, $B=8043$, we find that every integer between 8043 and 227, 297, 300 is a sum of eight cubes. This limit is nearly 4% larger than that obtained by Lemma 3.

* *Beiträge zum Waringschen Problem*, Dissertation, Göttingen, 1913.

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