AN EXTENSION OF PASCAL’S THEOREM*

BY

CHARLES A. RUPP

INTRODUCTION

In 1825 the Académie Royale de Bruxelles proposed as a prize topic the extension of Pascal’s theorem to space of three dimensions. The prize was won by Dandelin,† who showed that a skew hexagon formed of three lines from each regulus of a hyperboloid of revolution has the Pascal property that pairs of opposite planes meet on a plane, and the dual, or Brianchon property, that the lines joining pairs of opposite vertices meet in a point. Hesse‡ wrote several papers on the Dandelin skew hexagons, emphasizing the polar properties of the Pascal plane and the Brianchon point with respect to the quadric bearing the two reguli. Several of the older analytic geometries of three dimensions devote some space to the problem of the skew hexagon, as Salmon, and more notably Plücker§, who offered much original material. In a recent article, the present writer|| has discussed the skew hexagon from an elementary analytic approach.

The foregoing citations exhibit the idea that Pascal’s theorem deals with six elements of a quadratic curve, and that the space extension offered will deal with six elements of a quadric surface. It happens that an extension of this sort is valid in space of three dimensions; but in $S_n$, where $n > 3$, it is clear that a skew hexagon of six rulings of a hyperquadric must lie in an $S_3$ if it is to possess the Brianchon property. It seems, therefore, that the hexagon idea must be abandoned in the search for a valid extension of Pascal’s theorem in $S_n$.

In the plane, a variant of Pascal’s theorem affirms that if two triangles are in homology, then the six points of intersection of the sides of the one with the non-corresponding sides of the other lie upon a conic. From a consideration of the converses of this theorem and its dual it occurred in-
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dependently to Chasles* and to Weddle† that the Pascal configuration might conveniently be considered as a property of a pair of triangles whose sides meet on a conic, and that the space extension would concern a pair of tetrahedra and a quadric; in the plane opposite sides of the triangles meet in a triple of points on a line, and hence these geometers reasoned that in space opposite faces of the tetrahedra would meet in a quadruple of lines on a regulus. They readily devised synthetic proofs of the theorem. Chasles made the pregnant observation that the twelve points common to the edges of a given tetrahedron and quadric could be arranged in several ways to define a second tetrahedron which would be effective in the theorem. Weddle went on to study various properties possessed by a pair of effective tetrahedra, and thereby discovered several properties later used by Schläfli in discussing Schläfli simplexes.

Closely allied to the theorem generalized by Chasles and by Weddle is another stating that if two triangles are polar reciprocal with respect to a conic, they are perspective, and hence the lines joining non-corresponding vertices touch a conic, and dually. Schläfli‡ discussed the analogous situation in $S_n$. His most important discovery was that two simplexes, or complete $(n+1)$-points, of $S_n$ which are polar reciprocal with respect to a hyperquadric have what is now called the Schläfli property, i.e., the $n+1$ $S_{n-1}$'s of intersection of corresponding $S_{n-1}$'s (hyperplanes, faces) of the two simplexes lie in such a position that the lines of $S_n$ which meet $n$ of the $S_{n-1}$'s meet also the other. Following the suggestion of Berzolari,§ one now speaks of Schläfli simplexes, of a Schläfli set of lines or of $S_{n-1}$'s, or of lines or $S_{n-1}$'s in the position of Schläfli. Other contributions to the knowledge of the Schläfli situation are due to Brusotti|| and the present writer.¶ The extension of Pascal's theorem which this paper presents has a close connection with the Schläfli situation.

It will be shown that it is possible to construct, from the points common to a hyperquadric and the edges of a simplex, a certain number of auxiliary simplexes, each of which may be paired with the given simplex to make a Schläfli pair; the $\infty^{n-2}$ lines which meet the $S_{n-1}$'s of a Schläfli set thus

* Chasles, A perçu Historique, Note 32.
† Weddle, On theorems in space analogous to those of Pascal and Brianchon in a plane, Cambridge and Dublin Mathematical Journal, vol. 6 (1851), pp. 116–140.
defined lie upon a variety of order and dimension \( n-1 \), denoted by the symbol \( V^-_{n-1} \). On each \( V^-_{n-1} \) there are \( (n-1)! \) families of generators; since in \( S_3 \) the families are called reguli, we shall use the term hyperregulus to refer to a set of \( \infty \) lines which are rulings of a \( V^-_{n-1} \). The older literature about the variety appears to be scanty; Segre* twice mentions it casually in the Encyclopedia. Three papers on the variety have recently been published.†

I. The notation

The figure in \( S_n \) consisting of \( n+1 \) points, which do not lie in the same \( S_{n-1} \), the \( (n+1)n/2 \) lines joining the points in pairs, the \( (n+1)n(n-1)/6 \) planes joining the points in triples, \( \cdots \), and the \( n+1 \) \( S_{n-1} \)'s joining the points in \( n \)-tuples, is called a simplex. The points, lines, and hyperplanes \((S_{n-1})'s\) of a simplex are called its vertices, edges, and faces, respectively. Choose a given simplex \( F \) as the basis of a homogeneous coordinate system; the coordinates of the \( i \)th vertex \( P_i \), where \( i \) runs from 0 to \( n \), are all zero save at the \( i \)th place. The face of \( F \) that does not contain \( P_i \) is called the face opposite to \( P_i \), and will be denoted by \( P_i \); its equation is \( x_i = 0 \).

