PROPERTIES OF FUNCTIONS REPRESENTED BY THE DIRICHLET SERIES $\sum(a\nu+b)^{-s}$, OR BY LINEAR COMBINATIONS OF SUCH SERIES*

BY

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Introduction. The following paper deals with the functions

$$(1) \qquad Z(a, b, s) = Z(s) = \sum_{\nu=0}^{\infty}(a\nu + b)^{-s},$$

in which a and $b(\leqq a)$ represent real, positive numbers, and $s=\sigma+it$. They include as a special case $(a=b=1)$ the Riemann function $\zeta(s)$, characterized by Gram as "une des plus remarquables acquisitions de l'analyse moderne"; while simple linear combinations of $Z(a, b, s)$ for $b=1, 2, \cdots, a-1$, when a is a fixed integer, give the Dirichlet L-functions. All of these special functions, particularly $\zeta(s)$, are of the highest importance in the modern analytic theory of numbers and an immense amount of investigation has been lavished upon them.

Moreover, since $Z(s)=a^{-s}\,\zeta(s, b/a)$, the close relationship of $Z(s)$ to the generalized ζ-function $\zeta(s, w)$† lends an additional importance to these functions. Accordingly, §§1–8 may be regarded as a contribution to the theory of $\zeta(s, w)$. The function $Z(s)$, however, is more convenient for the purposes of the present paper.

The great value of the above mentioned and other functions for the theory of numbers has stimulated much activity during the past twenty years in the general theory of functions represented by Dirichlet series, quite apart from any question of applications, although, no doubt, with the idea of throwing more light on the properties of functions having such use. This is the point of view in what follows.

The main object of §§1–8 is to determine the real zeros of $Z(s)$ (formula (7))‡ and, approximately, the region of the plane in which the complex zeros are located (§4).

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† For $\zeta(s, w)$ see Bohr and Cramér, *Die neuere Entwicklung der analytischen Zahlentheorie*, Encyclopädie der Mathematischen Wissenschaften, II C 8, 1922, pp. 777–779. Recently $\zeta(s, w)$ has been adopted into the analytic theory of numbers as a basis of comparison. See E. Landau, *Vorlesungen über Zahlentheorie*, vol. 2, 1927, p. 9 ff.

‡ These are in striking contrast to the real zeros of $\zeta(s)$ which occur at $s=-2, -4, \cdots, -2n, \cdots$, so that $\zeta(s)$ contains the factor $1/\Gamma(1+s/2)$. For this reason these are called the "trivial zeros" of $\zeta(s)$. The L-functions, likewise, have trivial zeros on the negative real axis.

That complex zeros do occur is established (§5) by the actual calculation of these zeros for $Z(3, 1, s)$ and $Z(3, 2, s)$ between the limits $0<t<50$. The interesting fact is revealed in these two cases that the complex zeros (as far as determined) are scattered about irregularly in the narrow strip $0<\sigma<1$ and do not lie on a straight line as is presumably the case with the ζ- and L-functions.*

In §6 the formulas for numerical computation are developed and discussed, and in §7 the curves $C=0$, $S=0$, along which the real and imaginary parts of $Z(s)=C(\sigma, t)+iS(\sigma, t)$ vanish, are investigated. It is found that all these curves, excepting the curves C for which $b=1$, have a series of right-hand ($\sigma>0$) asymptotes occurring at equal distances, a result which is very useful in numerical calculation and in the determination of the number of imaginary zeros in the region $0<t<T$(§16). The behavior of these curves for $\sigma<0$ is much more complicated, the approximate variations of C and S being given by (18), and of $Z(s)$ by (19).

The remainder of the paper is devoted to the problem of determining linear combinations of $Z(a, b, s)$ which satisfy functional equations of a type similar to that of which $\zeta(s)$ is a solution. This problem, which was solved by Cahen† for a a prime, is completed for all integer values of a (Theorems 3 and 4). The functions $f(s)$ thus determined include the L-functions as very particular cases and form a large class of new functions which seem to be worthy of further study. The problem of functional equations is somewhat generalized (§18) so as to include the L-functions with imaginary characters.

In §15–17 we deal very briefly with the complex roots of the functions $f(s)$, the number of such roots being infinite (§17). It can be proved that

* The famous and as yet unproved "Riemann Hypothesis" (1859) supposes that all the complex zeros of $\zeta(s)$ lie on the line $\sigma=\frac{1}{2}$. Confidence in this hypothesis was first put on a solid basis by the aid of numerical computation. In particular, J. P. Gram, *Note sur les zéros de la fonction $\zeta(s)$ de Riemann*, Acta Mathematica, vol. 27 (1903), pp. 289–304, calculated the roots on $\sigma=\frac{1}{2}$ as far as $t=65$; and R. J. Backlund, *Ueber die Nullstellen der Riemannschen Zetafunktion*, Dissertation, Helsingfors, 1916, located within narrow limits all the remaining zeros up to $t=200$ and proved that all the roots within the limits $0<t<200$ lie on $\sigma=\frac{1}{2}$. Bohr and Cramér (loc. cit., p. 773, foot-note 133) remark that "this result is one of the most powerful arguments for the belief in the Riemann hypothesis." More recently the proof of this hypothesis has been extended as far as $t=300$ by J. I. Hutchinson, *On the roots of the Riemann zeta function*, these Transactions, vol. 27 (1925), pp. 49–60. Similar results have been obtained for some of the L-functions by Grossmann. The importance of this question is testified to by the increasing number of theorems in the theory of numbers whose proofs depend on an assumption of the Riemann hypothesis. See, for example, the long paper by Hardy and Littlewood quoted in the first foot-note to p. 343.

† See footnote, p. 329.

some of these functions have an infinity of roots on the line $\sigma=\frac{1}{2}$.*

1. **Fundamental formulas.** The series (1) is absolutely convergent for all values of s satisfying the inequality

$$\sigma > 1\,. \tag{2}$$

To convert (1) into a series valid over a larger part of the s-plane, we start with the formula

$$Z(s) = \sum_{\nu=0}^{n-1}(a\nu+b)^{-s} + \frac{1}{2(an+b)^s} + \frac{(an+b)^{1-s}}{a(s-1)} - as\int_n^\infty \frac{\bar{\phi}_1(x)dx}{(ax+b)^{s+1}} \tag{3}$$

in which $\bar{\phi}_1(x)=x-[x]-\frac{1}{2}$; that is, $\bar{\phi}_1(x)$ is a periodic function, with period 1, which coincides with the values of $x-\frac{1}{2}$ in the interval $0\leqq x<1$.

To prove (3), denote the right member by $F(s, n)$. Separate the integral into $\int_n^{n+1}+\int_{n+1}^\infty$ and substitute $x=\xi+n$ in the first of these, giving

$$\int_0^1 \frac{(\xi-\frac{1}{2})d\xi}{(a\xi+an+b)^{s+1}},$$

which may be integrated by parts. After substituting the result in the right member of (3) we obtain

$$F(s, n) = F(s, n+1) = F(s, n+2) = \cdots = \lim_{n\to\infty} F(s, n) = Z(s).$$

In taking the limit we must restrict s by (2); but having established the relation for $\sigma>1$ we extend it to other values of σ by the principle of analytic continuation.

Formula (3) shows that $Z(s)$ has a pole of the first order at $s=1$ with residue $1/a$. By successive integration of the last term by parts we obtain the more general formula†

$$Z(s) = \sum_{\nu=0}^{n-1}\frac{1}{(a\nu+b)^s} + \frac{1}{2(an+b)^s} + \frac{(an+b)^{1-s}}{a(s-1)} + \sum_{\nu=1}^{k} T_\nu + R_k, \tag{4}$$

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* This was first proved for $\zeta(s)$ by G. H. Hardy, *Sur les zéros de la fonction $\zeta(s)$ de Riemann*, Comptes Rendus, vol. 158 (1914), pp. 1012. Hardy's theorem was extended to the L-functions by E. Landau, *Ueber die Hardysche Entdeckung unendlich vieler Nullstellen der ζ-Funktion mit reellem Teil $\frac{1}{2}$*, Mathematische Annalen, vol. 76 (1915), pp. 212–243.

