ON A GENERALIZATION OF THE ASSOCIATIVE LAW*

BY

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1. In my investigations in group theory, I have observed that Lagrange’s theorem (that the order of a group is divisible by the order of any subgroup) does not use for its proof the Associative Law in its whole extent; this law can be replaced by a more general postulate, “Postulate A”, as I shall call it.

We shall represent our elements by capital italic letters; the operation upon them may be represented by a star \( \star \), so that \( A \star B \) signifies the result of this operation performed upon \( A \) and \( B \). A set of elements closed under any operation \( \star \) may be called “a group”; this word is thus used in a more general sense than is usual, since the operation \( \star \) is arbitrary. The ordinary groups with a special well known operation may be called “classic” to distinguish them from our generalised groups. Sets and groups will be denoted by capital German letters.

Postulate A. In the equation

\[
(1) \quad (X \star A) \star B = X \star C,
\]

the element \( C \) depends upon the elements \( A \) and \( B \) only and not upon \( X \). (We suppose here that \( X \) can be an arbitrary element of a finite group to which \( A, B \) and \( C \) belong also.)

The Associative Law is obviously a special case of this Postulate A, viz. if \( C = A \star B \).

I have investigated the finite groups that are obtained by replacing the Associative Law in the system of postulates of Frobenius* by Postulate A. I have found the following properties of these groups.

I. Besides our operation \( \star \) every group \( \mathfrak{G} \) of our type has another operation that will be denoted by a little circle \( \circ \) and defined as follows: the equation \( (1) \) being given, we write

\[
(2) \quad C = A \circ B.
\]

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† Frobenius (Über endliche Gruppen, Berliner Sitzungsberichte, 1895) defines the classic finite groups by the four following postulates: 1. The operation that will be considered is uniform (eindeutig) and applicable to any two elements. 2. This operation is uniformly reversible (eindeutig umkehrbar), i.e. from \( AB = AC \) or \( BA = CA \) it follows that \( B = C \). 3. The Associative Law is true for it. 4. The operation is “limited in its effect” (begrenzt in ihrer Wirkung); that signifies the possibility of forming finite groups of our elements.
A GENERALIZATION OF THE ASSOCIATIVE LAW

It is easy to see, that \( \mathfrak{G} \) is also a group relative to the operation \( \circ \); we express this fact by writing \( \mathfrak{G}(\circ) \); (analogously, \( \mathfrak{G}(\ast) \)). I shall prove that \( \mathfrak{G}(\circ) \) is classic.

II. The group \( \mathfrak{G}(\ast) \) has always a right unit (the same for all its elements).

III. If the group \( \mathfrak{G}(\ast) \) has also a single left unit for all its elements (that must necessarily coincide with the right unit), then the Associative Law is true for \( \mathfrak{G}(\ast) \); in this case \( \mathfrak{G}(\ast) \) is classic and the operations \( \ast \) and \( \circ \) are identical.

It follows that in the systems of postulates of Moore\(^*\) and Dickson\(^\dagger\) for the definition of classic groups the Associative Law can be replaced by Postulate A (or its left analogue).

IV. We associate with every element \( A \) of our group \( \mathfrak{G} \) a substitution

\[
\overline{A} = \begin{pmatrix} X \\ X \ast A \end{pmatrix}.
\]

whereby \( X \) runs over all elements of \( \mathfrak{G} \). I prove that all those substitutions \( \overline{A} \) (corresponding to each element \( A \) of \( \mathfrak{G} \)) form a substitution group \( \overline{\mathfrak{G}} \) which is obviously classic and simply isomorphic with \( \mathfrak{G}(\circ) \). Conversely, all such substitutions \( \overline{A} \) form a group only if the Postulate A is true for \( \mathfrak{G}(\ast) \).

V. All groups of our type will be obtained from classic groups by making any substitution in the head-line of Cayley’s table of a classic group. Moreover, it is sufficient to make only such substitutions as do not alter the unit of the classic group. Such a substitution may be denoted by \( \alpha \).

