THE BEHAVIOR OF A BOUNDARY VALUE PROBLEM AS THE INTERVAL BECOMES INFINITE*

BY

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The boundary value problems for linear self-adjoint differential equations of the second order with homogeneous linear boundary conditions at the ends of a finite interval have been extensively studied and the principal facts are well known. In this connection a problem of some interest arises if the ends of the interval at which the boundary conditions apply are allowed to recede to $-\infty$ and $+\infty$. The aim of this paper is to investigate the behavior of characteristic numbers, characteristic solutions, and oscillation properties, as the interval becomes infinite. A closely related problem for the differential equation

\[(d/dx)(p(x)du/dx) + (\lambda - q(x))u = 0\]

has been solved by Weyl† and further studied by Hilb‡ and Gray.§

In this paper some interesting results are obtained for the equation

\[d^2u/dx^2 + G(x, \lambda)u = 0\]

and are set forth in Theorems I and II.

It is planned to treat the degree of convergence of certain associated expansions for the infinite interval in a subsequent paper.

1. The differential equation under investigation is

\[d^2u/dx^2 + G(x, \lambda) = 0.\]

The function $G(x, \lambda)$ is assumed to be real and continuous and to possess a positive partial derivative with respect to $\lambda$ for all real values of $x$ and $\lambda$. It is further assumed that

\[\lim_{x \to -\infty} G(x, \lambda) = -\infty, \quad \lim_{x \to +\infty} G(x, \lambda) = +\infty,\]

and that

\[\lim_{x \to \pm \infty} G(x, \lambda) = -\infty.\]

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These conditions are all satisfied in the important special case
\[ G(x, \lambda) = \lambda - q(x) \]
provided that \( q(x) \) is real and continuous and \( \lim q(x) = +\infty \).

Associated with (1) we consider boundary conditions at \( x = a \) and \( x = b \),
\begin{align*}
\alpha_1 u(a) + \alpha_2 u'(a) + \alpha_3 u(b) + \alpha_4 u'(b) &= 0, \\
\beta_1 u(a) + \beta_2 u'(a) + \beta_3 u(b) + \beta_4 u'(b) &= 0.
\end{align*}
(4)

It is assumed of course that these conditions are linearly independent. Let
the determinant \( \alpha_i \beta_j - \alpha_j \beta_i \) be denoted by \( A_{ij} \), and let \( A_{12} = A_{34} \), so that
the conditions are self-adjoint. Then there exists* an infinite sequence of values
of \( \lambda, l_0, l_1, l_2, l_3, \ldots \), with limit point at \( +\infty \) only, and furnishing solutions
of (1) and (4). If we take these values in increasing order of magnitude, counting each double value twice, the solution \( u_n(x) \) corresponding to \( \lambda = l_n \)
vanishes at least \( n \) times and not more than \( n + 2 \) times in the interval
\( a < x \leq b \).

Now as \( a \) and \( b \) recede to \( -\infty \) and \( +\infty \) respectively, what becomes of
the \( l_n \), the \( u_n(x) \), and the roots of \( u_n(x) \)?

2. In order to answer the foregoing question we must recall some facts
concerning the solutions of equation (1), their roots and their behavior at
infinity. Multiply (1) by \( u(x) \) and integrate from \( x_1 \) to \( x_2 \), integrating the
first term by parts. After transposition we have
\[ u(x)u'(x) - u(x)u'(x_1) = \int_{x_1}^{x_2} u^2(x)dx - \int_{x_1}^{x_2} G(x, \lambda)u(x)dx. \]
(5)

From the hypothesis (3) it follows that there exist two numbers \( \alpha \) and \( \beta \)
such that \( G(x, \lambda) \) is negative when \( x > \beta \) and when \( x < \alpha \). Then if \( x_2 > x_1 > \beta \)
the right hand side of (5) is positive, which shows that the product \( u(x)u'(x) \)
does not vanish at both \( x_1 \) and \( x_2 \). Stated otherwise, \( u(x)u'(x) \) does not have
more than one root greater than \( \beta \) and does not have more than one root less
than \( \alpha \). The total number of roots of \( u(x) \) is therefore finite since the interval
between any two consecutive roots is not less than \( \pi/M \), where \( M^2 \) is a con-
stant such that \( G(x, \lambda) \leq M^2 \).

