THE DEGREE AND CLASS OF MULTIPLY
TRANSITIVE GROUPS, II*

BY

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1. The paper to which this is a sequel had to do only with those multiply
transitive groups of class $\mu(>3)$ in which at least one substitution of degree
$\mu$ is of even order.† Among other results, it was there proved that the degree
$n$ of triply transitive groups of class $\mu(>3)$, which contain a substitution of
even order on $\mu$ letters, does not exceed $2\mu$. Triply transitive groups of
class $\mu$ and degree $2\mu$ exist. This measure of success was due to the simplicity
of structure of diedral rotation groups of class $\mu$ generated by two similar
substitutions of degree $\mu$ and order 2. Up to the present time no better
limit than that of Bochert, $n \leq 3\mu(\mu > 6)$, has been found to apply to these
triply transitive groups of class $\mu(>3)$ in which all substitutions of degree $\mu$
are of odd order.‡ But it will here be proved that if such a group contains a
substitution of order $p^e$ ($p$ an odd prime) on $\mu$ letters,

$$\mu > \frac{n}{2} \left(1 - \frac{1}{p^e}\right) - \frac{2}{p^e}$$

For doubly transitive groups the lower limit for the class $\mu(>3)$ in terms
of the degree when there is a substitution of even order on $\mu$ letters present
in the group is given by

$$\mu > \frac{n}{2} - \frac{n^{1/2}}{2} - 1.$$  §

Whenever applicable, this replaces Bochert’s limit of $n/3 - 2n^{1/2}/3$.¶ We
can now prove that, when a doubly transitive group of class $\mu(>3)$ contains
a substitution of odd prime order $p$, and (when $p = 3$) $n$ is sufficiently large,

$$\mu > \frac{n}{2} \left(1 - \frac{1}{p}\right) - \frac{n^{1/2}}{2} \left(1 - \frac{1}{p^2}\right)^{1/2} - 1.$$

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cember, 1928.

† Manning, these Transactions, vol. 18 (1917), p. 463.


For $p=3$, this is only $\mu > n/3 - (2n)^{1/2}/3 - 1$. It was stated by Bochert without proof that $n/3 - (2n)^{1/2}/3$ is a limit to which his ascending series of inferior limits for $\mu$ approaches on repeated use of his inequality (19). That is a mistake. His inequality is satisfied by $n/3 - (2n)^{1/2}/3$ and also by $n/3 - (2n)^{1/2}/3 - 1/2$ while a true inferior limit for $\mu$ is a number that does not satisfy the given inequality and which therefore is less than $n/3 - (2n)^{1/2} - 1/2$.

2. Bochert's Lemma will be used in the following form:

If the substitutions $S$ and $T$ have exactly $m$ letters in common, and if $S$ replaces $q$, and $Tr$, common letters by common letters, the degree of $S^{-1}T^{-1}ST$ is not greater than $3m - q - r$.

This lemma remains indispensable in studying quadruply transitive groups of class $\mu(>3)$ and those doubly and triply transitive groups of class $\mu(>3)$ in which all the substitutions of degree $\mu$ are of order 3. But additional information is given in other cases by the following lemma:

**Lemma.** If $S$ and $T$ are two substitutions of degree $\mu$ and odd order $d$ which generate a group of class $\mu$, and if no power of $S$ is commutative with a power of $T$ (identity excepted), $S$ and $T$ have at least $\mu/2 - \mu/(2d)$ letters in common.

Let $S$ and $T$ have exactly $m$ letters in common, $m$ roman letters, say. Let the other letters of $S$ and $T$ be greek letters. From Bochert's Lemma it is known that $m \geq \lfloor \mu/3 \rfloor$, the integral part of $\mu/3$. Then no proof is needed when $d=3$. We assume that $d \geq 5$. Now we say that $S$ has $s_i$ cycles each of which contains $i$ roman letters, and that $T$ has $t_i$ cycles each of which contains $j$ romans. Then

$$s_0 + s_1 + \cdots + s_d = t_0 + t_1 + \cdots + t_d = \frac{\mu}{d},$$

the number of cycles in $S$ and $T$. Also

$$s_1 + 2s_2 + 3s_3 + \cdots + ds_d = t_1 + 2t_2 + 3t_3 + \cdots + dt_d,$$

$$= \mu/3 - \epsilon + k_1 = \mu/3 + k,$$

where $\epsilon=0$, $1/3$, or $2/3$, as required to make $\mu/3 - \epsilon = \lfloor \mu/3 \rfloor$, and $k_1$ is a positive integer or zero.