A hypersurface (variety of dimension \( n-1 \)) of the second order is the locus of points satisfying a quadratic equation; such a hypersurface is called a hyperquadric. Let there be given a hyperquadric of equation

\[
Q : \quad \sum a_{ij}x_i x_j = 0 \quad (i,j = 0, 1, \cdots, n; a_{ij} = a_{ji}).
\]

The edges of the fundamental simplex \( F \) meet the given hyperquadric \( Q \) in \( 2m \) piercing points \( P_{ij} \), where \( 2m = n(n+1) \), and the points \( P_{ij}, P_{ji} \) lie on the line \( P_i P_j \). To find the coordinates of these piercing points, solve the binary quadratic

\[
a_{ij}x^2 + 2a_{ij}x_i x_j + a_{jj}x^2 = 0.
\]

It will be convenient to use the quantities defined by

\[
\Delta_{ij} = a_{ij} - a_{ii} a_{jj},
\]

\[
b_{ij} = a_{ii}, \quad b_{ij} = a_{ij} + \Delta_{ij} = b_{ji},
\]

\[
c_{ij} = a_{ii}, \quad c_{ij} = a_{ij} - \Delta_{ij} = c_{ji},
\]

in exhibiting the coordinates of \( P_{ij} \) and \( P_{ji} \) in the form

---

† Wong, "On a certain system of \( \infty \) lines in \( r \)-space, and On the loci of the lines incident with \( k \) \( (r-2) \)-spaces in \( S_r \)," Bulletin of the American Mathematical Society, vol. 34 (1928), pp. 553-554, and pp. 715-717.
Note that the coordinates of either piercing point on an edge of $F$ may be expressed either in terms of $b_{ij}$ or $c_{ij}$. In handling the first effective simplex which we shall set up, we shall use only the quantities $b_{ij}$; we shall find that the other effective simplexes can be found from the first, or standard one, by interchanging the elements in certain pairs of the points $P_{ij}$ and $P_{ji}$. The corresponding change in the analytic work is accomplished by replacing the appropriate $b_{ij}$ by $c_{ij}$. If the hyperquadric $Q$ is a general one, the quantities $b_{ij}$ and $c_{ij}$ will all be different from zero.

The standard effective grouping of the $2m$ piercing points will be defined by the symbolic matrix

\[ G_i = (P_{ij}) \quad (i \neq j). \]

The elements of the matrix $(P_{ij})$ are the piercing points $P_{ij}$; there are $n+1$ rows and columns, each containing $n$ points. The points of the $i$th row share the first subscript $i$, and no pair of them share the second subscript. Geometrically, the points of the $i$th row lie one on each of the $n$ edges of $F$ which pass through the vertex $P_i$. Through the points of the $i$th row can be passed a unique $S_{n-1}$, which we call $\pi_i$, and make correspond to the hyperplane $P_i$ of $F$, which is the face of $F$ opposite $P_i$.

The standard grouping $G_i$ accordingly defines $n+1$ hyperplanes $\pi_i$, which constitute the faces of a simplex, $T_i$, said to be auxiliary to $F$, the fundamental simplex. The vertices of $T_i$ may be called the points $R_i$, $R_i$ being opposite to $\pi_i$. The points $R_i$ and $P_i$ are said to be corresponding points.

Consider the problem of determining all the groups of the $2m$ piercing points which have the geometrical property of $G_i$. In choosing the points of the first row, we may take either of the piercing points on each of the $n$ edges through the corresponding vertex, that is, there are $2^n$ possible ways to choose the first row. There are but $2^{n-1}$ ways to choose the second row, for on one of the edges there is but one available piercing point, the other having been already used. Proceeding in this fashion, we see that there are in all

\[ 2^n \times 2^{n-1} \times 2^{n-2} \times \cdots \times 2^2 \times 2 = 2^m \]
possible groupings of the $2m$ piercing points $P_i$, which have the same geometrical character that $G_1$ has.