† This formula (excepting the inequality for R_k) is given by J. Grossmann, *Ueber die Nullstellen der Riemannschen ζ-Funktion und der Dirichletschen L-Funktionen*, Dissertation, Göttingen, 1913. The inequality is derived by a method similar to that employed for the ζ-function by E. Lindelöf, *Quelques applications d'une formule sommatoire générale*, Acta Societatis Scientiarum Fennicae, vol. 31 (1903), and extended by R. J. Backlund, loc. cit., p. 18. The formula for $|R_k|$ is given here for the sake of completeness but is not used in the present paper.

$$T_\nu = \frac{(-1)^{\nu-1}B_\nu a^{2\nu-1}}{(2\nu)!}\cdot\frac{s(s+1)\cdots(s+2\nu-2)}{(an+b)^{s+2\nu-1}},$$

(5)

$$R_k = -\frac{a^{2k+2}s(s+1)\cdots(s+2k+1)}{(2k+2)!}\int_n^\infty \frac{\bar{P}_{2k+2}(x)dx}{(ax+b)^{s+2k+2}},$$

$\bar{P}_{2n}(x)$ being a periodic function with period 1,

$$|R_k| < \frac{|s+2k+1|}{\sigma+2k+1}|T_{k+1}|.$$

Formula (4) is valid for the half-plane $\sigma > 1-2k$.

2. **Functional equation for $Z(s)$.** By starting with the expression for $Z(s)$ as a definite integral

$$Z(s) = \frac{1}{\Gamma(s)}\int_0^\infty \frac{e^{(a-b)x}x^{s-1}dx}{e^{ax}-1},$$

and proceeding exactly as Riemann does with the ζ-function, we obtain the functional equation*

$$Z(s) = \left(\frac{2\pi}{a}\right)^s \frac{\Gamma(1-s)}{\pi}\sum_{n=1}^\infty \frac{\sin\left(\frac{s}{2}+\frac{2bn}{a}\right)\pi}{n^{1-s}}. \tag{6}$$

3. **The real zeros of $Z(s)$.** Suppose that $s=-\rho$ (ρ a real positive number) is a zero of $Z(s)$. If we substitute $s=-\rho$ in (6), the factor $\Gamma(1-s)$ does not vanish. When ρ is sufficiently large the sign of the series is determined by its first term and this term must be numerically small, if the series is to vanish. Hence, in order that $Z(-\rho)$ may be zero it is necessary for the angle $(-\rho/2+2b/a)\pi$ to take the form $(-\frac{1}{2}\epsilon-\lambda)\pi$, ϵ being a small positive or negative number and λ an integer. From this follows

THEOREM 1. *The real roots of $Z(s)$ are given by the formula*

$$-s = \rho = 2\lambda + \frac{4b}{a} + \epsilon(\lambda), \quad \epsilon(\lambda)\to 0 \quad as \quad \lambda\to\infty, \tag{7}$$

in which λ takes all positive integral values, including 0, and such negative integral values as will make the right member of (7) positive.

It is obvious that $Z(-\rho)$ will change sign when ρ passes through one of the values given by (7).

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* This formula was given, for integral values of a and b, by Ad. Hurwitz, *Einige Eigenschaften der Dirichletschen Funktionen* etc., Zeitschrift für Mathematik und Physik, vol. 27 (1882), p. 86.

For sufficiently large values of λ (say $\lambda > 5$), a very good approximation for $\epsilon(\lambda)$ is obtained by using the first two terms of the series in the right member of (6). Substituting $s=-2\lambda-4b/a-\epsilon$ and replacing $\sin(\epsilon\pi/2)$ in the first term by $\epsilon\pi/2$, dropping ϵ in the second term, equating the result to 0, and solving for ϵ, we obtain

$$\epsilon = \frac{\sin(2b\pi/a)}{\pi \cdot 2^{2\lambda+4b/a}}\cdot$$

This shows that when λ increases by 1, $\epsilon(\lambda)$ is multiplied by $\frac{1}{4}$ (approximately). Hence, *as λ increases by unit increments, the function $\epsilon(\lambda)$ decreases in a geometric series (approximately and asymptotically) with $\frac{1}{4}$ as the common ratio.*

To illustrate the preceding and to give some clue to the rate of change of $\epsilon(\lambda)$ for small values of λ, the following real roots have been calculated.

Roots of $Z(3, 1, s)$: -1.401, -3.357, -5.340, -7.335, -9.334, -11.3334, $-13.33336, \cdots$.

Roots of $Z(3, 2, s)$: $-.535$, -2.630, -4.657, -6.664, -8.666, -10.6665, $-12.666625, \cdots$.

4. **The imaginary zeros of $Z(s)$.** When a and b are real, the imaginary roots of $Z(s)$ occur in conjugate pairs. In order to determine the region of the s-plane in which such zeros are situated, we note that $\Gamma(1-s)$ cannot vanish or become infinite for complex values of s. Hence the imaginary zeros of $Z(s)$ are also zeros of the function

$$f(s) = \sum_{n=1}^{\infty} \frac{\sin(s/2 + 2bn/a)\pi}{n^{1-s}}\cdot$$

In this substitute $s=\sigma+it=1-p+it$, $p>1$, $t>0$. Then we have

$$\sin\left(\frac{s}{2}+\frac{2bn}{a}\right)\pi = \cos\pi\left(\frac{p}{2}-\frac{2bn}{a}-i\frac{t}{2}\right) = \frac{r}{2}\left(\lambda_n+\frac{1}{\lambda_n r^2}\right),$$

$$r = e^{\pi t/2}, \quad \lambda_n = \cos\theta_n + i\sin\theta_n, \quad \theta_n = \left(\frac{p}{2}-\frac{2bn}{a}\right)\pi.$$

Hence

$$f(s) = \frac{r}{2}\left(\lambda_1+\frac{1}{\lambda_1 r^2}\right)[1+\phi(s)], \ \phi(s) = \sum_{n=2}^{\infty} \frac{\lambda_n + 1/(\lambda_n r^2)}{\lambda_1 + 1/(\lambda_1 r^2)}\cdot\frac{1}{n^{1-s}}\cdot$$

From the relation

$$\left|\lambda_n + \frac{1}{\lambda_n r^2}\right| = (1 + 2r^{-2}\cos 2\theta_n + r^{-4})^{1/2}$$

we deduce

$$1 - r^{-2} \leqq \left| \lambda_n + \frac{1}{\lambda_n r^2} \right| \leqq 1 + r^{-2}. \tag{8}$$

From (8) follows the inequality

$$| \phi(s) | < \frac{r^2 + 1}{r^2 - 1} \sum_{n=2}^{\infty} \frac{1}{n^p} = \frac{r^2 + 1}{r^2 - 1} [\zeta(p \quad - 1].$$

This gives the following theorem:

THEOREM 2. *A necessary condition that $Z(s)$ may have imaginary zeros for $t>0$ is*

$$\frac{r^2 + 1}{r^2 - 1} [\zeta(p) - 1] \geqq 1, \ p > 1, \ r = e^{\pi t/2},$$

that is, the zero points s must be to the right of the curve

$$\zeta(1 - \sigma) = \frac{2}{1 + e^{-\pi t}}, \quad \sigma < 0. \tag{9}$$

In graphing (9) we assume the real axis $t=0$ to be horizontal and directed to the right. The graph starts at $t=0$, $\sigma=-\infty$ and rises slowly to $t=1$, $\sigma=-.722$. From there on it rises very rapidly and practically coincides with the line

$$\sigma = - .7143 \tag{10}$$

for t varying from 5 to $+\infty$.

In the same way we find that the critical strip for $t<0$ (the strip in the s-plane in which imaginary zeros may occur) is bounded on the left by the image of (9) below the real axis.