Consider a cycle of $S$ in which there are $i$ romans. It generates a regular cyclic group. Therefore in the $d-1$ powers of this cycle every sequence of two letters occurs once and only once; then in these $d-1$ powers of one cycle of $S$ there are $i(i-1)$ roman sequences. In the $d-1$ powers of $S$ the number of roman sequences is exactly $2s_2 + 6s_3 + \cdots + i(i-1)s_i + \cdots + d(d-1)s_d$, and the average number of roman sequences in the $d-1$ powers of $S$ is $[2s_2 + 6s_3 + \cdots + d(d-1)s_d]/(d-1)$. Similarly, in the powers of $T$ the
average number of roman sequences is \( [2t_2+6t_3+\cdots+d(d-1)t_d]/(d-1) \). From the powers of \( S \) and \( T \) let two substitutions \( S^u \) and \( T^v \) be chosen, each of which has the average or more than the average number of roman sequences. Since the number of roman letters is \( \mu/3+k \), the total number of roman sequences in \( S^u \) and \( T^v \) jointly cannot exceed \( 3k \). For Bochtet's Lemma asserts that the degree of \( S^{-u}T^{-v}S^uT^v \) is at most \( \mu+3k \) diminished by the number of roman sequences in \( S^u \) and \( T^v \). And since \( S^u \) and \( T^v \) are not commutative this degree is at least \( \mu \). Then

\[
\sum_{i=0}^{d} \frac{i(i-1)s_i}{d-1} + \sum_{j=0}^{d} \frac{j(j-1)t_j}{d-1} \leq 3k,
\]

and one of the two summation terms of this inequality is not greater than \( 3k/2 \), say the first. Thus we have simultaneously

\[
s_0 + s_1 + s_2 + \cdots + s_d = \frac{\mu}{d},
\]
\[
s_1 + 2s_2 + 3s_3 + \cdots + ds_d = \frac{\mu}{3} + k,
\]
\[
2s_2 + 6s_3 + 12s_4 + \cdots + d(d-1)s_d \leq \frac{3}{2}(d-1)k.
\]

Let two terms, \( s_x \) and \( s_y \), be eliminated \((x<y)\). The result is

\[
\frac{(x+y-1)d - 3xy}{3d} + \frac{1}{2}(3d-2x-2y-1)k.
\]

If now \( x+1 \) be put for \( y \),

\[
\sum_{i=x}^{d} (i-x)(i-x-1)s_i \geq 0,
\]

and therefore

\[
\mu \leq \frac{3d(3d-4x-3)}{2x(2d-3x-3)}k.
\]

We may choose the integer \( x \) to make the coefficient of \( k \) a minimum. The continuous curve

\[
y = \frac{3d(3d-4x-3)}{2x(2d-3x-3)}
\]

has a minimum between \( x=0 \) and \( x=2d/3-1 \). From \( D_s y = 0 \) we get \( x = \).
\[
\left\{3d - 3 - \left[(d+1)(d+3)\right]^{1/2}\right\}/4, \text{ lying between } (d-3)/2 \text{ and } (d-1)/2, \text{ and to these two integers correspond equal ordinates, } 6d/(d-3). \text{ Then}
\]
\[
k \geq \frac{d - 3}{6d},
\]
and
\[
m \geq \frac{d - 1}{2d}.
\]

It follows that the degree of \( \{S, T\} \) is at most \( 3\mu/2 + \mu/(2d) \).