VI. \( \mathfrak{G}(\ast) \) being any subgroup of \( \mathfrak{G}(\ast) \), \( \mathfrak{G}(\circ) \) is also a subgroup of \( \mathfrak{G}(\circ) \), i.e. relative to the operation \( \circ \). The converse is not true. Every subgroup \( \mathcal{G} \) of \( \mathfrak{G} \) relative to \( \circ \) is also a group relative to \( \ast \), if and only if the substitution \( \alpha \), which corresponds to \( \mathfrak{G}(\ast) \), has the following form:

\[
\alpha = \begin{pmatrix} X \\ X \ast \end{pmatrix},
\]

the numbers \( l \) being relatively prime to the orders of corresponding elements \( X \).

2. We shall prove now all the assertions of §1.

I. The group \( \mathfrak{G}(\circ) \) is obviously uniformly reversible. Again:

\[
[(X \ast A) \ast B] \ast C = [X \ast (A \circ B)] \ast C = X \ast [(A \circ B) \circ C];
\]

\(^*\) Moore, A definition of abstract groups, these Transactions, vol. 3 (1902).

\(^\dagger\) Dickson, Definition of a group and a field by independent postulates, these Transactions, vol. 6 (1905).

\(^\dagger\) The sign \( \simeq \) signifies that we denote a complicated expression more simply with a single letter.
and on the other hand
\[(X \star A) \star B] \star C = (X \star A) \star (B \circ C) = X \star [A \circ (B \circ C)];\]
and hence the Associative Law is true for \(\circ\) (o).

II. The classic group \(\circ\) (o) has always a unit \(E\); it is such that
\[(X \star E) \star A = X \star (E \circ A) = X \star A;\]
and therefore
\[X \star E = X\] for every \(X\);
\(E\) is thus the right unit for \(\circ\) (o).

III. Let \(E\) be a left unit of \(\circ\) (x); we have, then,
\[(E \star A) \star B = E \star (A \star B) = A \star B;\]
and hence by virtue of Postulate A for every element \(X\)
\[(X \star A) \star B = X \star (A \star B),\]
i.e. the Associative Law; hence \(\circ\) (x) is classic, and \(A \circ B = A \star B\).

IV. It follows from (1), by virtue of Postulate A, that \(\overline{AB} = C\); hence \(\overline{\circ}\) is a substitution group simply isomorphic with \(\circ\) (o) (see (2)).

Conversely, let \(\circ\) (x) be any finite uniformly reversible group and let \(\overline{\circ}\) be the set of corresponding substitutions, which form also a (classic) group. Let
\[\overline{AB} = C, \text{ or } \begin{pmatrix} X \\ X \star A \end{pmatrix} \begin{pmatrix} X \\ X \star B \end{pmatrix} = \begin{pmatrix} X \\ X \star C \end{pmatrix};\]

since
\[\begin{pmatrix} X \\ X \star B \end{pmatrix} = \begin{pmatrix} X \star A \\ (X \star A) \star B \end{pmatrix},\]
it follows that
\[(X \star A) \star B = X \star C\]
for each element \(X\) of \(\circ\) (x); hence Postulate A holds.

V. In the head-line of Cayley’s table of \(\circ\) (x) we make the following substitution:
\[\alpha = \begin{pmatrix} X \\ E \star X \end{pmatrix}\]
\((E\) being the right unit of \(\circ\) (x)). Let \(E \star X = X’\). We define the third operation \(\times\) as follows:
\[(3) \quad A \times B = A \star X’\].
The operation $\times$ is uniformly reversible and also associative; in fact we have from (1) and (3):

$$\tag{4} (X \times A') \times B' = X \times C',$$

$C'$ depending on $A'$ and $B'$ only but not on $X$; let $X = E$; then $(E \times A') \times B' = E \times C'$; but we have $E \times X' = E \times X = X'$; hence $C' = A' \times B'$, and (4) gives us the Associative Law for $\times$; thus $\mathfrak{S}(\times)$ is classic. Again it follows from (2) that $\alpha$ gives an isomorphism between $\mathfrak{S}(\circ)$ and $\mathfrak{S}(\times)$.