The right hand limit for the critical strip cannot be determined so precisely as the left hand boundary (9) and (10).

It is obvious that the right hand limit for all roots is $\sigma=\sigma_0$, in which σ_0 is not greater than the value σ_1 of σ determined by the equation

$$\sum_{\nu=1}^{\infty} (a\nu + b)^{-\sigma} = b^{-\sigma}. \tag{11}$$

Substituting $a=bp$, (11) becomes $\sum_{\nu=1}^{\infty} (p\nu+1)^{-\sigma}=1$. It is easy to see from this equation that σ_1 varies from 1 to ∞ as $p(=a/b)$ varies from ∞ to 0, or as b/a varies from 0 to ∞. When $b=a$, (11) reduces to $\zeta(\sigma)=2$, from which $\sigma_1=1.645$. Hence for $b<a$ we have $\sigma_0 \leqq \sigma_1 < 1.645$.

5. The imaginary zeros of Z_1 and Z_2. In order to obtain some clue to the distribution of imaginary roots, I have calculated several of them for each of the functions $Z_1=Z(3, 1, s)$, $Z_2=Z(3, 2, s)$ whose real roots have already been determined. Formula (11) shows that all of these zeros must be to the left of the line $\sigma=1.26$, for Z_1, and to the left of the line $\sigma=1.5$ for Z_2. Denoting the coördinates of the zero points by (σ, t) the results obtained are as follows.

Roots of Z_1: (.3, 11.45), (.25, 15.2), (.6, 20.7), (.25, 23.9), (.3, 28.55), (.4, 30.6), (.5, 33.65), (.5, 37.6), (.5, 39.7), (.15, 42.25), (.35, 43.65), (.6, 47.75).

Roots of Z_2: (.15, 10.8), (.6, 16.6), (.8, 24.3), (.7, 30.8), (.37, 34.13), (.46, 37.6), (.7, 42.9), (.35, 45.34), (.1, 47.67).

6. **Numerical computation.** The imaginary zeros of functions we are dealing with are comparatively easy to calculate for values of t not too great. For this purpose we separate (4) into real and imaginary parts and denote them respectively by $C=C(\sigma, t)$, $S=S(\sigma, t)$. Introducing (for §6 only) the abbreviations $\mu=an+b$, $s=\sin(t\log\mu)$, $c=\cos(t\log\mu)$, we have

$$C = \sum_{\nu=0}^{n-1} \frac{\cos[t\log(a\nu+b)]}{(a\nu+b)^\sigma} + \frac{\cos(t\log\mu)}{2\mu^\sigma} + C_0 + C_1 + \cdots, \tag{12}$$

$$C_0 = -\frac{\mu^{1-\sigma}[st+(1-\sigma)c]}{a[t^2+(1-\sigma)^2]}, \qquad C_1 = \frac{a(st+\sigma c)}{12\mu^{1+\sigma}};$$

$$S = -\sum_{\nu=0}^{n-1} \frac{\sin[t\log(a\nu+b)]}{(a\nu+b)^\sigma} + \frac{\sin(t\log\mu)}{2\mu^\sigma} + S_0 + S_1 + \cdots, \tag{13}$$

$$S_0 = \frac{\mu^{1-\sigma}[(1-\sigma)s-ct]}{a[t^2+(1-\sigma)^2]}, \qquad S_1 = \frac{a(ct-\sigma s)}{12\mu^{1+\sigma}}.$$

If t is fairly large in comparison with σ, the terms C_2 and C_3 in (12) may be computed by the simpler approximate formulas

$$C_2 = \frac{a^2}{60}\left(\frac{t}{\mu}\right)^2 C_1, \qquad C_3 = \frac{a^2}{42}\left(\frac{t}{\mu}\right)^2 C_2, \tag{14}$$

which are equally good for S_2 and S_3 in (13).

From (5) we obtain

$$\frac{T_{k+1}}{T_k} = -\frac{a^2 B_{k+1}(s+2k-1)(s+2k)}{B_k(2k+1)(2k+2)\mu^2}.$$

The factor $B_{k+1}/(B_k(2k+1)(2k+2))$ rapidly approaches the limit $1/(4\pi^2)$ as k increases. If t is not too small, a good approximation for the ratio T_{k+1}/T_k is $(at/(2\pi\mu))^2$. Consequently the terms in $\sum T_\nu$ are decreasing numerically, if $|at|<2\pi\mu$. In the computations made in connection with the present paper and carried out with three decimals, the value of n has been chosen in (12) and (13) so that $at \leqq \mu\pi$. The terms C_ν (or S_ν), $\nu = 1, 2, 3$, then diminish very rapidly and it is unnecessary to use any other of these terms to secure a sufficient degree of approximation for the values of C and S. It is also unnecessary to calculate the remainder (a laborious process), which will be in general too small to affect materially the few decimals used. In case of doubt, the value of C (or S) has been recalculated by using an increased value of n. This can be quickly done and furnishes a good idea of the error introduced by the approximate formulas and has justified their use.

7. **The C and S curves and their right-hand asymptotes.** In calculating the roots given in §5, certain properties common to all the curves

$$C(\sigma, t) = 0, \quad S(\sigma, t) = 0 \tag{15}$$

were found very helpful. Consider first the case $b = 1$. The values of C and S are readily computed by (12) and (13), the curves (15) roughly sketched, and their points of intersection located.

In order to reduce the work to a minimum, we consider what means there are of estimating where a horizontal line l, $t = k$, will intersect either curve. From (12) we observe that all the terms in C, excluding the first which is 1, diminish as σ increases and their sum becomes and remains less than 1. Accordingly, there is a point $\sigma = \sigma_0$ on the line l such that for all $\sigma > \sigma_0$, $t = k$, the function C is positive. We will speak of points on this part of l as being to the right of the curve $C = 0$. I have found it most convenient (unless there are indications to the contrary) to take $\sigma = .5$ and integral values of t. Having calculated C for these values, the sign of C determines on which side of the vertical line $\sigma = .5$ the graph is situated and the numerical magnitude of C gives some rough indication (as a rule) as to how far away the graph is.

While the portion of the s-plane to the right of the curve $C = 0$ is a region of positive values for the function C, it is quite otherwise with the function S whose first term is $-\sin\,[t \log (a+1)]/(a+1)^{-\sigma}$. As in the case of C, S has an invariable sign on the line l to the right of a certain point $\sigma = \sigma_0$, but when t passes through a value determined by the equation

$$t \log (a + 1) = n\pi, \tag{16}$$

n an integer, the sign of S is reversed on that part of the line l to the right of the graph. In other words, *the lines* (16) *are right-hand asymptotes for infinite branches of the curve $S = 0$ extending to $\sigma = +\infty$.*

If $b \neq 1$, then both the curves (15) *have an infinity of branches extending to*

$\sigma = +\infty$ *in the direction of the asymptotes* $t = (2n+1)\pi/(2 \log b)$ *for* $C=0$ *and* $t = n\pi/\log b$ *for* $S=0$.