3. We prove the following theorem:

**Theorem I.** The class \( \mu(> 3) \) of a triply transitive group of degree \( n \) that contains a substitution of degree \( \mu \) and of order \( \rho^c \) (\( \rho \) an odd prime) is greater than
\[
\frac{n}{2} \left(1 - \frac{1}{\rho^c}\right) - \frac{2}{\rho^c}.
\]

One of the substitutions of degree \( \mu \) in the group \( G \) is \( S = (a \cdots b \cdots) \cdots \), of order \( \rho^c \). In case the exponent \( c \) is greater than unity, \( b \) is one of the letters in the same cycle of \( S^{\rho^c-1} \) as \( a \), and therefore in the same cycle as \( a \) in every power of \( S \). There are in \( G \) substitutions similar to \( S \) which displace \( a \) and fix \( b \). Let \( S_1, S_2, \ldots, S_w \) be a complete set of \( w \) such substitutions, conjugate under the transitive subgroup \( G(a)(b) \) of \( G \) that fixes \( a \) and \( b \). No power of \( S_i (i = 1, 2, \ldots, w) \) is commutative with a power of \( S \). The substitutions of this complete set of conjugates displace exactly
\[
w + \frac{w(\mu - 1)(\mu - 2)}{n - 2}
\]
letters of \( S \).* For they all displace \( a \), and each of the other \( w(\mu - 1) \) letters in the set \( S_1, S_2, \ldots, S_w \) occurs as often as any other, that is, \( w(\mu - 1)/(n-2) \) times. Thus the \( \mu - 2 \) letters of \( S \), disregarding \( a \) and \( b \), occur in the set \( S_1, S_2, \ldots \) the number of times stated above. Then, by our lemma,
\[
w + \frac{w(\mu - 1)(\mu - 2)}{n - 2} \geq w\left(\frac{\mu}{2} - \frac{\mu}{2\rho^c}\right),
\]
whence
\[
\frac{n - \mu}{\rho^c} - (\mu - 2)\left(\frac{n}{2} - \frac{n}{2\rho^c} - \mu\right) \geq 0.
\]

* For a detailed explanation of this and of similar formulas which follow, see Bochert, *Mathematische Annalen*, vol. 40 (1892), p. 176.
If now we put $n/2 - n/(2p^c) - \delta$ for $\mu$ in the left member of this inequality, we get

$$-\frac{n}{2} \left( \delta - \frac{\delta}{p^c} - \frac{1}{p^c} - \frac{1}{2p^{2c}} \right) + \delta^2 + 2\delta - \frac{\delta}{p^c}.$$  

This is positive if $\delta = 1/p^c$, but if $\delta = 2/p^c$ it reduces to

$$-\frac{n}{2} \left( \frac{1}{p^c} - \frac{3}{p^{2c}} \right) + \frac{4}{p^c} + \frac{2}{p^{2c}},$$

negative if $n$ is greater than $8 + 28/(\mu^c - 3)$. It is known that there is no triply transitive group of class 5. Hence the theorem is true as stated without exception.

4. We prove the following theorem:

**Theorem II.** The class $\nu(>3)$ of a triply transitive group of degree $n$ in which every substitution of degree $\mu$ is of order 3 is not less than $(n+4)/3$.

Let $S = (abc) \cdots$, a substitution of degree $\nu$ of $G$. Let $S_1 = (a \cdots)$ \cdots $(b)$, similar to $S$. Under $G(a)(b)$, $S_1$ is one of $w$ conjugates, no one of which is commutative with $S$. If $S$ and $S_i$, have exactly $\mu/3$ common letters (their roman letters), $S_i$ can have no roman sequence. This fact is an immediate consequence of Bochert's Lemma. In other words, if $S_i$ has a roman sequence, it has at least $\mu/3 + 1$ letters in common with $S$. If $S$ and $S_i$ have $\mu/3 + 1$ letters in common, $S_i$ because it has only $\mu/3$ cycles, has some two letters of $S_i$ in sequence. Hence, if $S_i$ has a cycle of three roman letters, which means three roman sequences, $S$ and $S_i$ have $\mu/3 + 2$ or more common letters. Now in $w/(n-2)$ substitutions of the set $S_1, S_2, \cdots, S_w$ the letter $a$ is followed by a given letter of $G(a)(b)$. Then in $w(\mu-2)/(n-2)$ substitutions $S_i$, $a$ is followed by a letter of $S$, and similarly $a$ is preceded by a letter of $S$ in $w(\mu-2)/(n-2)$ substitutions $S_i$. Therefore

$$w + \frac{w(\mu - 1)(\mu - 2)}{n - 2} \geq \frac{w\mu}{3} + 2w \frac{\mu - 2}{n - 2}.$$  

This inequality reduces to

$$\frac{(\mu - 2)(\mu - 3)}{n - 2} \geq \frac{\mu - 3}{3},$$

whence, because by hypothesis $\mu > 3$,

$$n \leq 3\mu - 4.$$  

5. We prove the following theorem:
THEOREM III. The degree of a quadruply transitive group of class $\mu$ (>3) does not exceed $2\mu + 1$.