Conversely, let $\mathfrak{S}(\times)$ be now a given classic group; we make in the headline of Cayley's table of $\mathfrak{S}(\times)$ any substitution

$$\beta = \begin{pmatrix} X \\ X' \end{pmatrix}$$

and define a new operation $*$ as follows:

$$A \times B = A \star B.$$

The operation $*$ is obviously uniform and uniformly reversible; the Postulate $A$ is also true for $*$; in fact, if

$$(X \times A) \star B = (X \times A) \times B = X \times (A \times B);$$

we have

$$(X \times A) \star B = X \times C;$$

and

$$X \star C = X \times C;$$

hence $A \times B = C$ and thus $C$ depends upon $A$ and $B$ only.

$E$ being the unit of $\mathfrak{S}(\times)$, $E$ is the right unit for $\mathfrak{S}(\star)$; we have in fact

$$A \star E = A \times E = A.$$

I affirm that we can replace $\beta$ by another substitution $\alpha$, which does not alter $E$, and in this manner define a new operation, say $\square$, so that the group $\mathfrak{S} (\square)$ will be simply isomorphic with $\mathfrak{S}(\star)$ and have the right unit $E$. We take for $\alpha$

$$\alpha \simeq \begin{pmatrix} X \\ X \end{pmatrix} \begin{pmatrix} E \star X \\ X \end{pmatrix} = \begin{pmatrix} X \\ X \end{pmatrix} \begin{pmatrix} E \star X \\ X \end{pmatrix} = \begin{pmatrix} X \\ X \end{pmatrix} \begin{pmatrix} E \times X \\ X \end{pmatrix};$$

let $E \times X' = X'$; we can write then

$$\alpha = \begin{pmatrix} X \\ X \end{pmatrix} \begin{pmatrix} X' \\ X' \end{pmatrix} = \begin{pmatrix} X' \end{pmatrix};$$

and so we define

$$A \times B = A \square B'.$$
We shall prove that the substitution

$$\alpha_1 = \left( \bar{X} \right)$$

gives an isomorphism between the groups $\mathfrak{G} (\star)$ and $\mathfrak{G} (\square)$. Let

(5)  
$$\bar{A} \star \bar{B} = \bar{C} ;$$

we shall prove that we shall have also

(6)  
$$A' \square B' = C'.$$

It follows from (5) that $\bar{A} \times \bar{B} = \bar{C}$; but $\bar{A} = \bar{E} \times A'$, $\bar{C} = \bar{E} \times C'$; hence $(\bar{E} \times A') \times B = \bar{E} \times C'$; and since $\mathfrak{G} (\times)$ is classic,

$$\bar{E} \times (A' \times B) = \bar{E} \times C' ;$$

hence $A' \times B = C'$, and so (6) is established.

VI. Let $\mathfrak{H} = P_1 + P_2 + P_3 + \cdots$, $\mathfrak{H} (\star)$ being a subgroup of $\mathfrak{G} (\star)$. Let

$$(X \star P_1) \star P_2 = X \star P_2 ;$$

the elements $P_1$ and $P_2$ of $\mathfrak{H}$ being given, the element $P_\mu$ exists also in $\mathfrak{H}$; by virtue of Postulate A we have $P_\alpha P_\lambda = P_\mu$; hence $\mathfrak{H} (\circ)$ is also a group.

It follows, hence, that Lagrange's theorem is true for the groups $\mathfrak{G} (\star)$ of our type.