In keeping account of the individual members of this set of $2^m$ effective groupings it is convenient to use a multiple index system; the general grouping of the set will be denoted by

$$G_a = G_{\alpha_1\alpha_2\alpha_3 \cdots \alpha_{n-1}}$$

where the general subscript $\alpha_r$ ranges through 1, 2, 3, \ldots, to $2^{r+1}$ for all values of $r$ from 2 to $n - 1$ inclusive. The subscript $\alpha_1$ ranges from 1 to 8. Each of the subscripts controls a certain sub-set of the points $P_{ij}$; the nature of this control is indicated in the following display:

<table>
<thead>
<tr>
<th>$\alpha_1$</th>
<th>$\alpha_2$</th>
<th>$\alpha_3$</th>
<th>$\alpha_{n-1}$</th>
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</thead>
<tbody>
<tr>
<td>1</td>
<td>$P_{10}$</td>
<td>$P_{30}$</td>
<td>$P_{n0}$</td>
</tr>
<tr>
<td></td>
<td>$P_{20}$</td>
<td>$P_{31}$</td>
<td>$P_{n1}$</td>
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<td>2</td>
<td>$P_{10}$</td>
<td>$P_{03}$</td>
<td>$P_{0m}$</td>
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<td>$P_{02}$</td>
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<td>$P_{n1}$</td>
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<td></td>
<td>$P_{02}$</td>
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<tr>
<td>5</td>
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<td>$P_{30}$</td>
<td>$P_{n0}$</td>
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<td></td>
<td>$P_{20}$</td>
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<td>$P_{n0}$</td>
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<td></td>
<td>$P_{02}$</td>
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<td>$P_{14}$</td>
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<tr>
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<td>$P_{n0}$</td>
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<tr>
<td></td>
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<td>$P_{p_{n-1}}$</td>
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</tr>
<tr>
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<td>$P_{0n}$</td>
<td>$P_{1n}$</td>
<td>$P_{n-1,n}$</td>
</tr>
</tbody>
</table>
To get the lower half of the matrix which is the complete display of $G_{a_1 a_2 \cdots a_{n-1}}$, we adjoin the sets of points $P_{ij}$ controlled by the various indices, and fill in the upper half of the matrix by inversive symmetry. As an example we give the upper left hand corner of the matrix $G_{38} \ldots$:

$$
\begin{array}{ccc}
* & P_{01} & P_{02} & P_{30} \\
0 & * & P_{21} & P_{31} \\
1 & * & * & P_{32} \\
\end{array}
$$

Each of these groupings $G_a$ will define, in the same manner that $G_1$ did, a simplex, $T_a$, auxiliary to the fundamental simplex $F$. The $2^n$ simplexes $T_a$ are called the auxiliary simplexes of the extended Pascal configuration in $S_n$, or simply the simplexes of the $Kf_n$. The $2^n$ pairs of simplexes composed of the fundamental simplex $F$ and one of the auxiliary simplexes will be called the pairs of $Kf_n$. It will be shown that the pairs of $Kf_n$ are Schlafli pairs of simplexes, and hence each determines a hyperregulus; the $2^n$ hyperreguli will be called the hyperreguli of $Kf_n$; thus the symbol $Kf_n$ means the total configuration of geometric elements associated with the extended Pascal theorem in $S_n$.

II. THEOREMS IN $S_n$

**Theorem 1.** The intersections of corresponding faces (and the lines joining corresponding vertices) of the pairings of an extended Pascal configuration in $S_n$ are linearly dependent.

Consider first the pair of simplexes $F$ and $T_1$, corresponding to the grouping $G_1$. By the use of (4), it is seen that the equations of the faces of $T_1$ are

$$
\sum_i b_{ij} x_i = 0 \quad (i, j = 0, 1, \ldots, n)
$$

The equations of the $n+1$ $S_{n-2}$'s of intersection with the corresponding faces of $F$ may be written as

$$
x_i = \sum_j b_{ij} x_j = 0.
$$

The Plücker-Grassmann coordinates of these $S_{n-2}$'s are the two-rowed determinants formed from the matrices of coefficients in the equations (7). Each set of coordinates has exactly $n$ elements that are not zero; they are shown in the following display:
In each column there are two and only two elements, the sum of which is zero since $b_{ij} = b_{ji}$. The display shows that there is linear dependence between the sets of coordinates of the $n+1$ $S_{n-2}$'s of intersection of corresponding faces of the simplexes $F$ and $T_1$, which is what is meant when it is said that the $S_{n-2}$'s themselves are linearly dependent.

By duality it follows that the lines joining corresponding vertices of the two simplexes are also linearly dependent.