8. **The behavior of C and S on the line $t=k, \sigma<0$.** In equation (6) put $\sigma = -p$, $p>0$. Let $t(>0)$ have an arbitrary, fixed value and let p increase without limit. The sum in the right member of (6) may be written

$$\sin\left(\frac{\pi s}{2} + \frac{2b}{a}\right) + \epsilon(p), \quad \epsilon(p) \to 0 \quad \text{as} \quad p \to \infty .$$

Discard $\epsilon(p)$, replace the sine function by its exponential form, and drop the term containing $e^{-\pi t/2}$ which becomes very small for moderately large values of t. Writing $\Gamma(1+p-it) = \gamma_1 + i\gamma_2$, we finally obtain from (6) the approximate formula

$$Z(-p+it) = \left(\frac{2\pi}{a}\right)^{-p+it}\left(\frac{\gamma_1 + i\gamma_2}{2\pi}\right)e^{\pi t/2 + i\pi(1/2+p/2-2b/a)} = C + iS.$$

From the theory of the Gamma function we have*

$$\gamma_1 + i\gamma_2 = \Gamma(1+p)e^{-P(1+p,-t)+i\omega},$$

$$P = \frac{1}{2}\sum_{\nu=0}^{\infty}\log\left[1 + \frac{t^2}{(1+p+\nu)^2}\right], \ \omega = \Theta(1+p, -t) - t\Psi(1+p),$$

$$\Theta = \sum_{\nu=0}^{\infty}\left[\frac{-t}{1+p+\nu} - \arctan\frac{-t}{1+p+\nu}\right],$$

$$\Psi = \frac{d \log \Gamma(1+p)}{dp} = -\gamma + \sum_{\nu=0}^{\infty}\frac{p}{(\nu+1)(1+\nu+p)},$$

$$\gamma = \text{Euler's const.}$$

From these formulas we deduce

$$P(1+p, -t) \to 0 \quad \text{and} \quad \Theta(1+p, -t) \to 0 \quad \text{as} \quad p \to \infty .$$

Dropping the functions P and Q, we obtain

$$C = R\cos\theta, \qquad S = R\sin\theta,$$

$$R = \frac{\Gamma(1+p)}{2\pi}e^{\pi t/2 - p\log(2\pi/a)}, \ \theta = t\log\frac{2\pi}{a} + \pi\left(\frac{1}{2} + \frac{p}{2} - \frac{2b}{a}\right) - t\Psi(1+p).$$

Replacing $\Gamma(1+p)$ by Stirling's formula we see that $R \to \infty$ as $p \to \infty$. As to the angle θ, the terms containing p may be written

$$p\left[\frac{\pi}{2} - tf(p)\right], \ f(p) = \sum_{\nu=0}^{\infty}\frac{1}{(\nu+1)(\nu+1+p)} .$$

* See Nielsen, *Handbuch der Gammafunktion*, p. 23.

The positive function $f(p)$ evidently decreases to 0 as a limit as p increases. Hence θ eventually takes the form $\pi p/2+\tau$,

$$\tau = t\left(\gamma + \log\frac{2\pi}{a}\right) + \frac{\pi}{2} - \frac{2\pi b}{a} + \epsilon(p), \quad \epsilon(p) \to 0. \tag{17}$$

We arrive finally at the formulas

$$C = R\cos\left(\frac{\pi p}{2} + \tau\right), \quad S = R\sin\left(\frac{\pi p}{2} + \tau\right), \quad p > p_0, \tag{18}$$

p_0 being a positive number sufficiently large.

These formulas show that, when $k(>0)$ is not too small, the curves (15) cross the line $t=k$ an infinity of times as σ approaches $-\infty$, the functions C and S oscillating between positive and negative values that increase numerically without limit.

The function $Z(s)$ may be replaced by the simpler asymptotic expression

$$Z(s) = Re^{i(\tau-\sigma\pi/2)} \tag{19}$$

in the part of the plane that we are considering.

If t is negative, we derive in the same way formulas exactly similar to (18) and (19), with appropriate changes of sign.

9. **Conjugate functions.** The two functions $Z(a, b, s)$ and $Z(a, a-b, s)$ will be called *conjugate*. Their sum and difference satisfy respectively the simpler functional equations

$$Z_b + Z_{a-b} = x_b(s) = \eta\sum_{n=1}^{\infty}\frac{\cos(2bn\pi/a)}{n^{1-s}}, \tag{20}$$

$$\eta = \frac{2}{\pi}\left(\frac{2\pi}{a}\right)^s \sin\frac{\pi s}{2}\Gamma(1-s),$$

$$Z_b - Z_{a-b} = y_b(s) = \lambda\sum_{n=1}^{\infty}\frac{\sin(2bn\pi/a)}{n^{1-s}}, \tag{21}$$

$$\lambda = \frac{2}{\pi}\left(\frac{2\pi}{a}\right)^s \cos\frac{\pi s}{2}\Gamma(1-s).$$

It is obvious that all functions x_b vanish for $s=-2, -4, \cdots, -2n, \cdots$; and that all functions y_b are integral functions (being regular at $s=1$) and vanish for $s=-1, -3, \cdots, -(2n-1), \cdots$. Hence x_b has the factor $1/\Gamma(s/2+1)$ and y_b has the factor $1/\Gamma((s+1)/2)$.

On account of the simplification of the real zeros of x_b and y_b as compared with those of $Z(s)$, the question arises as to whether there is also a simplification in the positions of the imaginary zeros, as, for example, if they may lie on a vertical line $\sigma=\sigma_0$. That this is not the case in general may be shown by a single illustration. Namely, the first two imaginary zeros of $Z(5, 1, s) - Z(5, 4, s)$ are found by computation to be at the points $(\sigma, t) = (.63, 8.94)$ and $(.3, 12.15)$.

10. **Linear functions of x_b, or y_b, and their functional equations.** Suppose, now, that a is a fixed, positive integer. We propose to consider the different functions $x_1, x_2, \cdots, x_m$ (or $y_1, \cdots, y_m$), $m=\frac{1}{2}(a-1)$ or $\frac{1}{2}(a-2)$, according as a is odd or even. Let

$$f(s) = a_1x_1 + a_2x_2 + \cdots + a_mx_m \tag{22}$$

be any linear function with constant coefficients. On account of (20), $f(s)$ satisfies the relation

$$f(s) = \eta \sum_{n=1}^{\infty} \frac{a_1 \cos(2\pi n/a) + a_2 \cos(4\pi n/a) + \cdots + a_m \cos(2\pi mn/a)}{n^{1-s}}. \tag{23}$$

The question that we now propose to consider is this: *Can we determine the constants $a_1, a_2, \cdots, a_m$ so that the sum in* (23) *takes the form*

$$k(a_1x_1' + a_2x_2' + \cdots + a_mx_m'), \cdot\ x_\mu' = x_\mu(1-s)?$$

When this is possible the functional equation reduces to the very simple form

$$f(s) = \eta\cdot kf(1-s). \tag{24}$$

Three essentially distinct cases need to be considered:

1. a even, $a=2p$, p odd;
2. a even, $a=4q$;
3. a odd, $a=2m+1$.

11. **Case 1.** Use the abbreviation $c_\mu = \cos(2\pi\mu/a)$. Then the equations of condition for (24) are:

$$(\mathrm{A}_1)\qquad \begin{aligned} c_1a_1 + c_2a_2 + \cdots + c_ma_m &= ka_1, \\ c_2a_1 + c_4a_2 + \cdots + c_{2m}a_m &= ka_2, \\ \cdots\cdots\cdots\cdots&\cdots \\ c_ma_1 + c_{2m}a_2 + \cdots + c_{mm}a_m &= ka_m; \end{aligned}$$

$$(\mathrm{A}_2)\qquad \begin{aligned} a_1 + a_2 + \cdots + a_m &= 0, \\ a_1 - a_2 + \cdots + (-1)^{m-1}a_m &= 0. \end{aligned}$$

In the present case $m=p-1$. In order to handle these equations we use the following trigonometric relations which hold for $\mu<p$:

$$
\begin{aligned}
&c_\mu + c_{2\mu} + \cdots + c_{(p-1)\mu} = \begin{cases} 0, & \mu \text{ odd}, \\ -1, & \mu \text{ even}, \end{cases} \\
(25) \qquad &c_\mu - c_{2\mu} + \cdots - c_{(p-1)\mu} = \begin{cases} 1, & \mu \text{ odd}, \\ 0, & \mu \text{ even}, \end{cases} \\
&c_\mu^2 + c_{2\mu}^2 + \cdots + c_{(p-1)\mu}^2 = \frac{a}{4} - 1.
\end{aligned}
$$

These are easily verified by first observing that, if $\omega = e^{2\pi i/a}$, any odd power of ω satisfies the equation $1-\omega^\mu+\omega^{2\mu}- \cdots -\omega^{(p-1)\mu}=0$, and any even power satisfies $1+\omega^\mu + \cdots +\omega^{(p-1)\mu}=0$. Again, group the terms thus,

$$(c_\mu \pm c_{p\mu-\mu}) \pm (c_{2\mu} \pm c_{p\mu-2\mu}) + \cdots \text{ (upper, or lower, signs together).}$$

The sum of the subscripts in each pair is $p\mu$ and hence the two cosines in the same group are equal, if μ is even, or opposite in sign, if μ is odd. The last expression in (25) may be put in the form $\frac{1}{2}[(1-c_{2\mu})+(1-c_{4\mu})+ \cdots]=a/4-1$.

