ON EXTENDED STIELTJES SERIES*

BY

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1. Let

\[ c_0 - c_1 z + c_2 z^2 - \cdots \]

be a power series with real coefficients such that the determinants

\[
A_n = \begin{vmatrix}
  c_0, & c_1, & \cdots, & c_{n-1} \\
  c_1, & c_2, & \cdots, & c_n \\
  \vdots & \vdots & \ddots & \vdots \\
  c_{n-1}, & c_n, & \cdots, & c_{2n-2}
\end{vmatrix}, \quad
B_n = \begin{vmatrix}
  c_1, & c_2, & \cdots, & c_n \\
  c_2, & c_3, & \cdots, & c_{n+1} \\
  \vdots & \vdots & \ddots & \vdots \\
  c_{n-1}, & c_n, & \cdots, & c_{2n-2}
\end{vmatrix},
\]

\( n = 1, 2, 3, \ldots \), are all positive. Then we define a \textit{kth extension} of (1) to be a series

\[
(-1)^k \frac{c_{-k}}{z^k} + (-1)^{k-1} \frac{c_{-k+1}}{z^{k-1}} + \cdots - \frac{c_{-1}}{z} + c_0 - c_1 z + c_2 z^2 - \cdots
\]

such that all the determinants formed from the \( A_n \) and \( B_n \) by replacing throughout \( c_i \) by \( c_{i-k} \), \( i = 0, 1, 2, 3, \ldots \), are positive.

In a previous paper† in these Transactions the present writer gave a necessary and sufficient condition for the existence of a first extension of (1), and gave examples to show that for any \( k \) there are series possessing a \( k \)th but not a \( (k+1) \)st extension, and others possessing extensions of infinite order. The condition there given is as follows. Let

\[
\frac{1}{a_1 + a_2 + a_3 + \cdots}
\]

be the Stieltjes‡ continued fraction corresponding to the Stieltjes series (1). Then if \( \sum a_{2i} = a_2 + a_4 + \cdots \) converges, and only then, a first extension exists and we may choose \( c_{-1} \geq \sum a_{2i} \) at pleasure. If \( c_{-p} \) exists then \( c_{-p-1} \) exists if and only if the series§ \( \sum a_{2i}^p \) in the continued fraction

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† H. S. Wall, On the Padé approximants associated with the continued fraction and series of Stieltjes, these Transactions, vol. 31 (1929), pp. 91–116, Chapter III.
§ Here and hereafter I write the superscripts without parentheses.
corresponding to the Stieltjes series

\[ c_{-p} - c_{-p+1}z + c_{-p+2}z^2 - \cdots \]

converges. The minimum value of \( c_{-p-1} \) is \( \sum a_{2i}^p, \ p = 0, 1, 2, \ldots, a_0^p = a_n \).

It will be convenient to make the following definition. The \( k \)th extension of (1) in which every \( c_{-p}, \ p = 1, 2, 3, \ldots, k \), has its minimum value is the \textit{minimal} \( k \)th extension of (1).

In the following article I shall give a necessary and sufficient condition for a minimal \( k \)th extension of (1), and then show that throughout a large class of Stieltjes series, including among others all those for which \( \sum a_i = a_1 + a_2 + a_3 + \cdots \) converges, minimal extensions of infinite order exist. Furthermore, if in this case we form the Stieltjes series

\[ \frac{c_{-1}}{z} - \frac{c_{-2}}{z^2} + \frac{c_{-3}}{z^3} - \cdots \]

with corresponding Stieltjes continued fraction

\[ \frac{1}{\alpha_1z} + \frac{1}{\alpha_2} + \frac{1}{\alpha_3z} + \cdots \]

then the latter converges over any finite region not containing a part of the negative half of the real axis, and its limit is the limit of the even convergents of (3). The series (6) converges without a circle of known radius \( R \) to this same limit.

The next paragraph contains preliminaries.