Let $\mathfrak{H}$ be now a subgroup of $\mathfrak{G}$ relative to $\circ$; we shall analyse the conditions by which $\mathfrak{H}$ is also a group relative to $\star$. Let $\alpha$ be the same substitution as in V, and

$$\mathfrak{H}' = P_1' + P_2' + P_3' + \cdots .$$

($P_1', P_2', P_3', \cdots$ are elements in $\mathfrak{G}$ corresponding to $P_1, P_2, P_3, \cdots$, by virtue of $\alpha$.) Since $\alpha$ gives an isomorphism between $\mathfrak{G} (\circ)$ and $\mathfrak{G} (\times)$ ($\times$ being the operation defined by (3)), $\mathfrak{H}' (\times)$ is also a group (relative to $\times$).

Let $\mathfrak{H} (\star)$ be also a group; then

$$P_\star P_\lambda = P_\star \times P_\lambda = P_\mu .$$

If $P_\mu$ runs over all elements of $\mathfrak{H}'$, then $P_\mu$ runs over all elements of $\mathfrak{H}$, and conversely. Hence

$$P_\star \times \mathfrak{H}' = \mathfrak{H}$$

(for each $P_\star$ of $\mathfrak{H}$). Consequently $\mathfrak{H}$ is one of the partitions of $\mathfrak{G} (\times)$ relative to $\mathfrak{H}' (\times)$†. This condition is obviously also sufficient for $\mathfrak{H} (\star)$ to be a group.

* The sign $+$ signifies that the elements $P_1, P_2, \cdots$ form a set $\mathfrak{H}$.

Since the substitution $\alpha$ does not alter the unit $E$ of $\mathfrak{G}(\times)$, $\mathfrak{H}$ and $\mathfrak{H}'$ must be identically equal to each other, because both of them have a common element $E$.

We shall now analyse the conditions by which every subgroup $\mathfrak{H}(\circ)$ of $\mathfrak{G}(\circ)$ is also a group relative to $\star$. Then we must have $\mathfrak{H}' = \mathfrak{H}$ (our notation remains as above) for every subgroup $\mathfrak{H}(\circ)$. We take $\mathfrak{H}' = \mathfrak{H}(\times) = \{P\}$, a cyclic group, $P$ being an arbitrary element of $\mathfrak{G}$. Since $\{P\}$ must be also a group relative to $\star$, we have

$$P^* \star P = P^* \times P^1,$$

consequently for each element $X$ of $\mathfrak{G}$ also,

$$X \star P = X \times P^1.$$

More generally,

$$X^k P^* = X \times P^k.$$

To every exponent $k$ in (8) there corresponds one and only one exponent $\lambda$ and vice versa. This must be true for each element $P$ of $\mathfrak{G}$; if we take $P^*$ instead of $P$, we obtain, in the same manner as in (8),

$$X^k P^{k \mu} = X \times P^{k \nu};$$

for every $\mu$ there is a definite $\nu$ and vice versa. Let $m$ be the order of $P$, and $d$ the greatest common divisor of $k$ and $m$; then $m/d$ is the order of $P^*$ and each exponent $k \mu$ and $k \nu$ in (9) is divisible by $d$. Conversely, if one of the exponents $k, \lambda$ in (8) is prime to $m$, the other is also prime to $m$. Consequently the exponent $l$ in (7) or (7') must be prime to $m$. Thus $\alpha$ has in this case the following form:

$$\alpha = \left( \begin{array}{c} X \\ X' \end{array} \right),$$

where the numbers $l$ are prime to the orders of corresponding elements $X$. This condition is not only necessary but also sufficient: if it holds, then every cyclic subgroup $\{P\}$ of $\mathfrak{G}(\times)$ is also a group relative to $\star$. But hence every subgroup $\mathfrak{H}(\times)$ of $\mathfrak{G}(\times)$ is also a group relative to $\star$. $Q$ and $P$ being any two elements of $\mathfrak{H}$, we have in fact $Q^* P = Q \times P^1$; thus $Q^* P$ belongs also to $\mathfrak{H}$.