Suppose now the members of certain pairs of piercing points are interchanged. The effect will be to replace the standard grouping $G_i$ by some particular one of the set $G_a$; if we know which pairs of piercing points are interchanged, we know which $G_a$ is represented by the modified $G_i$. For example, suppose that $P_{04}$ and $P_{40}$ are interchanged; the standard grouping $G_1$ is replaced by $G_{112111 \ldots}$. If now we interchange $b_{04}$ and $c_{04}$ in the equations (7) and display (8), we have converted the proof that $A$ is an effective grouping in Theorem 1 into a proof that the modified grouping is also effective. Since all of the $2^m$ groupings $G_a$ were obtained from the standard $G_1$ by such inversions of subscripts among the points $P_{ij}$, it follows that all the groupings $G_a$ are effective in the theorem.

**Theorem 2. The pairs of the extended Pascal configuration in $S_n$ are pairs of Schläfli simplexes.**

This follows at once from the theorem* that a set of $n+1$ linearly dependent $S_{n-2}$'s in $S_n$ are in the position of Schläfli. A quite different proof is based on the fact that the simplexes of a pair are polar reciprocal with respect to a hyperquadric, which, it will be remembered, was Schläfli's original point of departure. The form of equations (6) show that the hyperplanes $\pi_i$ are the polar hyperplanes of the points $P_i$ with respect to the hyperquadric of equation

\begin{equation}
\sum b_{ij} x_i x_j = 0, \tag{9}
\end{equation}

---

and accordingly that the points $R_i$ are the poles of the faces $\rho_i$, that is, the simplexes $F$ and $T_1$ are polar reciprocal with respect to the hyperquadric in question.

It is a fundamental property of Schlaffi simplexes that all the lines which meet $n$ of the $S_{n-3}$'s of intersection of corresponding faces meet also the remaining $S_{n-3}$. In the author's note on the $V_{n-1}^*$ previously cited is given a method of obtaining the point equation of the variety made up of the lines meeting $n$ given $S_{n-3}$'s of $S_n$. The use of that method here will give the equations of the varieties bearing the $2^m$ hyperreguli of the extended Pascal configuration.

By the use of the incidence criterion for line and $S_{n-2}$, as expressed by Grassmann-Plücker coordinates, the condition that $x$ be the coordinates of a point such that the line joining $x$ to a point of the $S_{n-2} \pi_0 \rho_0$ meets also the $n-1$ $S_{n-2}$'s $\pi_i \rho_i$, where $i$ runs from 1 to $n-1$, is found to be the equation

$$0 = \begin{vmatrix} b_{01} & b_{02} & b_{03} & \cdots & b_{0,n-1} & b_{0n} \\ -\sum_{i}^{ij=1} b_{1j}x_j & b_{12}x_1 & b_{13}x_1 & \cdots & b_{1,n-1}x_1 & b_{1n}x_1 \\ b_{21}x_2 & -\sum_{i}^{ij=2} b_{2j}x_j & b_{23}x_2 & \cdots & b_{2,n-1}x_2 & b_{2n}x_2 \\ b_{31}x_3 & b_{32}x_3 & -\sum_{i}^{ij=3} b_{3j}x_j & \cdots & b_{3,n-1}x_3 & b_{3n}x_3 \\ & & & \ddots & & \\ b_{n-1,1}x_{n-1} & b_{n-1,2}x_{n-1} & \cdots & -\sum_{i}^{ij=n-1} b_{n-1,j}x_j & b_{n-1,n}x_{n-1} \\ \end{vmatrix}$$

It should be observed that two options were exercised in writing the foregoing equation, one in choosing the variable line through $x$ to pass through a point of $\pi_0 \rho_0$, and one in considering $\pi_n \rho_n$ as the last $S_{n-3}$ of the Schlaffi set. The consequence of the first choice is the absence of a variable in the top row of the determinant, and the consequence of the second choice is that the coordinate $x_n$ appears only in the summations. A different ordering of the Schlaffi set gives rise to a superficially different form of equation (10), but it is easy to prove the two forms are equivalent.

If we replace, in equation (10), certain of the symbols $b_{ij}$ by the corresponding $c_{ij}$, the resulting equation will be that of the $V_{n-1}^*$ bearing the hyperregulus associated with one of the groupings $G_n$.