2. In the above mentioned article I gave formulas† which may be used to connect the numbers \( a_{-p} \) of (4) with the \( a_{-p-1} \) and also with the \( a_{-p+1} \). They run as follows:

\[ a_{-p} = a_{-p-1} \left( \sum_{i=0}^{i-1} d_{2i+1}^{-p-1} \right) \left( \sum_{i=0}^{i-1} d_{2i+1}^{-p-1} \right), \]

\[ a_{-p} = a_{-p-1} \left( \sum_{i=0}^{i-1} d_{2i+1}^{-p-1} \right)^2, \]

• This case was treated in my article, loc. cit., p. 112, Theorem 5. The extensions there obtained were not minimal extensions.

† Wall, loc. cit., formulas (49), (50), (65), (67).
\[ a_{2i}^{-p} = a_{2i-1}^{-p+1} \left( c_p - \sum_{i=1}^{i-1} a_{2i}^{-p+1} \right)^2, \]

\[ a_{2i+1}^{-p} = a_{2i}^{-p+1} / \left( c_p - \sum_{i=1}^{i-1} a_{2i}^{-p+1} \right) \cdot \left( c_p - \sum_{i=1}^{i} a_{2i+1}^{-p+1} \right). \]

If we solve (9) for \( a_{2i}^{-p} \), replace \( p \) by \( p-1 \) and equate the value of \( a_{2i}^{-p} \) so found to that given by (10) we will obtain, after simple reductions,

\[ c_p = \sum_{i=1}^{i-1} a_{2i}^{-p+1} + 1 / \sum_{i=1}^{i} a_{2i+1}^{-p}. \]

Stieltjes* showed that the sequences of even and odd convergents of the continued fraction

\[ \frac{1}{a_1z + a_2 + a_3z + \cdots} \]

always converge to limit functions \( F_1(z) \) and \( F_2(z) \) respectively, and that these limits are expressible as Stieltjes† integrals

\[ F_1(z) = \int_0^\infty \frac{d\phi_1(u)}{z + u}, \quad F_2(z) = \int_0^\infty \frac{d\phi_2(u)}{z + u}, \]

where \( \phi_1(u) \) and \( \phi_2(u) \) are non-decreasing real functions such that \( \phi_1(0) = \phi_2(0) = 0, \phi_1(\infty) = \phi_2(\infty) = 1/a_i \). The formal expansion of either integral into a power series \( P(1/z) \) gives the Stieltjes series corresponding to (13), namely

\[ \frac{c_0}{z} - \frac{c_1}{z^2} + \frac{c_2}{z^3} - \cdots, \]

and accordingly \( \phi_1(u) \) and \( \phi_2(u) \) are functions \( \phi(u) \) satisfying the equations

\[ \int_0^\infty u^i d\phi(u) = c_i \quad (i = 0, 1, 2, \cdots). \]

When \( \sum a_i \) diverges, \( F_1(z) = F_2(z) \), and all functions \( \phi(u) \) satisfying (16) are equivalent, i.e. equal at all points of continuity. On the other hand, when \( \sum a_i \) converges, \( F_1(z) \neq F_2(z) \) and there is an infinite number of non-equivalent functions \( \phi(u) \) satisfying (16). In this case the integrals (14) reduce to infinite series of the form

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* Stieltjes, loc. cit., §§47–48. Note that (13) becomes (3) if we replace \( z \) by \( 1/z \) and then drop the factor \( z \).

† Stieltjes, loc. cit., §38. Cf. also O. Perron, *Die Lehre von den Kettenbrüchen*, 1913, Chapter IX, for the definition and essential properties of Stieltjes integrals, and the chief results of Stieltjes.
\begin{align*}
F_1(z) &= \sum_{i=1}^{\infty} \frac{\mu_i}{z + \lambda_i}, \quad F_2(z) = \sum_{i=1}^{\infty} \frac{\nu_i}{z + \theta_i},
\end{align*}

in which \(\mu_i, \lambda_i, \nu_i, \theta_i\) are all real and positive; and (16) for \(\phi(u) = \phi_1(u)\) become

\begin{align*}
\sum_{i=1}^{\infty} \lambda_i \mu_i &= c_p \quad (p = 0, 1, 2, \cdots),
\end{align*}

with similar equations for \(\phi = \phi_2\).