We can take, in particular, a substitution $\alpha$ of the following form:

$$\alpha = \left( \begin{array}{c} X \\ X' \end{array} \right),$$

where $r$ is the same for each element $X$ and relatively prime to the order of our group $\mathfrak{G}$.
3. We shall consider now a special case of Postulate A, that is, however, more general than the Associative Law.

Postulate B. In the equation

\[(X \ast A) \ast B = X \ast (A \ast B_1),\]

the elements \(B\) and \(B_1\) depend only upon each other; every \(B\) is completely defined by the corresponding \(B_1\), and conversely.

This postulate can be expressed in another form as follows:

Postulate B'. If

\[A \ast B = C \ast D\]

and if \(K\) is an arbitrary element, then

\[A \ast (B \ast K) = C \ast (D \ast K)\].

We prove first that Postulate B' follows from Postulate B. Let \(R\) be an element such that

\[A \ast (B \ast K) = (A \ast B) \ast R ;\]

\(R\) depends upon \(K\) only (by Postulate B). Again it follows from (11) that

\[(A \ast B) \ast R = (C \ast D) \ast R ;\]

and by Postulate B it follows from (13) that

\[C \ast (D \ast K) = (C \ast D) \ast R ;\]

hence, from (11), (13), (14), (15) it follows that (12) holds.

Second, we prove that Postulate B follows also from the Postulate B'. For that purpose we shall prove the following lemma:

**Lemma:** If Postulate B' is true for a (uniformly reversible) group \(\mathcal{G}\) (\(*\)), there exists in \(\mathcal{G}\) (\(*\)) a right unit (for all elements of \(\mathcal{G}\) (\(*\))).

If \(B\) is a given element, there always exists in \(\mathcal{G}\) (\(*\)) an element \(E\) such that

\[B \ast E = B.\]

Let \(D\) be an arbitrary element of \(\mathcal{G}\) (\(*\)); if \(A\) is also a given element, there always exists an element \(C\), for which

\[A \ast B = C \ast D ;\]

by virtue of Postulate B' we have, then,

\[A \ast (B \ast E) = C \ast (D \ast E) ;\]
and by virtue of (16) and (11) it follows from (17) that $E$ is the right unit for every element $D$.

Assume now that $A \star B = C = C \star E$; in the hypothesis of Postulate $B'$ we have, $K$ being an arbitrary element,

$$A \star (B \star K) = C \star (E \star K) = (A \star B) \star (E \star K);$$

and thus we have in (13) $E \star K = R$; this shows that Postulate $B$ holds for our group.

Since the groups with Postulate $B$ form a special case of groups with Postulate $A$, they can be obtained in the same manner as groups with Postulate $A$ ($\S 1, V$). We must now examine what must be the substitution $\alpha$ ($\S 1, V$), in order that we may obtain a group $\mathfrak{G} (\star)$ with Postulate $B$. The answer is given by the following theorem:

Theorem. If the group $\mathfrak{G} (\star)$ is obtained from the classic group $\mathfrak{G} (\times)$ by means of the substitution $\alpha$, Postulate $B$ is true for $\mathfrak{G} (\star)$ if and only if $\alpha$ is an automorphism of the group $\mathfrak{G} (\times)$. In this case $\alpha$ is also an automorphism for $\mathfrak{G} (\star)$, and the operations $\circ$ and $\times$ coincide with each other.