**Theorem 3.** The section of the extended Pascal configuration associated with a given hyperquadric $Q$ and fundamental simplex $F$ by a space of the simplex $F$ is itself an extended Pascal configuration plus certain residual flat spaces.
Consider first the nature of the intersection of a hyperplane of \( F \), say \( \rho_n \), with the \( V_{n-1} \) given by equation (10); we shall call the \( V_{n-1} \)'s of the \( Kf_n \) simply the \( V_n \), in which case the one given by equation (10) is \( V_1 \), for it corresponds to \( G_1 \). The equation of \( \rho_n \) is \( x_n = 0 \); part of its intersection with \( V_1 \) is the \( S_{n-2} \rho_n \), as we may see by a manipulation of the equation of \( V_1 \). If the top row of the determinant in (10) is multiplied by \( x_0 \), and the other rows added to it, the new top row has the form

\[
\begin{bmatrix}
  b_1 x_n & b_2 x_n & b_3 x_n & \cdots & b_{n-1,n} x_n & b_0 x_0 + b_1 x_1 + \cdots + b_{n-1,n} x_{n-1},
\end{bmatrix}
\]

whose elements are identically zero if \( x \) satisfies the restrictions

\[
x_n = \sum_i b_{i,n} x_i = 0,
\]

that is, if \( x \) lies on \( \rho_n \). We remark that this incidentally furnishes an analytic proof of the theorem of Wong that the \( S_{n-2} \)'s defining a \( V_{n-1} \) lie on it; in this connection it may be said that the same theorem follows at once from the present writer's geometric interpretation of linear dependence in \( S_n \), for, just as \( n+2 \) points which are linearly dependent lie in, and determine, a flat space, an \( S_n \), so do \( n+1 \) linearly dependent \( S_{n-2} \)'s lie in, and determine, a curved manifold, a \( V_{n-1} \).

To return to the discussion of Theorem 3, we have seen that one factor of the intersection of \( V_1 \) and \( \rho_n \) is an \( S_{n-2} \); it remains to find the residual portion. Suppose a point of \( \rho_n \) is such that its coordinates annul the determinant which is the cofactor of \( b_{n-1,n} x_{n-1} \) in the left member of (10); it will lie on a \( V_{n-2} \), for it satisfies an equation of the same form as (10). We do not mean to say that the \( V_{n-2} \) and the \( S_{n-2} \) of intersection of \( V_1 \) with a hyperplane of \( F \) have no common points; they actually meet in a \( V_{n-3} \). For our present purposes, nevertheless, it is the \( V_{n-2} \) and the \( S_{n-2} \) which are the important spaces of intersection.

We have seen that an \( S_{n-1} \) of \( F \) meets \( V_1 \) in a \( V_{n-2} \). Consider now the intersection of the \( 2^n V_{n-1} \)'s whose symbols are

\[
V_{11 \cdots 11 a_{n-1}} \quad (a_{n-1} = 1, 2, \cdots, 2^n),
\]

with the \( S_{n-1} \rho_n \). They all share the \( V_{n-2} \) whose equation, in \( \rho_n \), is obtained by setting the cofactor of \( b_{n-1,n} x_{n-1} \) in the determinant in the left member of (10) equal to zero, because the index \( a_{n-1} \) by definition controls only the symbols \( b_{n,i} \), where \( i \) runs from 0, 1, to \( n-1 \), which appear exclusively, if we keep to the \( S_{n-1} \rho_n \) in the last column of the determinant of (10). Varying the first \( n-2 \) indices of \( V_a \) is equivalent to replacing certain of the \( b_{ij} \) by the corresponding \( c_{ij} \). Thus \( \rho_n \) meets the \( 2^n V_{n-1} \)'s which bear the \( Kf_n \) in \( 2^{n+1} V_{n-2} \)'s, and these \( V_{n-2} \)'s bear the \( Kf_{n-1} \) of the simplex \( P_0 P_1 \cdots P_{n-1} \).
set up by means of the hyperquadric which is the intersection of $Q$ with $p_n$. The $S_{n-1} p_n$ also meets the $V_{n-1}^{n-1}$'s of the $Kf_n$ in the $S_{n-3}$'s in which it meets its corresponding faces among the auxiliary simplexes $T_{11 \ldots 11a_{n-1}}$. We may clearly reduce the $Kf_{n-1}$ to a $Kf_{n-3}$ in a similar manner. Had we desired to ascertain the nature of the intersections of the $Kf_n$ with another $S_{n-1}$ of $F$, we could clearly have chosen the appropriate form of equation (10) and proceeded as above.

An example of the successive reduction of the $Kf_n$ is given below where we show that the planes of a tetrahedron meet the 64 $V_1^{2}$'s bearing the $Kf_3$ in a $Kf_2$ and certain residual spaces.