3. These preliminary remarks having been made, I shall prove the following theorem.

**Theorem 1.** The Stieltjes series (1) admits a first extension when and only when the integral

\begin{align*}
\int_0^{\infty} \frac{d\phi_1(u)}{u}
\end{align*}

converges. When this condition is fulfilled we may choose \(c_{-1}\) equal to (19) or any greater number.

For the proof of this theorem the following lemmas will be needed.

**Lemma 1.** If the Stieltjes integrals

\begin{align*}
\int_0^{\infty} u^k \phi(u) \, du = c_k \quad (k = 0, 1, 2, \cdots), \quad \text{and} \quad \phi_1(u) = \int_0^{u} \frac{\phi(u)}{u^n} \, du,
\end{align*}

where \(u\) is real and positive and \(n\) is a positive integer, exist, then \(\phi_1(u)\), which is real, non-negative, and non-decreasing, satisfies the equations

\begin{align*}
\int_0^{\infty} u^{n+k} \phi_1(u) \, du = c_k \quad (k = 0, 1, 2, 3, \cdots).
\end{align*}

**Lemma 2.** If

\begin{align*}
\phi_1(u) = \int_0^{u} u^n \phi(u) \, du,
\end{align*}

where \(n\) is a positive or negative integer or 0, and \(\phi(u)\) satisfies the equations

\begin{align*}
\int_0^{\infty} u^{n+k} \phi(u) \, du = c_k \quad (k = 0, 1, 2, 3, \cdots),
\end{align*}

is convergent, then

\begin{align*}
\int_0^{\infty} u^{k} \phi_1(u) \, du = c_k \quad (k = 0, 1, 2, 3, \cdots).
\end{align*}
According to the definition of a Stieltjes integral, divide the interval \((0,b), b>0,\) in \(m\) sub-intervals by the points \((x_0 = 0 < x_1 < x_2 < \cdots < x_m = b)\), and let the norm of the division be \(\delta\). Then if \(x_{i-1} \leq \xi_i \leq x_i\),

\[
\int_0^b u^{n+k} d\phi_1(u) = \lim_{\delta \to 0} \sum_{i=1}^m \xi_i^{n+k} \left[ \int_0^{x_i} \frac{d\phi(u)}{u^n} - \int_0^{x_{i-1}} \frac{d\phi(u)}{u^n} \right]
\]

\[
= \lim_{\delta \to 0} \sum_{i=1}^m \xi_i^{n+k} \int_{x_{i-1}}^{x_i} \frac{d\phi(u)}{u^n}
\]

\[
= \lim_{\delta \to 0} \sum_{i=1}^m \xi_i^{n+k} \frac{1}{\xi_i^n} \left[ \phi(x_i) - \phi(x_{i-1}) \right],
\]

where \(\xi_i^*\) is a properly chosen point between \(x_{i-1}\) and \(x_i\). But since \(\phi_1(u)\) is a non-decreasing, non-negative, real function, and \(u^{n+k}\) is continuous in the interval \((0, b)\), the integral \(\int_0^b u^{n+k} d\phi_1(u)\) exists. Consequently we may take \(\xi_i = \xi_i^*\) and the above limit becomes

\[
\int_0^b u^{n+k} d\phi_1(u) = \int_0^b u^k d\phi(u) \quad (k = 0, 1, 2, \cdots).
\]

Now the integral on the right has a limit for \(b = \infty\). Hence the integral on the left has a limit for \(b = \infty\) and these limits are equal. This proves Lemma 1.