Let $\mathfrak{G} (\star)$ be a group with Postulate $B$. The equation (10) gives a dependence of $B$ and $B_1$ upon each other; this dependence is given by a substitution, that we denote symbolically by $(\star)$. Let $A = E$ (the right unit of $\mathfrak{G} (\star)$) in (10); then

$$X \star B = X \star (E \star B_1);$$

and hence

(18)

$$B = E \star B_1.$$

Let

$$\alpha = \left( \begin{array}{c} X \\ X' \end{array} \right);$$

we have then

$$B = E \star B_1 = E \times (B_1)' = (B_1)' ;$$

and hence

$$\alpha = \left( \begin{array}{c} X_1 \\ X \end{array} \right) = \left( \begin{array}{c} X \\ X_1 \end{array} \right)^{-1}.$$

Moreover it follows from (10), if we use the notation $A \star B_1 \simeq C$, that

$$C = A \circ B = A \star B_1 = A \times (B_1)' = A \times B;$$

thus the operations $\circ$ and $\times$ coincide.
Conversely, suppose that the operations $\circ$ and $\times$ coincide. Let

$$\alpha = \begin{pmatrix} X \\ X' \end{pmatrix} = \begin{pmatrix} X_1 \\ X \end{pmatrix};$$

we have then

$$X\alpha A = X \times A';$$

and

$$(X\alpha A)\alpha B = X\alpha(A \times B) = X\alpha(A\alpha B_1),$$

and that is Postulate B, because $B$ and $B_1$ depend only on each other. Again, by (19),

$$(X\alpha A)\alpha B = X\alpha C = (X \times A') \times B' = X \times C';$$

and hence

$$A \times B = C, \quad A' \times B' = C';$$

this shows us that $\alpha$ is an automorphism of $\mathfrak{G}(X)$. Conversely, let $\alpha$ be an automorphism of $\mathfrak{G}(X)$; then

$$(X\alpha A)\alpha B = (X \times A') \times B' = X \times (A' \times B')$$

$$= X \times (A' \times B') = X\alpha(A \times B) = X\alpha(A\alpha B_1);$$

and thus Postulate B holds.

It remains to prove that $\alpha$ is in this case an automorphism of $\mathfrak{G}(\ast)$ also. We have in fact

$$(A\ast B)' = (A' \times B')' = A' \times (B')' = A'\ast B'.$$

4. In the theory of uniformly reversible groups we can consider the operations inverse to the operation of a given group. Since the operation of our group is performed upon two elements (viz. $X\ast Y$), two inverse operations exist according as the left or the right of these two elements is unknown to us.

If the commutative law is true for our group, such a group has only one inverse operation and only one “inverse group” (i.e. the group relative to the inverse operation). But although a general classic group has two “inverse groups,” it has only one inverse operation (abstractly considered), because the properties of the operation of a classic group are “symmetric,” i.e. the same on both sides; two “inverse groups” of a classic group are simply isomorphic to each other (if our notations are conveniently chosen); this follows from the fact that a classic group is always “anti-isomorphic” to itself, i.e. there always exists such a substitution $\frac{\lambda}{\lambda}$ of elements of a classic group, that if $A, B$ correspond respectively to $\overline{A}, \overline{B}$, then $AB$ corresponds to $\overline{BA}$; we can take, for example, $\overline{X} = X^{-1}$. 

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The operation of a finite classic group $\mathcal{G}$ may be denoted by $\times$, the two inverse operations by $\triangle$ and $\triangledown$; more precisely,

if $A \times B = C$, then $C \triangle B = A$, $C \triangledown A = B$.

Both inverse groups $\mathcal{G} (\triangle)$, $\mathcal{G} (\triangledown)$ are finite and uniformly reversible but not associative. Let us consider what influence the associative law of the operation $\times$ makes on the operations $\triangle$ and $\triangledown$. Let $(A \times B) \times C = A \times (B \times C) \cong R$; $A \times B \equiv P$; $B \times C \equiv Q$; then $P \times C = A \times Q = R$. Hence $P \triangle B = A$, $Q \triangle C = B$, $R \triangle C = P$, $R \triangle Q = A$; consequently

$$(R \triangle C) \triangle B = R \triangle Q, \quad \text{and} \quad B \times C = Q.$$  

[Or $P \triangledown A = B$, $Q \triangledown B = C$, $R \triangledown P = C$, $R \triangledown A = Q$, $(R \triangledown A) \triangledown B = R \triangledown P$, and $P = A \times B$.] This is Postulate A, that is true for the operation $\triangle$ (and for $\triangledown$). But the operations $\triangle$ and $\triangledown$ are subject to still another postulate, viz.:

**Postulate J.** Every element $X$ satisfies the equation

$$X \triangle X = E \quad (\text{or} \quad X \triangledown X = E),$$

where $E$ is a determined element (the unit of the direct operation $\times$).