The characteristic equation for the system (A_1) is

$$
(26) \qquad D(k) \equiv \begin{vmatrix} c_1 - k, & c_2, & \cdots, & c_{p-1} \\ c_2, & c_4 - k, & \cdots, & c_{2p-2} \\ \cdot & \cdot & \cdot & \cdot \\ c_{p-1}, & c_{2p-2}, & \cdots, & c_{(p-1)^2} - k \end{vmatrix} = 0.
$$

I shall speak of this relation, and similar relations elsewhere, as the necessary and sufficient condition for a solution of the given system of equations, meaning, thereby, a solution different from $a_1=a_2= \cdots =0$.

We first prove the relation $D(-k)=D(k)$. For this purpose write $D(k)$ in the form

$$
(27) \qquad \begin{vmatrix} c_{(p-1)(p-1)} - k, & c_{(p-2)(p-1)}, & \cdots, & c_{p-1} \\ c_{(p-1)(p-2)}, & c_{(p-2)(p-2)} - k, & \cdots, & c_{p-2} \\ \cdot & \cdot & \cdot & \cdot \\ c_{p-1}, & c_{p-2}, & \cdots, & c_1 - k \end{vmatrix}
$$

obtained by writing rows and columns in the reverse order. The νth element of the μth row satisfies the relation

$$(28) \qquad c_{(p-\nu)(p-\mu)} = c_{p(p-\mu-\nu)+\mu\nu} = (-1)^{\mu+\nu-1} c_{\mu\nu}.$$

By means of this, (27) takes the form

$$\begin{vmatrix} -c_1-k, & c_2 \quad, & -c_3, \quad, & c_4, \cdots, & c_{p-1} \\ c_2 \quad, & -c_4-k, & c_6 \quad, & -c_8, \cdots, & -c_{p-2} \\ -c_3 \quad, & c_6 \quad, & -c_9-k, & c_{12}, \cdots, & c_{p-3} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ c_{p-1} \quad, & -c_{2p-2} \quad, & & \cdots, & -c_{(p-1)^2}-k \end{vmatrix}.$$

This reduces to $D(-k)$ by changing the signs of all the elements in the odd numbered columns and the even numbered rows.

Consider now the product determinant $D(k)\,D(-k)$. If $\mu \neq \nu$, the element of the μth row and the νth column is

$$\sum_{b=1}^{p-1} c_{\mu b} c_{\nu b} = \frac{1}{2} \sum_{b=1}^{p-1} \left[c_{(\mu-\nu)b} + c_{(\mu+\nu)b} \right] = \frac{1}{2}\left(\sum\nolimits_1 + \sum\nolimits_2 \right).$$

If $\mu-\nu$ is odd, so also is $\mu+\nu$ and each sum $\sum_1$, or $\sum_2$, is 0, according to (25). If $\mu-\nu$ is even, each sum is -1. If $\mu=\nu$, we obtain an element of the main diagonal of $D(k)D(-k)$, viz.,

$$\sum_{b=1}^{p-1} c_{\nu b}^2 - k^2 = \frac{a}{4} - 1 - k^2 = A.$$

The result is

$$D(k)D(-k) = \begin{vmatrix} A, & 0, & -1, & 0, & -1, \cdots, & -1, & 0 \\ 0, & A, & 0, & -1, & 0, \cdots, & 0, & 1 \\ -1, & 0, & A, & 0, & -1, \cdots, & -1, & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0, & -1, & 0, & -1, & \cdots, & 0, & A \end{vmatrix}.$$

To reduce this determinant, add to the first column the sum of all the other columns. The elements of the first column will then all be equal to $A-\frac{1}{2}(p-3)=\frac{1}{2}-k^2$ and we have $D(k)\,D(-k)=(\frac{1}{2}-k^2)D_1$,

$$D_1 = \begin{vmatrix} 1 & 0 & -1 & 0 & -1 \cdots -1 & 0 \\ 1 & A & 0 & -1 & 0 \cdots 0 & -1 \\ 1 & 0 & A & 0 & -1 \cdots -1 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & -1 & 0 & -1 & 0 \cdots 0 & A \end{vmatrix}.$$

Now subtract the first row from each of the others and D_1 reduces at once to $(A+1)^{(p-3)/2}\,D_2$,

$$D_2 = \begin{vmatrix} A & -1 & -1 \cdots & -1 \\ -1 & A & -1 \cdots & -1 \\ \cdot & \cdot & \cdot \quad \cdot \quad \cdot & \cdot \\ -1 & -1 & -1 \cdots & A \end{vmatrix}.$$

Add to the first column of D_2 the sum of the other columns, take out the common factor $\frac{1}{2}-k^2$, and modify the resulting determinant by adding the first column to each of the others. It then reduces to $(A+1)^{(p-3)/2}$. Collecting the results we have

$$D(k)D(-k) = D^2(k) = \left(\frac{1}{2} - k\right)^2\left(\frac{a}{4} - k^2\right)^{p-3},$$

and hence

$$D(k) = \pm\left(\frac{1}{2} - k^2\right)\left(\frac{a}{4} - k^2\right)^{(p-3)/2}.$$

The solutions $k = \pm 1/2^{1/2}$ must be discarded, since the complete system (A_1) and (A_2) has no solution for these values of k. For, adding the $p-1$ equations (A_1) we obtain

$$-(a_2 + a_4 + \cdots + a_{p-1}) = k(a_1 + a_2 + \cdots + a_{p-1}). \tag{29}$$

Again, changing the signs of every other equation of (A_1) beginning with the second, and adding, we have

$$a_1 + a_3 + a_5 + \cdots + a_{p-2} = k(a_1 - a_2 + a_3 - \cdots - a_{p-1}). \tag{30}$$

Denoting the left members of (29) and (30) by $-\alpha_2$ and α_1 respectively, these equations may be written

$$\begin{aligned} k\alpha_1 + (k+1)\alpha_2 &= 0, \\ (k-1)\alpha_1 + k\alpha_2 &= 0. \end{aligned} \tag{31}$$

The condition for a non-vanishing solution of (31) is $k^2 = \frac{1}{2}$. We have thus separated out the two roots $k = \pm 1/2^{1/2}$ of (26).

Suppose now the system (A), consisting of (A_1) and (A_2), be subjected to the linear transformation

$$\begin{aligned} a_1 + a_3 + \cdots + a_{p-1} &= \alpha_1, \\ a_2 + a_4 + \cdots + a_{p-2} &= \alpha_2, \\ a_\mu &= \alpha_\mu, \quad \mu = 3, 4, \cdots, p-1. \end{aligned}$$

Suppose, further, that the system (A_1) in the new unknowns $\alpha_1, \alpha_2, \cdots$ be replaced by the two equations (31) together with $p-3$ of the equations

(A_1) so selected that the new system (A_1') of $p-1$ equations in $\alpha_1, \cdots, \alpha_{p-1}$ are linearly independent. The two equations (A_2) may be replaced by $\alpha_1=0$, $\alpha_2=0$. Consequently, if we drop the terms containing α_1 and α_2 in the new system (A_1'), it is obvious that the $p-3$ equations other than (31) will have as characteristic determinant one of degree $p-3$ in k whose roots will be $k=\pm\frac{1}{2}a^{1/2}$ since the characteristic equation is unaltered by linear transformation. These $p-3$ equations of (A_1') will consequently have no solution, if $k=\pm 1/2^{1/2}$, and hence the complete system (A) has no solution for these values of k.