To prove the second lemma, we choose \(b\) and \(x_0, x_1, x_2, \cdots, x_m\) as above and form the sum

\[
\sum_{i=1}^m \xi_i^k \left[ \int_0^{x_i} u^n d\phi(u) - \int_0^{x_{i-1}} u^n d\phi(u) \right] = \sum_{i=1}^m \xi_i^k \int_{x_{i-1}}^{x_i} u^n d\phi(u),
\]

which is equal to

\[
\sum_{i=1}^m \xi_i^{n+k} \xi_i^n \left[ \phi(x_i) - \phi(x_{i-1}) \right],
\]

where \(\xi_i^*\) is a properly chosen point between \(x_{i-1}\) and \(x_i\). But since \(\xi_i\) is an arbitrary point in this interval we may take \(\xi_i = \xi_i^*\). Hence the last sum is equal to

\[
\sum_{i=1}^m \xi_i^{n+k} \left[ \phi(x_i) - \phi(x_{i-1}) \right]
\]

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* The theorem here used, which corresponds to the mean value theorem for Riemannian integrals and is proved similarly, is as follows. If \(f(x)\) is continuous for \(a \leq x \leq b\), and \(\phi(x)\) is non-decreasing and non-negative then there exists some point \(\xi, a \leq \xi \leq b\), such that

\[
\int_a^b f(x) d\phi(x) = f(\xi) [\phi(b) - \phi(a)].
\]

If \(f(x)\) is continuous only for \(a < x \leq b\), and \(\lim_{x \to b^+} f(x) = +\infty\), the same equation holds with \(a < \xi \leq b\).
which by hypothesis has the limit $c_k$ for $\delta = 0$, $b = \infty$. Consequently the left member of (20) has the limit $c_k$ for $\delta = 0$, $b = \infty$, and this limit is the integral $\int_0^x u^k d\phi(u)$. This proves Lemma 2.

We now prove that the condition of Theorem 1 is sufficient for a first extension of (1). Assume that (15) converges and set

$$\frac{\phi^{-1}(u)}{u} = \int_0^u \frac{d\phi_1(u)}{u}, \quad \phi^{-1}(0) = 0.$$

Since $\phi_1(u)$ is a solution of (16) we have, by Lemma 1 with $n = 1$,

$$\int_0^\infty u^{i+1} d\phi^{-1}(u) = c_i \quad (i = 0, 1, 2, \ldots).$$

Thus if

$$\int_0^\infty d\phi^{-1}(u) = \int_0^\infty \frac{d\phi_1(u)}{u} = c_0' ; \quad c_{i-1} = c_i' \quad (i = 1, 2, 3, \ldots),$$

the following equations hold:

$$\int_0^\infty u^i d\phi^{-1}(u) = c_i' \quad (i = 0, 1, 2, \ldots).$$

It then follows from the work of Stieltjes that $c_0'$, $c_0$, $c_1$, $\ldots$ are coefficients in a Stieltjes series. The sufficiency of the condition is thus proved.

To prove the necessity of the condition, assume that a first extension of (1) exists, and consider separately the cases $\sum a_i$ diverges, $\sum a_i$ converges, respectively.

(a) If $\sum a_i$ diverges, then $c_{i-1} = \sum a_{2i-1} + \delta$, where $\delta \geq 0$ ($\S 1$). If $\delta = 0$ it follows from (12) with $p = 1$, that $\sum a_{2i-1}$ must diverge; and if $\delta > 0$, we see from (10) with $p = 1$ that $\sum a_{2i-1}$ diverges. Hence in either case $\sum a_i^{-1}$ diverges, and consequently the continued fraction (4) with $p = 1$ converges to the limit

$$\frac{1}{z} \int_0^\infty \frac{d\phi^{-1}(u)}{z^{-1} + u}$$

and

$$\int_0^\infty u^{i+1} d\phi^{-1}(u) = c_i \quad (i = 0, 1, 2, \ldots).$$

Therefore by Lemma 2 with $n = 1$, $\phi(u) = \phi^{-1}(u)$, the function

$$\psi_1(u) = \int_0^u u d\phi^{-1}(u)$$
is a solution of (16), and since $\sum a_i$ diverges this function is equivalent to $\phi_1(u)$.