**Theorem 1.** A finite uniformly reversible group $\mathcal{G} (\ast)$ is an “inverse” to a classic group, if and only if it is subject to the postulates A and J.

Only one part of this theorem remains for us to prove. Let $\mathcal{G} (\ast)$ be subject to the postulates A and J. We use the same notation as before; if $A \ast B = C$, then $C \triangle B = A$. We must prove that $\mathcal{G} (\triangle)$ is classic. Obviously the operation $\triangle$ is uniform and uniformly reversible. Again, we have $(X \ast A) \ast B = X \ast C = Z$; $C$ depends upon $A$ and $B$ only; let $X \ast A = Y$; then $Y \ast B = X \ast C = Z$; $Z \triangle C = X$, $Z \triangle B = Y$, $Y \triangle A = X$; thus

$$(Z \triangle B) \triangle A = Z \triangle C,$$

which is Postulate A for the operation $\triangle$. It follows from Postulate J, that the group $\mathcal{G} (\triangle)$ has a left unit $E$; and hence (see §1, III) $\mathcal{G} (\triangle)$ is classic.

We consider a special case, when our classic group is abelian. We obtain then

**Theorem 2.** A finite uniformly reversible group $\mathcal{G} (\ast)$ is an “inverse” to an abelian group, if and only if it is subject to the postulates B and J.

Let $\mathcal{G} (\ast)$ be subject to the postulates B and J; by the preceding theorem the inverse group $\mathcal{G} (\triangle)$ is classic; it remains for us to show that $\mathcal{G} (\triangle)$ is commutative. We have

$$(X \ast A) \ast B = X \ast (A \ast B),$$

(10)
B and $B_i$ depending only upon each other. Let $A \star B_i = C$; then (as in the preceding theorem) $B \triangle A = C$, $C \triangle B_i = A$; hence

\[(20) \quad B \triangle A \triangle B_i = A.\]

We write this without brackets, because the Associative Law is true for $\triangle$; (20) is true for each element $A$; we take $A = E$ (unit); then $B \triangle B_i = E$; $B_i = B^{-1}$; and thus from (20) it follows that $A \triangle B = B \triangle A$; i.e., the Commutative Law holds for $\triangle$.

Conversely, let $\otimes (\triangle)$ be an abelian group; we must prove that $B$ and $B_i$ in (10) depend only upon each other. But (20) gives $A \star B^{-1} = B \triangle A = C$; hence $B_i = B^{-1}$ in (10), and Postulate B holds for $\star$.

The postulates $B$ and $J$ are characteristic for the operation of division. Thus it is possible, for example, to construct an abstract theory of proportions.

Supplement

Example I. A group with Postulate $A$ but not classic (see Table 1). This group is obtained from the symmetric group of 6th order by making in the head-line of Cayley's table of this group (see Table 2) the following substitution:

\[
\begin{pmatrix}
E & A & B & C & D & F \\
E & C & D & A & F & B
\end{pmatrix}.
\]

Example II. A group with Postulate $B$ but not classic (see Table 3). This group is obtained from the same symmetric group by making in the head-line of Table 2 the following substitution:

\[
\begin{pmatrix}
E & A & B & C & D & F \\
E & B & A & C & F & D
\end{pmatrix}.
\]

This substitution gives an automorphism of the symmetric group of 6th order.

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