For quadruply transitive groups of class $\mu$ Bochert’s limit is $2\mu + 2$, while the limit found in the preceding paper under the restriction that one of the substitutions of degree $\mu$ is of even order is $2\mu - 1$. In the following proof it is assumed that all the substitutions of degree $\mu$ in $G$ are of odd order.

Two cases are to be distinguished: (1) At least one substitution of degree $\mu$ is of order $>3$. (2) All substitutions of degree $\mu$ are of order 3.

Case 1. Let $S = (a \cdots b \cdots)$ be a substitution of degree $\mu$ and of order $>3$. The letters $a$ and $b$ are in the same cycle of $S$ but are not adjacent. There are $w$ substitutions $S_1 = (b) (a \cdots), S_2, \cdots$, conjugate under $G(a)(b)$, a doubly transitive subgroup of $G$. It was shown in §3 that the number of times the letters of $S$ are to be found in the set $S_1, S_2, \cdots$ is $w + w(\mu - 1)(\mu - 2)/(n - 2)$. The number of times letters of $S$ occur just before or just after $a$ in the set $S_1, S_2, \cdots$ is $2w(\mu - 2)/(n - 2)$. The number of sequences in $S_1, S_2, \cdots$ is $w(\mu - 2)$, if we exclude those of which one letter is $a$. Since $G(a)(b)$ is doubly transitive each sequence occurs as often as any other and therefore exactly $w(\mu - 2)/[(n - 2)(n - 3)]$ times. Now there are $(\mu - 2)(\mu - 3)$ permutations two at a time of the letters of $S$ (excluding $a$ and $b$). Therefore $2w(\mu - 2)/(n - 2) + w(\mu - 3)/[(n - 2)(n - 3)]$ is the number of sequences in $S_1, S_2, \cdots$ of which both letters are letters of $S$. The number of times the letter before $a$ (and the letter after $a$) in $S$ occurs in the set $S_1, S_2, \cdots$ is $2w(\mu - 1)/(n - 2)$. Excluding $a$ and $b$, $S$ has $\mu - 4$ sequences. Excluding $a$, $S_i$ has $(\mu - 1)(\mu - 2)$ permutations of letters two at a time, and therefore the set $S_1, S_2, \cdots$ contains $w(\mu - 1)(\mu - 2)$ such permutations of $\mu - 1$ letters two at a time, each occurring $w(\mu - 1)(\mu - 2)\cdot[(n - 2)(n - 3)]^{-1}$ times, so that each sequence of $S$ (not including $a$ or $b$) is in $w(\mu - 1)(\mu - 2)/[(n - 2)(n - 3)]$ substitutions of the set $S_1, S_2, \cdots$. Therefore $2w(\mu - 1)/(n - 2) + w(\mu - 1)(\mu - 2)(\mu - 4)/[(n - 2)(n - 3)]$ is the number of times permutations two at a time of letters of $S_1, S_2, \cdots$ are a sequence in $S$. Therefore the sum of the degrees of the $w$ commutators $S^{-1}S_i^{-1}SS_i$ ($i = 1, 2, \ldots, w$) gives

$$3w + \frac{3w(\mu - 1)(\mu - 2)}{n - 2} - \frac{2w(\mu - 2)}{n - 2} - \frac{w(\mu - 2)(\mu - 3)}{(n - 2)(n - 3)}$$

$$\quad - \frac{2w(\mu - 1)}{n - 2} - \frac{w(\mu - 1)(\mu - 2)(\mu - 4)}{(n - 2)(n - 3)} \geq w\mu.$$

Whence, if $z$ is put for $n - 2\mu - 3$,
\[ (\mu - 3)[z^2 + (\mu + 5)z] + 2(\mu^2 - 10) \leq 0, \]
or
\[ z^2 + (\mu + 5)z + 2\mu + 6 \leq 0 \quad (\mu > 5). \]

Therefore
\[ n - 2\mu - 3 \leq -2, \quad n \leq 2\mu + 1. \]