Since each of the roots $\pm\frac{1}{2}a^{1/2}$ of (26) is of multiplicity $\frac{1}{2}(p-3)$, it follows that for either of these values of k the system (A_1) has only $p-1-\frac{1}{2}(p-3)-2=\frac{1}{2}(p-3)$ linearly independent equations, since there are two relations (29) and (30) that are satisfied by (A_2). We accordingly obtain the following result:

THEOREM 3. *There are $\frac{1}{2}(p-3)$ linearly independent functions $f(s)$ that satisfy the functional equation*

$$f_1(s) = a^{1/2}\left(\frac{2\pi}{a}\right)^s \sin\frac{\pi s}{2}\,\frac{\Gamma(1-s)}{\pi} f_1(1-s), \tag{I}$$

and likewise $\frac{1}{2}(p-3)$ linearly independent functions that satisfy the equation

$$f_2(s) = -a^{1/2}\left(\frac{2\pi}{a}\right)^s \sin\frac{\pi s}{2}\,\frac{\Gamma(1-s)}{\pi} f_2(1-s). \tag{II}$$

12. **Case 1, the y-relations.** We next consider the function

$$f(s) = b_1y_1 + b_2y_2 + \cdots + b_my_m \tag{32}$$

and seek to determine $b_1, \cdots, b_m$ so that formula (21) reduces to

$$f(s) = \lambda\cdot kf(1-s). \tag{33}$$

Using for brevity $s_\mu=\sin(2\mu\pi/a)$, the equations of condition are

$$\begin{aligned} s_1b_1 + s_2b_2 + \cdots + s_mb_m &= kb_1,\\ s_2b_1 + s_4b_2 + \cdots + s_{2m}b_m &= kb_2,\\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad&\cdot\quad\cdot\quad\cdot\\ s_mb_1 + s_{2m}b_2 + \cdots + s_{mm}b_m &= kb_m. \end{aligned} \tag{B}$$

Denoting the determinant of this system by $D(k)$, we prove, just as in §11, $D(-k)=D(k)$, since $s_{(p-\nu)(p-\mu)}$ satisfies (28). Using the relations

$$\sum_{b=1}^{m} s_{\mu b}s_{\nu b} = \tfrac{1}{2}\sum_{b=1}^{m}\left[c_{(\nu-\mu)b} - c_{(\nu+\mu)b}\right] = 0, \quad \mu \neq \nu,$$

$$\sum_{b=1}^{m} s_{\nu b}^{2} = \tfrac{1}{2}\sum(1 - c_{2\nu b}) = \frac{a}{4}, \tag{34}$$

the product $D(k)D(-k) = D^2(k)$ reduces at once to $(a/4-k^2)^{p-1}$. From this we deduce the following result:

THEOREM 4. *There are* $\frac{1}{2}(p-1)$ *linearly independent functions* (32) *satisfying the relation*

$$f_3(s) = a^{1/2}\left(\frac{2\pi}{a}\right)^s \cos\frac{\pi s}{2}\,\frac{\Gamma(1-s)}{\pi} f_3(1-s), \tag{III}$$

and $\frac{1}{2}(p-1)$ *linearly independent functions that satisfy*

$$f_4(s) = -a^{1/2}\left(\frac{2\pi}{a}\right)^s \cos\frac{\pi s}{2}\,\frac{\Gamma(1-s)}{\pi} f_4(1-s). \tag{IV}$$

13. **Case 2,** $a=4q$. Consider first the function (32) and the conditions (B) with $m=2q-1$. We find, just as in §12, $D(k)D(-k)=(a/4-k^2)^{2q-1}$, by using (34) which holds in this case on account of (35). But we no longer have the relation $D(-k)=D(k)$ to determine the multiplicity of each root. The fact appears to be that $\frac{1}{2}a^{1/2}$ is a root of $D(k)$ of multiplicity q, and $-\frac{1}{2}a^{1/2}$ is of multiplicity $q-1$. I have not succeeded in obtaining a complete proof for this statement, but the following considerations will indicate the reasons for it.

Denote by $(B_1), (B_2), \cdots, (B_m)$ the separate equations of (B) in the order in which they are written. Multiply these by $s_\mu, s_{2\mu}, \cdots, s_{m\mu}$ respectively and add. We obtain $\frac{1}{4}ab_\mu = k(s_\mu b_1 + \cdots + s_{m\mu}b_m)$, which, by using (B_μ) in the right member, reduces to

$$\left(\frac{a}{4} - k^2\right) b_\mu = 0 \qquad (\mu = 1,2,\cdots, q-1). \tag{B_μ'}$$

It further seems to be always possible to form, by additions or subtractions, a linear combination of the odd numbered equations (B_μ) in case $q\equiv 1$ or 2 (mod 4), or of the even numbered equations in case $q\equiv 0$ or 3 (mod 4), which reduce to the form

$$\begin{array}{ll} & (\epsilon_1 s_1 + \epsilon_3 s_3 + \cdots - k)(\epsilon_1 b_1 + \epsilon_3 b_3 + \cdots) = 0, \\ (B_q') \quad \text{or} & \\ & (\epsilon_2 s_2 + \epsilon_4 s_4 + \cdots - k)(\epsilon_2 b_2 + \epsilon_4 b_4 + \cdots) = 0, \end{array}$$

the ϵ's having the values 1, -1, or 0 ($\epsilon_q = \pm 2$, or 0), such that the first factor reduces by Gauss' sums (or otherwise) to $(\frac{1}{2}a^{1/2}-k)$. The system of $2q-1$ equations (B_μ), (B_μ'), (B_q'), $\mu=1, \cdots, q-1$, appears to be equivalent to the original system (B) and evidently has a determinant whose factors in k are $(a/4-k^2)^{q-1}\,(a^{1/2}/2-k)$.

To each of the two values of k correspond functions $f(s)$ having as many independent parameters b_μ as the multiplicity of the corresponding root k and satisfying one or the other of the relations (III), (IV).

Consider now the functions (23) and the system (A) that their coefficients must satisfy. In evaluating the product $D(k)D(-k)$, in this case, we use the relations

$$
\begin{aligned}
c_\mu + c_{2\mu} + \cdots + c_{(2q-1)\mu} &= \begin{cases} 0, & \mu \text{ odd,} \\ -1, & \mu \text{ even,} \end{cases} \\
c_\mu - c_{2\mu} + \cdots + c_{(2q-1)\mu} &= \begin{cases} 0, & \mu \text{ odd,} \\ 1, & \mu \text{ even,} \end{cases} \\
c_\mu^2 + c_{2\mu}^2 + \cdots + c_{(2q-1)\mu}^2 &= \frac{a}{4} - 1 = A, \\
\sum_{b=1}^{2q-1} c_{\mu b} c_{\nu b} &= \begin{cases} 0, & \mu - \nu \text{ odd,} \\ -1, & \mu - \nu \text{ even.} \end{cases}
\end{aligned} \tag{35}
$$

The product $D(k)D(-k)$ is of the same form as in §11 except that, the determinant being of odd order, the elements 0, -1 in the last row and column will interchange.

To reduce this determinant, add all the remaining odd numbered columns to the first column, and all the other even numbered columns to the second one. We can then divide $k^2(k^2-1)$ out of the first two columns. The resulting determinant at once reduces, after adding the first column to each of the other odd numbered columns and the second column to each of the other even numbered columns, and we obtain

$$D(k)D(-\,k) = k^2(k^2 - 1)(\tfrac{1}{4}a - k^2)^{2q-3}.$$

Continuing with the methods of Case 1, it is easy to show that $D(k)$ has the roots 0 and -1, which must be excluded. The remaining $2q-3$ roots, apparently of multiplicity $q-1$ for $\frac{1}{2}a^{1/2}$ and $q-2$ for $-\frac{1}{2}a^{1/2}$, determine functions $f(s)$ having as many independent parameters a_μ as the multiplicity of the corresponding root k and satisfying (I), or (II). The facts that indicate the degree of multiplicity are similar to those given at the beginning of this article.