Let now $a, b$ be real and positive and points of continuity* of $\phi_1(u)$. Then if $b > a$ it follows that

$$\int_a^b \frac{d\phi_1(u)}{u} = \int_a^b \frac{d\phi_1(u)}{u} = \lim_{t \to 0} \sum_{i=0}^{\infty} \frac{1}{\xi_i'} \left[ \phi^{-1}(x_i) - \phi^{-1}(x_{i-1}) \right],$$

where $\xi_i'$ is a properly chosen point between $x_{i-1}$ and $x_i$, $i = 1, 2, \ldots, m$, $x_0 = a, x_m = b$. Thus if $b' > b$,

$$\int_a^{b'} \frac{d\phi_1(u)}{u} = \int_a^b d\phi^{-1}(u) + \int_b^{b'} \frac{d\phi_1(u)}{u}.$$ 

Now since $\int_a^b d\phi_1(u)$ converges, $\int_a^b \frac{d\phi_1(u)}{u}$ will surely converge if $b \geq 1$. Hence for any $\epsilon > 0$, there exists a number $B > 0$ such that if $b > B, b' > b$,

$$\left| \int_b^{b'} \frac{d\phi_1(u)}{u} \right| < \epsilon,$$

and consequently

$$\lim_{b' \to \infty} \int_a^{b'} \frac{d\phi_1(u)}{u} = \int_a^\infty d\phi^{-1}(u) = \phi^{-1}(\infty) - \phi^{-1}(a),$$

or

$$(21) \quad \int_a^\infty \frac{d\phi_1(u)}{u} = c_{-1} - \phi^{-1}(a).$$

If now $a$ approaches 0, over points of continuity of $\phi_1(u)$, the left member of (21) will have the limit†

$$(22) \quad \lim_{a \to 0^+} \int_a^\infty \frac{d\phi_1(u)}{u} = c_{-1} - \frac{1}{\sum a_{-1}}.$$ 

Let $a_1$ be another point of continuity of $\phi_1(u)$ and let $0 < a_1 < a' < a$. Then

$$\int_{a'}^{a_1} \frac{d\phi_1(u)}{u} = \int_{a_1}^\infty \frac{d\phi_1(u)}{u} - \int_{a_1}^{a'} \frac{d\phi_1(u)}{u},$$

or simply

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* Note that $\phi_1(u)$, being monotone, has points of continuity everywhere dense in the interval $(0, \infty)$.
† Cf. Stieltjes, loc. cit., §58.
Now
\[ \int_{a_i}^{a'} = \frac{1}{\xi} \left[ \phi_1(a') - \phi_1(a_i) \right], \quad a_i \leq \xi \leq a', \]
and since \( \phi_1(u) \) is continuous at \( a_i \) we may make
\[ \int_{a_i}^{a'} < \frac{\epsilon}{2}, \quad \text{if } \epsilon > 0, \quad a' - a_i < \delta. \]

Then by (22), (23), (24),
\[ \int_{a_i}^{a'} = c_{-1} - \frac{1}{\sum a_{2i-1}} + \epsilon, \quad \text{if } a_i < \eta, \quad a' - a_i < \delta. \]

Consequently
\[ \lim_{a' \to 0^+} \int_{a_i}^{a'} \frac{d\phi_1(u)}{u} = c_{-1} - \frac{1}{\sum a_{2i-1}}. \]

But by (12) with \( p = 1 \), \( c_{-1} = \sum a_{2i} + 1/\sum a_{2i-1} \), and therefore
\[ \lim_{a' \to 0^+} \int_{a_i}^{a'} \frac{d\phi_1(u)}{u} = \int_{0}^{\infty} \frac{d\phi_1(u)}{u} = \sum a_{2i} \leq c_{-1}. \]

This completes the proof of the theorem for the case that \( \sum a_i \) is divergent.