III. The Extended Pascal Configuration in the Plane

For the case that $n$ is 2, we consider a fundamental triangle $F$ and conic $Q$. Paired with $F$ are the triangles

$$T_{\alpha_1}, \quad (\alpha_1 = 1, 2, \ldots, 8),$$

each pairing of triangles possessing the property that the points of intersection of corresponding sides are linearly dependent, i.e., collinear. The line of collinearity is an axis of perspectivity for the pair of triangles, and also the ordinary Pascal line of the Pascal hexagon whose sides are the sides of the two triangles, arranged in proper order. The $2^n V_{n-1}$'s here reduce to the eight lines $V_{\alpha_1}$; the line $V_{\alpha_1}$ associated with an effective grouping $G_{\alpha_1}$ constitutes the entire class of lines meeting any two of the three linearly dependent Schläfli points of the pair of triangles, and as a member of such a class it has the Schläfli property of meeting the third point. The $Kf_2$ consists of these eight $V_{\alpha_1}$; the reciprocals of the line coordinates of these are shown in the array below:

$$
\begin{array}{cccccccc}
V_1 & V_2 & V_3 & V_4 & V_5 & V_6 & V_7 & V_8 \\
1/u_1 & b_{12} & c_{12} & c_{12} & b_{12} & c_{12} & c_{12} & c_{12} \\
1/u_2 & b_{20} & c_{20} & b_{20} & c_{20} & b_{20} & c_{20} & c_{20} \\
1/u_3 & b_{30} & b_{01} & b_{01} & c_{01} & c_{01} & c_{01} & c_{01} \\
(11)
\end{array}
$$

We notice that the pairs of lines whose indices sum to nine intersect upon the line of equation

$$
\sum_k \left( \frac{1}{b_{ij}} + \frac{1}{c_{ij}} \right) x_k = 0 \quad (i, j, k \text{ a permutation of } 0, 1, 2),
$$
an equation which may be rewritten, by the use of (3), in the form

$$
\sum_k \frac{a_{ij}}{a_{ij} a_{jj}} x_k = 0.
$$
This line is a Steiner-Plücker line of the Pascal hexagon whose vertices are the six points $P_{ij}$; the three hexagons formed of the triangle pairs $FT_1$, $T_1T_5$, and $T_8F$, are readily seen to be three hexagons in the Steinerian relation, i.e., three vertices are fixed, and the remaining three permuted cyclically. It follows that the lines $V_1$ and $V_8$ meet in a Steiner point; since the same argument applies to all pairs of the $Kf_2$ whose indices sum to nine, the four points which define the line of equation (12) must be Steiner points, and hence their line of collinearity is a Steiner-Plücker line.

An independent check is found by using a table due to Cayley* which shows the nature of the intersection of the pairs of the 60 Pascal lines. In correlating the two notations, let the triangles $F$ and $T_1$ be the hexagon denoted by the Cayley letters $AE$. The sides are arranged in the order $P_0P_1P_2P_3P_4P_5$, and the vertices in the order $P_0P_1P_2P_3P_4P_5$, or, for brevity, in the order 123456. If the symbols $p$, $h$, and $g$ denote Pascal, Kirkman, and Steiner points respectively, Cayley’s table can be correlated with the present paper as below:

<table>
<thead>
<tr>
<th>Group</th>
<th>Hexagon</th>
<th>Cayley Letters</th>
<th>Nature of Intersection</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>123456</td>
<td>$AE$</td>
<td>$V_1$ $V_2$ $V_3$ $V_4$ $V_5$ $V_6$ $V_7$</td>
</tr>
<tr>
<td>2</td>
<td>132456</td>
<td>$EL$</td>
<td>$V_2$ $p$</td>
</tr>
<tr>
<td>3</td>
<td>123546</td>
<td>$EF$</td>
<td>$V_3$ $p$ $h$</td>
</tr>
<tr>
<td>4</td>
<td>132546</td>
<td>$EG$</td>
<td>$V_4$ $h$ $p$ $p$</td>
</tr>
<tr>
<td>5</td>
<td>162345</td>
<td>$DE$</td>
<td>$V_5$ $p$ $h$ $h$ $g$</td>
</tr>
<tr>
<td>6</td>
<td>163245</td>
<td>$EI$</td>
<td>$V_6$ $h$ $p$ $g$ $h$ $p$</td>
</tr>
<tr>
<td>7</td>
<td>162354</td>
<td>$EM$</td>
<td>$V_7$ $g$ $p$ $h$ $p$ $h$</td>
</tr>
<tr>
<td>8</td>
<td>163254</td>
<td>$EH$</td>
<td>$V_8$ $g$ $h$ $p$ $h$ $p$ $p$</td>
</tr>
</tbody>
</table>

This display shows that the pairs $V_1V_8$, $V_2V_7$, $V_3V_6$, and $V_4V_5$ meet in Steiner points, but it does not show that the Steiner points thus garnered lie on a line.