14. **Case 3**, $a=2m+1$. Considering first the conditions (B) that the function (32) shall satisfy (33), we readily find $D(k)D(-k)=(a/4-k^2)^m$. The same considerations as in the preceding case lead to the belief that $D(k)$ has the two roots $\pm\frac{1}{2}a^{1/2}$ with the same degree of multiplicity, if m is even, and the root $\frac{1}{2}a^{1/2}$ with multiplicity one higher, in case m is odd.*

When m is odd, the relation corresponding to (B_q') of §13 is more easily found as it consists of a sum of the equations (B_μ) multiplied by $\epsilon_\mu=1, -1$, or 0, $\mu=1, \cdots, m$.

The conditions that the function (22) shall satisfy (24) are the same as for Case 1, except that (A_2) has only one equation $a_1+a_2+\cdots=0$. The trigonometric relations in this case are

$$c_\mu + c_{2\mu} + \cdots + c_{m\mu} = -\frac{1}{2},$$

$$c_\mu^2 + c_{2\mu}^2 + \cdots + c_{m\mu}^2 = \frac{a}{4} - \frac{1}{2}.$$

The product $D(k)D(-k)$ consists of $a/4-\frac{1}{2}-k^2$ for each element in the main diagonal and $-\frac{1}{2}$ for every other element. By adding all the other rows to the first, the common element $\frac{1}{4}-k^2$ divides out. By adding one-half the elements of the first row of the resulting determinant to each of the other rows we obtain

$$D(k)D(-k) = \left(\frac{1}{4} - k^2\right)\left(\frac{a}{4} - k^2\right)^{m-1}$$

By adding all the equations (A_1) we get $(-\frac{1}{2}-k)(a_1+a_2+\cdots+a_m)=0$, which shows that $D(k)$ has the root $-\frac{1}{2}$. This root does not satisfy the complete system (A).

Apparently $k=\frac{1}{2}a^{1/2}$ is a root of $D(k)$ of multiplicity one greater than that of the root $-\frac{1}{2}a^{1/2}$, if m is even, and of the same multiplicity if m is odd. The results of Cases 2 and 3 may be thus summarized:

The number of linearly independent functions satisfying (I) *and* (III), *or* (II) *and* (IV), *is equal to the multiplicity of the root* $\frac{1}{2}a^{1/2}$ *or* $-\frac{1}{2}a^{1/2}$, *of the determinant of the corresponding system of equations* (A_1) *and* (B).

15. **The argument of** $f_\mu(s)$ **on the line** $\sigma=\frac{1}{2}$. The functional equation (I) may be put in the form

* Case 3, for a a prime number, has been completely solved by E. Cahen, *Sur la fonction* $\zeta(s)$ *de Riemann et sur des fonctions analogues*, Annales de l'Ecole Normale Supérieure, 1894, pp. 75–164 His method, which is particularly suitable when a is a prime, does not seem to be applicable to any other case. We might notice here some errors in Cahen's memoir. On p. 151, last line, the coefficients of the functions that we have denoted by y_1 and y_2 should be interchanged. A similar interchange should be made in line 4, p. 152.

$$\frac{f_1(1-s)}{f_1(s)} = \frac{2}{a^{1/2}}\left(\frac{a}{2\pi}\right)^s \cos\frac{\pi s}{2}\Gamma(s). \tag{36}$$

Let $s=\frac{1}{2}+it$ and write $f_1(\frac{1}{2}+it)=\rho_1(t)e^{i\phi_1(t)}$. If we take the logarithm of (36), the left member becomes $-2\phi_1(t)$. By using the procedure of Gram* as developed for $\zeta(s)$, we obtain

$$\frac{-\phi_1(t)}{\pi} = \frac{t}{2\pi}\left[\log\left(\frac{at}{2\pi}\right) - 1\right] - \frac{1}{8}\cdot \tag{37}$$

The functional equations (II), (III), (IV) yield similar formulas in which ϕ_1 is replaced by ϕ_2, ϕ_3, ϕ_4 respectively, and the term $-\frac{1}{8}$ is replaced by $\frac{3}{8}$, $\frac{1}{8}$, $\frac{5}{8}$ respectively.

Let $f(s)$, without subscript, denote any one of the functions $f_\mu(s)$. Write $f(\frac{1}{2}+it)$ in the form

$$f(\tfrac{1}{2}+it) = \rho\cos\phi + i\rho\sin\phi = C(t) + iS(t)\,.$$

Denote by γ any value of t for which $\sin\phi(t)$ is 0, and by β any value of t for which $\cos\phi(t)$ is 0. If there are any real values $t=\alpha\neq\gamma$ which make $S(t)$ vanish, they must be zeros of $\rho(t)$ and hence zeros of $f(\frac{1}{2}+it)$ as well as of $C(t)$. The result obtained by Gram for $\zeta(\frac{1}{2}+it)$ holds equally well for our functions $f(\frac{1}{2}+it)$, viz.:

THEOREM 5. *If $\gamma_n(>2\pi/a)$ and γ_{n+1} are two consecutive roots of* $\sin\phi(t)$, *and if $C(\gamma_n)$ and $C(\gamma_{n+1})$ have the same sign, then an odd number of roots α of $f(\frac{1}{2}+it)$ occur between γ_n and γ_{n+1} on the line $\sigma=\frac{1}{2}$, and hence there is at least one root in this interval. Similarly, if $S(\beta_n)$ and $S(\beta_{n+1})$ have the same sign, there is at least one root α in the interval (β_n, β_{n+1}).*

A simpler proof of Theorem 5 than that given by Gram would be this. Since γ_n and γ_{n+1} are two consecutive roots of $\sin\phi(t)=0$ and since $-\phi(t)$ is an increasing function for $at>2\pi$, then $\phi(\gamma_n)$ and $\phi(\gamma_{n+1})$ are two consecutive multiples of π and hence $\cos\phi(t)$ takes opposite signs at $t=\gamma_n$ and γ_{n+1}. Hence, if $C(t)=\rho(t)\cos\phi(t)$ preserves the same sign for two consecutive values of γ, the factor $\rho(t)$ must change sign in the interval (γ_n, γ_{n+1}).

16. The number of imaginary zeros of $f(s)$. The functional equations (I), $\cdots$, (IV) may be written

$$\text{(I}'\text{)}\quad \chi_1(s) = \chi_1(1-s),$$

$$\chi_\mu(s) = f_\mu(s)\Gamma\left(\frac{s}{2}\right)\left(\frac{a}{\pi}\right)^{s/2}, \quad \mu = 1,2\,;$$

$$\text{(II}'\text{)}\quad \chi_2(s) = -\chi_2(1-s),$$

* J. P. Gram, loc. cit., pp. 298–304.

(III′) $\chi_3(s) = \chi_3(1 - s)$,

$$\chi_\mu(s) = f_\mu(s)\Gamma\left(\frac{1+s}{2}\right)\left(\frac{a}{\pi}\right)^{s/2}, \quad \mu = 3,4.$$

(IV′) $\chi_4(s) = -\chi_4(1 - s)$,

We propose to determine an upper limit $N(T)$ for the number of imaginary zeros of the functions $f_\mu(s)$, whose coefficients we assume to be all real, in the region $0<t<T$.