**Case 2.** The substitutions of degree \( \mu \) are all of order 3. We have \( S = (abc) \cdots \) and the set of \( w \) conjugates under \( G(a)(b) \): \( S_1 = (b)(a \cdots) \cdots, S_2, \cdots \). Since \( G(a)(b) \) is doubly transitive we can choose the two letters in one of these substitutions, in \( S_1 \) say, at will from the \( n - 2 \) letters of \( G(a)(b) \). If \( S_1 = (ac\alpha)(b) \cdots, S^{-1}S^{-1}SS_1 = (a\alpha)(ab) \cdots \). If this substitution is of degree \( \mu \), \( n \leq 2\mu - 1 \), and the theorem is true. Hence the degree of this commutator is not less than \( \mu + 1 \). Also if \( S_1 = (a\alpha)(b) \cdots, S^{-1}S^{-1}SS_1 = (ac)(b\alpha) \cdots, \) which must be of degree \( \geq \mu + 1 \).

In the first cycle of each of the substitutions \( S_1, S_2, \cdots \) there occurs every possible sequence of the letters of \( G(a)(b) \), and any such sequence occurs \( w \times [(n - 2)(n - 3)] \) times. Then the \( 2(\mu - \mu) \) sequences in which the letter \( c \) of \( S \) is followed or preceded by one of the \( n - \mu \) letters of \( G \) fixed by \( S \) occur \( 2w(n - \mu)/[(n - 2)(n - 3)] \) times in the first cycle of the substitutions \( S_1, S_2, \cdots \). Thus the commutator of \( S \) and \( 2w(n - \mu)/[(n - 2)(n - 3)] \) substitutions of the set \( S_1, S_2, \cdots \) is of degree \( \geq \mu + 1 \).

In the set \( S_1, S_2, \cdots, a \) is a letter of \( 2w(\mu - 2)/(n - 2) \) sequences in which the other letter is in \( S \), and \( c \) occurs in \( w(\mu - 1)/(n - 2) \) of the substitutions \( S_1, S_2, \cdots \). In the set \( S_1, S_2, \cdots \) there are \( w(\mu - 2)^2(\mu - 3)/[(n - 2)(n - 3)] \) sequences, not involving \( a \), whose two letters are the \( \mu - 2 \) letters \( c, d, \cdots \) of \( S \). There are \( \mu - 3 \) sequences in \( S \) (not including \( a \) or \( b \)) and each such permutation of letters two at a time occurs in \( S_1, S_2, \cdots, w(\mu - 1)(\mu - 2) \cdot [(n - 2)(n - 3)]^{-1} \) times. Hence
\[
3w + \frac{3w(\mu - 1)(\mu - 2)}{n - 2} - \frac{2w(\mu - 2)}{n - 2} - \frac{w(\mu - 1)}{n - 2} - \frac{w(\mu - 2)^2(\mu - 3)}{(n - 2)(n - 3)} \geq w\mu + \frac{2w(n - \mu)}{(n - 2)(n - 3)},
\]
or, with \( z = n - 2\mu - 3 \),
\[ z^2 + (\mu + 4)z + \mu + 4 \leq 0, \]
and therefore
\[ n \leq 2\mu + 1. \]

6. We prove the following theorem:
Theorem IV. Let \( \mu \) be the class and \( n \) the degree of a doubly transitive group in which one of the substitutions of degree \( \mu > 3 \) is of prime order \( p > 3 \); then
\[
\mu > \frac{n}{2} \left( 1 - \frac{1}{p} \right) - \frac{n^{1/2}}{2} \left( 1 - \frac{1}{p^2} \right)^{1/2} - 1.
\]

There is a substitution \( S = (ab \cdots) \cdots \) of prime order \( p > 3 \) in the given doubly transitive group \( G \), and \( S \) is one of \( w \) conjugates under \( G \). Any sequence of two letters, as \( ab \), occurs \( w\mu/[n(n-1)] \) times in this set. The number of possible sequences in \( n \) letters, in which one letter belongs to \( S \) and the other does not, is \( 2\mu(n-\mu) \). Then in the complete set of \( w \) conjugate substitutions such sequences as \( aa \) and \( aa \) occur in all \( 2\mu \cdot [n(n-1)]^{-1} \) times. There can be at most \( p-1 \) such sequences in a cycle. Let us say that \( S_1, S_2, \ldots, S_y \) are \( y \) substitutions of the set not commutative with \( S \). Then
\[
y \leq \frac{2w\mu(n-\mu)}{n(n-1)(1-1/p)}.
\]