**Theorem 4.** The $Kf_2$ determines a Steiner-Plücker line.

In the plane, the $Kf_2$ is apparently a less rich configuration than the Hexagrammaticum Mysticum, for it has but 8 lines where the other has 60. Consider, however, a different approach to the general situation in $S_n$ by taking, as did Schläfli, two simplexes polar reciprocal with respect to a hyperquadric. We have already observed that $F$ and $T_1$ are polar reciprocal

---

with respect to the hyperquadric given by equation (9). Let us take this hyperquadric as basic, and try to determine the hyperquadric \( Q \) whose matrix involves the numbers \( a_{ij} \). We now define the set of \( 2^n \) points \( P_{ij} \) as the intersection of \( P_i P_j \) with \( \pi_i \), and an analogous set of \( 2^n \) points \( R_{ij} \) as the intersection of \( R_i R_j \) with \( \rho_i \); it is an easily proved property of Schl"afli simplexes that the points \( P_{ij} \) and \( R_{ij} \) lie upon hyperquadrics, which are in general different, coinciding only when \( n \) is 2. In the plane there is associated with a fundamental triangle \( F \) by means of a conic \( Q \) a set of eight auxiliary triangles; associated with any one of these auxiliary triangles by means of the same conic \( Q \) is a set of eight more auxiliary triangles, which set partially overlaps the first. Indeed, the six points \( P_{ij} \) where the sides of \( F \) meet \( Q \) determine 15 lines, which may be grouped, by triples, into 15 distinct triangles. Any one of these 15 triangles pairs with eight others to make a Pascal hexagon; there are \( (15 \times 8)/(1 \times 2) = 60 \) different Pascal hexagons. It is because the points \( R_{ij} \) coincide with the points \( P_{ij} \) in the plane case that there exists a configuration of more than \( 2^n \) lines. The richness of the usual Pascal configuration is from this point of view due to the coincidence of the hyperquadrics (in the plane case, conics) determined by the sets of points \( P_{ij} \) and \( R_{ij} \).

The body of theorems about the Pascal configuration can be established from a consideration of the 15 triangles instead of the more usual approach by way of the 60 Pascal hexagons.

IV. The extended Pascal configuration in space

The edges of a fundamental tetrahedron \( F \) pierce a given quadric (1) in the 12 points \( P_{ij} \), which are grouped in the 64 effective groupings \( G_{\alpha_1 \alpha_2} \) \((\alpha_1 \alpha_2 = 1, 2, \ldots, 8)\), each of which groupings defines an auxiliary tetrahedron \( T_{\alpha_1 \alpha_2} \). The lines of intersection of corresponding faces of the fundamental tetrahedron \( F \) and an auxiliary tetrahedron \( T_{\alpha_1 \alpha_2} \) are linearly dependent, hence these four lines lie on a regulus. The \( K_F \) is made up of the 64 conjugate reguli, that is, of the 64 one-parameter families of lines which meet the 64 sets of linearly dependent lines. A pairing of the \( K_F \), as \( F \) and \( T_{\alpha_1 \alpha_2} \), is a Schl"afli pair of tetrahedra, that is, such that all the lines meeting three of the lines of intersection of corresponding faces meet also the fourth.

The theorem of Chasles and Weddle asserts that the four lines of intersection of corresponding faces lie on a regulus. Chasles further remarked that there are several effective groupings of the 12 piercing points, but apparently he did not consider how many effective groupings there are, nor what incidence relations exist among the various reguli.
The $V_{n-1}^{*}$ defined by a pairing is here a $V_{2}^{*}$, or quadric; it contains the regulus of lines which meet the four linearly dependent lines, and also the lines themselves.

**Theorem 5.** The fundamental tetrahedron $F$ circumscribes each of the quadrics $V_{a_{1}a_{2}}$.

In the derivation of $V_{a_{1}a_{2}}$, it appears that it contains the lines $\rho_{i}x_{i}$ ($i=0, 1, 2, 3$), that is, it has a generator in each of the planes $\rho_{i}$, hence it is tangent to each of these planes.

**Theorem 6.** The 128 lines common to a face of $F$ and the 64 quadrics $V_{a_{2}a_{2}}$ consist of two $Kf_{2}$'s, each line being counted eight times.

The face $\rho_{3}$ has the equation $x_{3}=0$. It meets $V_{11}$ in a conic of equation

$$0 = x_{3} = \left( \sum_{i} b_{i}^{0} x_{i} \right) \left( \sum_{i} b_{i}^{2} x_{i} \right),$$

where $i, j, k$ is a permutation of $0, 1, 2$. The eight quadrics $V_{1a_{2}}$ have equations differing from that of $V_{11}$ only in the coefficients controlled by $a_{2}$, i.e., the coefficients $b_{03}, b_{13},$ and $b_{23}$, which appear segregated in one factor of (13). It follows that the eight quadrics $V_{1a_{2}}$ share the ruling of equations

$$0 = x_{3} = \sum_{i} b_{i}^{0} x_{i}.$$  

In like manner the eight quadrics $V_{2a_{2}}$ share the ruling of equations

$$0 = x_{3} = b_{01} x_{0} + b_{10} x_{1} + c_{20} x_{2}.$$  

The eight lines obtained by varying the first index constitute the $Kf_{2}$ of $P_{0}P_{1}P_{2}$ with respect to the conic cut from $Q$ by $x_{3}=0$, as may be seen by comparing their equations with the display (11).