Now it is quite important to show that as \( x \) becomes infinite†
\[ \lim_{x \to \infty} u'(x)/u(x) = \infty. \]

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* See for example Birkhoff, these Transactions, vol. 10 (1909), p. 264.
† For the behavior of solutions at infinity see Wiman, Arkiv för Matematik, Astronomi och
To see this we note first of all that when \( x \) is greater than the greatest root of \( u(x) \) the function \( R = u'(x)/u(x) \) is continuous. Let \( N \) be any positive number, as large as we please, and let \( x \) be chosen so large that \( -G(x, \lambda) \) remains greater than \( 2N^2 \). By differentiation and substitution from (1) we get
\[
\frac{dR}{dx} = -G(x, \lambda) - R^2,
\]
so that if \( R \) ever comes within the interval \( -N < R < N \), the derivative \( dR/dx \) will be greater than \( N^2 \). Consequently \( R \) will increase beyond \( N \), and will not return, since \( R \) is continuous and \( dR/dx \) is positive for \( R = N \). Thus (6) is established.

Let the principal solutions of equation (1) at the origin be denoted by \( u_1(x) \) and \( u_2(x) \) so that
\[
(7) \quad u_1(0) = u_2'(0) = 1, \quad u_2(0) = u_1'(0) = 0.
\]
These solutions satisfy the well known identity
\[
(8) \quad u_1(x)u_2'(x) - u_2(x)u_1'(x) = 1.
\]
Then the general solution of equation (1) can be written
\[
(9) \quad u(x) = C(u_1^2 + u_2^2)^{1/2} \sin \left[ \phi(x) - \phi_1 \right],
\]
in which \( C \) and \( \theta \) are arbitrary constants and \( \phi(x) \) is defined by the equation
\[
(10) \quad \phi(x) = \tan^{-1} \frac{u_2(x)}{u_1(x)}.
\]
By differentiating (10) and using (8) we get
\[
\frac{d\phi}{dx} = [u_1^2 + u_2^2]^{-1},
\]
so that \( d\phi/dx \) is positive and \( \phi \) is an increasing function of \( x \). In view of the fact that \( u(x) \) has a finite number of roots, \( \phi(x) \) cannot increase indefinitely, and therefore must approach a limit. The same conclusion applies as \( x \) becomes negatively infinite. We therefore may define \( \phi_1 \) and \( \phi_2 \) as follows:
\[
\phi_1 = \lim_{x \to -\infty} \phi(x), \quad \phi_2 = \lim_{x \to -\infty} \phi(x).
\]

It is now convenient to define two pairs of independent particular solutions of equation (1) as follows:
\[
(11) \quad V_1(x) = (u_1^2 + u_2^2)^{1/2} \sin \left[ \phi(x) - \phi_1 \right],
\]
\[
W_1(x) = (u_1^2 + u_2^2)^{1/2} \cos \left[ \phi(x) - \phi_1 \right],
\]
and another pair \( V_2 \) and \( W_2 \) similarly defined with \( \phi_2 \) in place of \( \phi_1 \). As \( x \) approaches \( -\infty \) it can be shown that
3. It is now necessary to consider how $\phi_1$ and $\phi_2$ vary with $\lambda$. If $u(x)$ is a solution expressed in the form (9) in which $C$ and $\theta$ are independent of $\lambda$ we may derive in the usual manner* the equation

$$u' \frac{\partial u}{\partial \lambda} - u \frac{\partial u'}{\partial \lambda} = \int_0^z (\frac{\partial G}{\partial \lambda})u^2 \, dx,$$

since at the origin $\partial u/\partial \lambda = \partial u'/\partial \lambda = 0$ in view of (7). Let $r$ be a root of $u(x)$, so that $-u_1(r) \sin \theta + u_2(r) \cos \theta = 0$. By differentiating this equation with respect to $\lambda$ and making some simplifications by means of (8) and (14) we obtain

$$\frac{\partial r}{\partial \lambda} = - \left[ u_1^2(r) + u_2^2(r) \right] \int_0^z (\frac{\partial G}{\partial \lambda})u^2 \, dx.$$