We recall that if one of the \( w \) conjugate substitutions has exactly \( x \) letters in common with \( S \),
\[
\sum_x x = \frac{w\mu^2}{n},
\]
\[
\sum_x x(x-1) = \frac{w\mu^2(\mu - 1)^2}{n(n-1)},
\]
and
\[
\sum_x \left( x - \frac{\mu^2}{n} \right)^2 = \frac{w\mu^2(n-\mu)^2}{n^2(n-1)}.
\]

Now by our lemma, if \( S \) and \( S_i \) \( (i = 1, 2, \ldots, y) \) have \( x_i \) letters in common,
\[
x_i \geq \frac{\mu}{2} \left( 1 - \frac{1}{p} \right),
\]
and therefore
\[
\sum_{i=1}^y x_i \geq \frac{y\mu}{2} \left( 1 - \frac{1}{p} \right).
\]

Let us now restrict our attention to those doubly transitive groups for which
\[
\mu \leq \frac{n}{2} \left( 1 - \frac{1}{p} \right).
\]

For such groups
\[
\frac{\mu}{2} \left( 1 - \frac{1}{p} \right) - \mu \frac{2\mu}{n} \geq 0,
\]
and therefore
\[
\sum_{i=1}^{y} \left( x_i - \frac{\mu^2}{n} \right)^2 \geq y \left( \frac{\mu}{2} - \frac{\mu}{2p} - \frac{\mu^2}{n} \right)^2.
\]

Next,
\[
\sum_{w-y} \left( x - \frac{\mu^2}{n} \right)^2 \leq \frac{w\mu^2(n - \mu)^2}{n^2(n - 1)} - y \left( \frac{\mu}{2} - \frac{\mu}{2p} - \frac{\mu^2}{n} \right)^2.
\]

Also
\[
\sum_{w-y} \left( x - \frac{\mu^2}{n} \right)^2 \geq \frac{1}{w - y} \left[ \sum_{w-y} \left( x - \frac{\mu^2}{n} \right) \right]^2
\]
\[
\geq \frac{1}{w - y} \left( \frac{y\mu}{2} - \frac{y\mu}{2p} - \frac{y\mu^2}{n} \right),
\]

so that
\[
\frac{w\mu^2(n - \mu)^2}{n^2(n - 1)} - y \left( \frac{\mu}{2} - \frac{\mu}{2p} - \frac{\mu^2}{n} \right)^2 \geq \frac{y^2}{w - y} \left( \frac{\mu}{2} - \frac{\mu}{2p} - \frac{\mu^2}{n} \right),
\]

or
\[
\frac{w\mu^2(n - \mu)^2}{n^2(n - 1)} - \frac{wy}{w - y} \left( \frac{\mu}{2} - \frac{\mu}{2p} - \frac{\mu^2}{n} \right)^2 \geq 0,
\]

or
\[
\left( \frac{w}{y} - 1 \right) \mu^2(n - \mu)^2 - \left( \frac{\mu}{2} - \frac{\mu}{2p} - \frac{\mu^2}{n} \right)^2 \geq 0.
\]

Substituting for \( w/y \),
\[
\frac{\mu(n - \mu)(1 - 1/\rho)}{2n} - \frac{\mu^2(n - \mu)^2}{n^2(n - 1)} - \frac{\mu^2(n - n - \mu)}{n^2(2 - 2n - \mu)^2} \geq 0,
\]

or finally,
\[
\frac{n(n - \mu)(1 - 1/\rho)}{2\mu} - \frac{(n - \mu)^2}{n - 1} - \left( \frac{n - n - \mu}{2 - 2n - \mu} \right)^2 \geq 0.
\]