Octuples of quadrics sharing a common second index likewise have a common generator. The coördinates, in $x_{3}=0$, of the eight lines thus obtained are shown below:

$$L_1, L_2, L_3, L_4, L_6, L_7, L_8$$

$$u_{1}, b_{03}, c_{03}, b_{03}, c_{03}, b_{03}, c_{03}, u_{2}, b_{13}, c_{13}, b_{13}, c_{13}, b_{13}, c_{13}, u_{3}, b_{23}, b_{23}, c_{23}, b_{23}, c_{23}, u_{4}, b_{23}, b_{23}, c_{23}, b_{23}, c_{23}$$

The general similarity of (11) and (15) suggests that the second set of eight
lines is the $Kf_3$ of $P_0P_1P_2$ with respect to some conic. To find this conic, solve (3) for $a_{ij}$ in terms of $b_{ij}$, obtaining

$$a_{ij} = b_{ij}, \quad 2b_{ij}a_{ij} = b_{ij}^2 + b_{ii}b_{jj}.$$  

We next consider how we might pass from the lines of (11) to the conic whose matrix involves the numbers $a_{ij}$, and use the same method to obtain the conic of matrix

$$
\begin{pmatrix}
2a_{00} & \beta_{23} + a_{00}a_{11}b_{23} & \beta_{13} + a_{00}a_{22}b_{13} \\
\beta_{23} + a_{00}a_{11}b_{23} & 2a_{11} & \beta_{03} + a_{11}a_{22}b_{03} \\
\beta_{13} + a_{00}a_{22}b_{13} & \beta_{03} + a_{11}a_{22}b_{03} & 2a_{22}
\end{pmatrix},
$$

where $\beta_{ij}b_{ij} = 1$, by means of which the lines of coordinates (15) are the $Kf_3$ of $P_0P_1P_2$. This conic appears to have no simple geometrical relation to the original quadric $Q$, whereas the corresponding conic associated with the first set of eight lines is the intersection of $Q$ by $x_3 = 0$.

It is clear that $\rho_3$ is in no wise an exceptional face of $F$, hence the situation on it is duplicated on the other planes of the tetrahedron. This completes the proof of Theorem 6.

**Theorem 7.** A tetrahedron and a quadric determine two sets of Steiner-Plücker lines, each set lying on a regulus.

On each face of the fundamental tetrahedron there are two Steiner-Plücker lines, one deriving from each of the $Kf_3$'s. One $Kf_3$ consists of lines belonging to the reguli which form the $Kf_3$ of the tetrahedron and quadric; the four associated Steiner-Plücker lines have the equations

$$(17) \quad 0 = x_i = \sum_{j \neq i} \frac{a_{km}}{a_{kk}a_{mm}} x_j \quad (ijkl \text{ a permutation of 0123}),$$

equations which show by their symmetry that the four lines in question lie on a regulus. The same is true of the equations of the four Steiner-Plücker lines deriving from the $Kf_3$'s whose lines lie on the conjugate reguli of the $Kf_3$; the equations are

$$(17') \quad 0 = x_i = \sum_j (b_{ji} + c_{ij}) x_j.$$
fundamental simplex $F$ with the auxiliary simplexes $T_{\alpha_1\alpha_2\alpha_3}$ ($\alpha_1, \alpha_2 = 1, 2, \ldots, 8, \alpha_3 = 1, \ldots, 16$), these auxiliary simplexes being defined by the 1024 effective groupings $G_{\alpha_1\alpha_2\alpha_3}$ of the 20 points in which the edges of $F$ pierce a given hyperquadric $Q$. The two-parameter families of lines, or hyperreguli, lie on a set of $V_3$'s; the $V_3$ is a $V_{n-1}^1$ which has been thoroughly studied by Segre* and Castelnuovo,† its co-discoverers, Berzolari,‡ and many others.

The 16 $V_3$'s denoted by $V_{11\alpha_3}$ meet the $S_3$ whose equation is $x_4 = 0$ in the same quadric; its equation in $\rho_4$ is that of $V_{11}$ in three-space. It follows that the 1024 $V_3$'s meet $\rho_4$ in $64V_2$'s, each counted 16 times; this is a special case of Theorem 3.