This shows that as $\lambda$ increases all roots of $u(x)$ move toward the origin. In the same manner if $\lambda$ is constant and $\theta$ varies

$$\frac{\partial r}{\partial \theta} = u_1^2(r) + u_2^2(r),$$

from which we see that as $\theta$ increases all roots of $u(x)$ move to the right. Finally if $\lambda$ and $\theta$ vary in such a manner as to keep the root $r$ fixed, we get from (15) and (16)

$$\frac{\partial \theta}{\partial \lambda} = \int_0^r (\frac{\partial G}{\partial \lambda})u^2 \, dx.$$

We conclude that as $\lambda$ increases $\theta$ also increases when $r$ is positive, but decreases when $r$ is negative. Now at a root of $u(x)$ we have $\theta = \phi(x) + k\pi$ ($k = 0, \pm 1, \pm 2, \cdots$) so that

$$\frac{\partial \phi}{\partial \lambda} = \int_0^r (\frac{\partial G}{\partial \lambda})u^2 \, dx.$$

From this we see that $\phi_2$ increases as $\lambda$ increases while $\phi_1$ decreases as $\lambda$ increases.

4. Now let $\lambda$ increase from $-\infty$ to $+\infty$ and note the change in the quantity $\phi_2 - \phi_1$. Since there cannot be two roots of $u(x)u'(x)$ when $\lambda$ is

large and negative because $G(x, \lambda)$ is negative, it follows that, for such values of $\lambda, \phi_2 - \phi_1 < \pi$. But as $\lambda$ increases there will be an infinite sequence of values of $\lambda$ for which the increasing quantity $\phi_2 - \phi_1 = \pi, 2\pi, 3\pi, \cdots$, since $\phi_2 - \phi_1$ must increase without limit. We may denote these values of $\lambda$ in order of increasing magnitude by $\lambda_0, \lambda_1, \lambda_2, \cdots$. For $\lambda = \lambda_n$, it is at once apparent that the solutions $V_1(x)$ and $V_2(x)$ are identical, except perhaps for sign, and the same is true of $W_1(x)$ and $W_2(x)$. We may therefore define a solution $U_n(x)$ corresponding to $\lambda_n$ as follows:

$$U_n(x) = V_1(x) = \pm V_2(x), \text{ when } \lambda = \lambda_n.$$  

We have therefore

$$\lim_{x \to \pm \infty} U_n(x) = \lim_{x \to \pm \infty} U_1'(x) = 0.$$  

If we form equation (5) for the solution $U_n(x)$ and let $x_2$ approach $+\infty$ and $x_1$ approach $-\infty$, we see at once from (20) that the integrals

$$\int_{-\infty}^{+\infty} U_1^2(x)dx \quad \text{and} \quad \int_{-\infty}^{+\infty} G(x, \lambda) U_1^2(x)dx$$

both converge. From the second of these we may conclude that

$$\int_{-\infty}^{+\infty} U_1^2(x)dx$$

also converges.

Since the number of roots of $U_n(x)$ is equal to $k - 1$ when $\phi_2 - \phi_1 = k\pi$, we see that $U_n(x)$ has exactly $n$ roots.

The foregoing conclusions may be summarized as follows:

**Theorem I.** There exists an infinite set of critical values of $\lambda, \lambda_0, \lambda_1, \lambda_2, \cdots$, with limit point at $+\infty$ only, corresponding to which equation (1) has solutions (unique except for a constant factor) satisfying the conditions

$$\lim_{x \to \pm \infty} U_n(x) = \lim_{x \to \pm \infty} U_1'(x) = 0.$$  

The solution $U_n(x)$ vanishes $n$ times in the interval $-\infty < x < \infty$, and the integrals

$$\int_{-\infty}^{+\infty} U_1^2(x)dx, \quad \int_{-\infty}^{+\infty} U_1'(x)dx, \quad \text{and} \quad \int_{-\infty}^{+\infty} G(x, \lambda) U_1^2(x)dx$$

all exist.