Consider first the function

$$\xi_\mu(s) = \tfrac{1}{2}s(s-1)\chi_\mu(s), \quad \mu = 1,2.$$

The zeros of $\xi_\mu(s)$ (excepting $s=\frac{1}{2}$) are the imaginary zeros of $f_\mu(s)$, $\mu=1, 2$. They are symmetrical with respect to the lines $t=0$ and $\sigma=\frac{1}{2}$. Let $f(s)$ denote any one of the functions under consideration and write

$$f(s) = \sum_{n=n_0}^{\infty} c_n n^{-s},$$

the c_n all real. Let σ_0 be the least positive value of σ such that

$$|c_0 n_0^{-s}| > \sum_{n=n_0+1}^{\infty} |c_n n^{-s}| .$$

Then there are no imaginary zeros of $f(s)$ in the half-plane $\sigma>\sigma_0$ and, on account of the functional equations, none in the half-plane $\sigma<1-\sigma_0$.

The method of evaluating $N(T)$ is very similar to that employed for $\zeta(s)$. I follow as closely as possible the treatment of Backlund* and refer to his paper for details.

Consider the rectangle whose vertices are $\alpha\pm iT$, $1-\alpha\pm iT$, α real $>\sigma_0$. Then we have

$$\pi N(T) = \Delta_{\alpha\beta\gamma} \arg \xi_\mu(s),$$

the right member denoting the increment that the argument of $\xi_\mu(s)$ acquires as s describes the broken line $\alpha\beta\gamma$, $\beta=\alpha+iT$, $\gamma=\frac{1}{2}+iT$.

The only variations from the steps given by Backlund arise in the calculation of $\Delta_{\alpha\beta\gamma} \arg f(s)$. Denoting by $C(\sigma, t)$ the real part of $f(s)$, then $\Delta_{\alpha\beta\gamma} \arg f(s)/\pi$ will not exceed the number of times that $C(\sigma, t)$ vanishes on the broken line $\alpha\beta\gamma$. The results of §7 tell us that, if $f(s)$ contains the term

* R. J. Backlund, *Ueber die Nullstellen der Riemannschen Zetafunktion*, Acta Mathematica vol. 41 (1918), p. 348.

$Z(a, 1, s)$, the curve $C(\sigma, t)=0$ has no right-hand asymptotes and the vertical line $\sigma=\alpha>\sigma_0$ does not meet the curve. In this case $\Delta_{\alpha\beta} \arg f(s)<\pi/2$.

In other cases the lines $2t \log n_0=(2n+1)\pi$ are right-hand asymptotes of the curve $C=0$ and hence $C(\sigma, t)$ vanishes on the line $\sigma=\alpha$ at the points

$$t = \frac{(2n+1)\pi}{2 \log n_0} + \epsilon(\alpha,n), \quad \lim_{\alpha\to\infty} \epsilon(\alpha,n) = 0. \tag{38}$$

Moreover, when α is sufficiently large, the change of sign of $C(\alpha, t)$, as t increases through one of the values (38), is due solely to the first term $c_{n_0}[\cos (t \log n_0)]n_0^{-\alpha}$ and this term obviously changes sign but once in the vicinity of each point (38).

Let n_1 be the largest value of n in (38) for which $t<T$. Then we obtain

$$n_1 < \frac{T \log n_0}{\pi} + b_1$$

and hence

$$\Delta_{\alpha\beta} \arg f(s) < T \log n_0 + b_2,$$

b_1 and b_2 being constants.

We obtain as the final result,

$$N(T) = \frac{T}{2\pi}\left[\log \frac{an_0^2T}{2\pi} - 1 + \epsilon(T)\right], \quad \lim_{T\to\infty} \epsilon(T) = 0. \tag{39}$$

The functions $\chi_\mu(s)$, $\mu=3, 4$, are regular at $s=0, 1$. Treating them in the same way as $\xi_1(s)$, $\xi_2(s)$, we arrive at (39) as before.

17. **Proof that the functions $\chi_\mu(s)$ have an infinity of roots.** Denoting by ρ_n the roots of any of the functions $\chi_\mu(s)$ which are situated in the half-plane $t>0$, we find from (39) (Backlund, p. 354) that $\sum |\rho_n{}^{-1-\epsilon}|$ is convergent, ϵ being a fixed positive number as small as we please. Hence the exponent of convergence* of the roots of $\chi_\mu(s)$ does not exceed 1.

If we substitute $s=\frac{1}{2}+it$ in the functions $\xi_\mu(s)$, $\chi_\mu(s)$, the functional equations (I′), $\cdots$, (IV′) show that the functions ξ_1, ξ_2/t, χ_3, χ_4/t are uniform integral functions of $t^2=z$. The argument of Hadamard† relative to $\zeta(s)$ applies at once to these functions and shows that each of them is of genus zero as a function of z and hence has an infinity of zeros, since it is obviously not a polynomial.

Moreover, we can prove the following:

* E. Borel, *Leçons sur les Fonctions Entières*, p. 18.

† Borel, loc. cit., 2d edition, pp. 84–88.

THEOREM 6. *Those functions $f_\mu(s)$ that contain the term $Z(a, 1, s)$ have an infinity of roots on the line $\sigma = \frac{1}{2}$.*

The proof will be omitted as it is very similar to the Hardy and Littlewood* proof for $\zeta(s)$. We have followed the details as given by E. Landau.† They apply with no change for the functions $f_\mu(s)$, $\mu = 1, 2$, except in the values of some of the multiplying constants, and in using for $f_\mu(s)$ a series of the form

$$\sum_{n=1}^{\infty} c_n n^{-s}, \quad c_1 \neq 0, \tag{40}$$

absolutely convergent for $\sigma = 1$, in place of the series used for $\zeta(s)$.

As to the functions $f_\mu(s)$, $\mu = 3, 4$, which we also represent by (40), on account of the presence of the factor $\Gamma\{\frac{1}{2}(1+\beta)\}$ in the corresponding functional equations (III′) and (IV′), all the formulas depending on this factor must be subjected to a re-evaluation, the general effect of which is to increase the exponent of t by $\frac{1}{2}$ in all these formulas. This extra power of t divides out when we come to solve formula (526) of Landau, giving the same result $O(t^{7/8})$ as for $\zeta(s)$.

It is to be remarked that $\xi_2(\frac{1}{2}+it)$ and $\chi_4(\frac{1}{2}+it)$ have the root $t = 0$ and accordingly the functions $f_2(s)$ and $f_4(s)$ have the real root $s = \frac{1}{2}$ in addition to the infinity of real roots already determined in §9.

18. **Generalization of the functional equation.** Instead of the forms assumed for (24) and (33) we could make a more general assumption in which $f(1-s)$ in the right member is replaced by $\bar{f}(1-s)$, the notation $\bar{f}$ meaning that the coefficients $a_1, \cdots, a_m$ (or $b_1, \cdots, b_m$) in f are replaced by their conjugate imaginary values $\bar{a}_1, \cdots$.

Instead of the equations of condition (A) or (B), we would have similar equations $(\bar{\text{A}})$ or $(\bar{\text{B}})$ in which the letters $a_1, \cdots$ in the right members are replaced by $\bar{a}_1, \cdots$. Each case could be treated as in §13. For example, half the $(\bar{\text{B}})$ equations of Case 1 could be replaced by the equations obtained by multiplying the given equations by $s_\mu, s_{2\mu}, \cdots$, respectively, adding, and using the conjugate of the μth equation $(\bar{\text{B}})$. This would give

$$(a/4 - k\bar{k})b_\mu = 0 \qquad (\mu = 1, 2, \cdots, (p-1)/2).$$

From this we see that the system $(\bar{\text{B}})$ is satisfied if $k = \frac{1}{2}a^{1/2}e^{i\theta}$, θ arbitrary.

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* *The Riemann zeta-function and the distribution of primes*, Acta Mathematica, vol. 41 (1917), pp. 177–184.

† *Vorlesungen über Zahlentheorie*, vol. 2, 1927, pp. 78–85.

It is obvious that the resulting functional equation, multiplied by its conjugate, will give the relation

$$F(s) = \lambda^2 \cdot \frac{a}{4} F(1-s)\,, \quad F(s) = f(s)\,\bar{f}(s)\,.$$

Functions $f(s)$ satisfying equations obtained in this way include as special cases the L-functions with imaginary characters.

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