Let \( n/2 - n/(2\rho) - \epsilon n^{1/2}/2 - \delta \) be substituted for \( \mu \) in the final inequality. In the resulting polynomial in \( n^{1/2} \) it will be seen that the coefficient of \( n^3 \) vanishes if \( \epsilon = (1 - 1/\rho^2)^{1/2} \). With this value of \( \epsilon \) agreed upon, the polynomial may be written
\[
\left( 2 + \frac{2}{\rho^2} - 4\delta + \frac{4\delta}{\rho} \right) \epsilon n^{5/2} + \left( \frac{2\epsilon^2}{\rho} + 8\delta - 4\delta^2 + \frac{4\delta^2}{\rho} \right) n^2
\]
\[
+ \left( 8\delta + \frac{8\delta}{\rho} + 12\delta^2 - 2 + \frac{2}{\rho} \right) \epsilon n^{3/2} + \left( 8\delta^2 + 8\delta + \frac{8\delta}{\rho} - 4 + \frac{4}{\rho} \right) \delta n.
\]

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If \( \delta = \frac{1}{2} \), this polynomial is positive; if \( \delta = 1 \), it reduces to

\[
- \left( 1 - \frac{2}{p} - \frac{1}{p^2} \right) n^{3/2} + \left( 2 + \frac{3}{p} - \frac{1}{p^3} \right) n + \left( 9 + \frac{5}{p} \right) n^{1/2} + 6 + \frac{6}{p},
\]

which is negative for \( n > 57 \). Any possible cases of exception are covered by the known theorems on non-alternating primitive groups which contain substitutions of order \( p \) and degree \( qp \) \((q = 1, 2, 3, 4)\). *

7. The above proof is valid for \( p = 3 \) until we come to the discussion of the polynomial

\[
- n^{3/2} + 10 \cdot 2^{1/2} n + 48n^{1/2} + 27 \cdot 2^{1/2},
\]

which is negative when \( n > 292 \). If however we put \( \delta = 4/3 \), the polynomial is

\[
- 3n^{3/2} + 11 \cdot 2^{1/2} n + 77n^{1/2} + 58 \cdot 2^{1/2},
\]

and this is negative if \( n > 73 \). If \( n = 73, 70, 69, 60, 59 \), \([n - (2n)^{1/2} - 4]/3 = 18.9, 18.1, 17.8, 15.0, 14.7\), respectively. It can be shown that this limit holds for \( n \leq 73 \). It is known to be true for doubly transitive groups of class 6, 9, and 12†. For groups of class 15 or 18, we can use the following theorem:

A primitive group that contains a substitution of order \( p \) and degree \( pq \) \((p \text{ an odd prime, } p < q < 2p + 3)\), contains a transitive subgroup the degree of which is not greater than the larger of the two numbers \( pq + q^2 - q \) and \( 2q^2 - p^2 \). ‡

Thus our doubly transitive group of class 18 contains a transitive subgroup \( H \) of degree \( \leq 63 \), and we know from the proof of the theorem cited that the latter is generated by substitutions of order 3 and degree 18. We are concerned only with \( n = 70, 71, 72, 73 \). Then \( G \) is more than doubly transitive unless the transitive subgroup \( H \) is imprimitive. Since \( H \) is generated by similar substitutions of order 3 and degree 18, its systems of imprimitivity are of two, three, or six letters. If \( H \) has systems of imprimitivity of six letters each, its degree is not greater than 60, and \( G \) has a doubly transitive subgroup \( H' \) of degree \( \leq 67 \); and if the systems are of two or three letters only, the same is true.§ Thus \( G \) is more than doubly transitive and we should have (by §4) \( \mu > n/3 \).

If \( G \) is of class 15, \( H \) is of degree \( \leq 41 \), and the doubly transitive subgroup \( H' \) is of degree \( \leq 46 \).

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‡ Manning, these Transactions, vol. 12 (1911), p. 382, §12.
§ Manning, these Transactions, vol. 7 (1906), p. 499; Primitive Groups, part 1, 1921, p. 93.
Our result for \( p = 3 \) leaves much to be desired but is at any rate of the same form as Theorem IV:

**Theorem V.** Let \( \mu \) be the class and \( n (>292) \) the degree of a doubly transitive group in which one of the substitutions of degree \( \mu (>3) \) is of order 3; then

\[
\mu > \frac{n}{3} - \frac{(2n)^{1/2}}{3} - 1.
\]

If \( n \leq 292 \),

\[
\mu > \frac{n}{3} - \frac{(2n)^{1/2}}{3} - \frac{4}{3}.
\]

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