5. We return to the consideration of the conditions (4). The general solution of (1) may be written

$$u(x) = V_1(x)h + W_1(x)k,$$
so that the conditions (4) are equivalent to

\[
\begin{align*}
[\alpha_1 V_1(a) + \alpha_2 V_1'(a) + \alpha_3 V_1(b) + \alpha_4 V_1'(b)] k & = 0, \\
[\beta_1 V_1(a) + \beta_2 V_1'(a) + \beta_3 V_1(b) + \beta_4 V_1'(b)] k & = 0,
\end{align*}
\]

(21)

The determinant of these equations is

\[
\Delta = \begin{vmatrix}
[\alpha_1 V_1(a) + \alpha_2 V_1'(a) + \alpha_3 V_1(b) + \alpha_4 V_1'(b)] & [\alpha_1 W_1(a) + \alpha_2 W_1'(a) + \alpha_3 W_1(b) + \alpha_4 W_1'(b)] \\
[\beta_1 V_1(a) + \beta_2 V_1'(a) + \beta_3 V_1(b) + \beta_4 V_1'(b)] & [\beta_1 W_1(a) + \beta_2 W_1'(a) + \beta_3 W_1(b) + \beta_4 W_1'(b)]
\end{vmatrix}.
\]

We shall treat only the case for which \( A_{42} \) is not zero, as the modifications to be made when \( A_{42} = 0 \) are sufficiently obvious. Let \( \epsilon \) be a positive constant, as small as we please. We may choose \( b \) so large that \( |V_1(b)/V_1'(b)| < \epsilon \), and then choose \( a \) so large (and negative) that \( |V_1(a)| < \epsilon, |V_1'(a)| < \epsilon, |W_1(a)/W_1'(a)| < \epsilon, |W_1'(b)/W_1'(a)| < \epsilon \), uniformly with respect to \( \lambda \) in an interval \( \lambda_{n-1} + \epsilon \leq \lambda \leq \lambda_n - \epsilon \). Then the determinant may be written

\[
\Delta = V_1'(b)W_1'(a)[A_{42} + \epsilon E],
\]

in which \( E \) denotes a function that is bounded for \( \lambda \) in the given interval. Since \( W_1'(a) \) does not vanish when \( a \) is large and negative and \( V_1'(b) \) does not vanish for \( \lambda \) in the given interval when \( b \) is large, we see at once that \( \Delta \) does not vanish in this interval. But in the next interval \( \lambda_n + \epsilon \leq \lambda \leq \lambda_{n+1} - \epsilon \) the sign is changed and therefore \( \Delta \) vanishes between \( \lambda_n - \epsilon \) and \( \lambda_n + \epsilon \). Therefore the roots of \( \Delta \) approach \( \lambda_n \) \((n = 0, 1, 2, \ldots)\). Moreover we can easily show that \( \Delta \) does not vanish more than once in the interval \( \lambda_n - \epsilon < \lambda < \lambda_n + \epsilon \), so that the characteristic numbers \( l_n \) of §1 are ultimately all simple and

\[
\lim l_n = \lambda_n.
\]

From (12), (13), (19), and (20) it will be seen that (21) will be satisfied by taking \( h \neq 0, k = 0 \) when \( a \) and \( b \) are infinite. Therefore the characteristic functions \( u_n(x) \) of §1 approach the functions \( U_n(x) \). We therefore have

**Theorem II.** As the ends of the interval recede to \( -\infty \) and \( +\infty \) the characteristic numbers all become simple and

\[
\lim l_n = \lambda_n \quad \text{and} \quad \lim u_n(x) = U_n(x) \quad (n = 0, 1, 2, \ldots),
\]

the limits being entirely independent of the boundary conditions (4).

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