(b) When \( \sum a_i \) converges, \( \int_{0}^{\infty} \frac{d\phi_1(u)}{u} \) reduces to the first series (17) and therefore if \( 0 < a < \lambda_1 \), supposing \( \lambda_1 < \lambda_2 < \cdots \), this integral is equal to \( \int_{0}^{\infty} \frac{d\phi_1(u)}{u} \). It then follows by a known theorem* that this integral represents an analytic function for any \( z \) not contained in the interval \( (-\infty, -a) \). Consequently \( \int_{0}^{\infty} \frac{d\phi_1(u)}{u} \) converges. Furthermore,
\[ \int_{0}^{\infty} \frac{d\phi_1(u)}{u} = \sum_{i=1}^{\infty} \frac{\mu_i}{\lambda_i} = \lim_{n \to \infty} \frac{P_{2n}(0)}{Q_{2n}(0)} = \sum a_{2i} \leq c_{-1}. \]

inasmuch as \( P_{2n}(z)/Q_{2n}(z) \), the \( 2n \)th convergent of (13), has the value \( \sum_{i=1}^{\infty} a_{2i} \) when \( z = 0 \). This completes the proof of Theorem 1.

**Theorem 2.** The Stieltjes series (1) admits a minimal \( k \)th extension when and only when the integral
\[ \int_{0}^{\infty} \frac{d\phi_1(u)}{u^k} \]
converges.

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* Perron, loc. cit., p. 369.
Suppose first that (25) converges. Then \( f_0^x d\phi_1(u)/u^p, \ p < k, \) converges. For if \( 0 < x < x' < \delta < 1, \)
\[
\int_z^{x'} \frac{d\phi_1(u)}{u^p} < \int_z^{x'} \frac{d\phi_1(u)}{u^k} < \epsilon,
\]
if \( \delta \) is sufficiently small.

Taking \( p = 1 \) it follows from Theorem 1 that a first extension exists, and if
\[
c_{-1} = \int_0^\infty \frac{d\phi_1(u)}{u} = \sum a_{2i},
\]
\( \sum a_{-1} \) must diverge by (12). Consequently
\[
\phi^{-1}(u) = \int_0^u d\phi_1(u)/u.
\]
Then taking \( p = 2 \) we find that
\[
\int_0^\infty \frac{d\phi^{-1}(u)}{u} = \int_0^\infty \frac{d\phi_1(u)}{u^2}
\]
converges and again by Theorem 1, a second extension exists and we take \( c_{-2} \) equal to (26), etc. Continuing this argument one will finally arrive at a minimal \( \nu \)th extension of (1).

On the other hand suppose that (1) admits a minimal \( k \)th extension, \( k \geq 1. \) Then by Theorem 1,
\[
\int_0^\infty \frac{d\phi_1(u)}{u} = \sum a_{2i}
\]
converges, and \( c_{-1} \) has this value. Then by (12) \( \sum a_{-1} \) diverges and therefore
\[
\phi^{-1}(u) = \int_0^u d\phi_1(u)/u.
\]
If \( k \geq 2 \) it follows from Theorem 1 that
\[
\int_0^\infty \frac{d\phi^{-1}(u)}{u} = \int_0^\infty \frac{d\phi_1(u)}{u^2} = \sum a_{-1}^{-1}
\]
converges and is equal to \( c_{-2}. \) Hence
\[
\phi^{-2}(u) = \int_0^u \frac{d\phi_1(u)}{u^2},
\]
and if \( k \geq 3, \)
\[
\int_0^\infty \frac{d\phi_1(u)}{u^3} = \sum a_{-2}^{-2}
\]
converges, etc. This argument may evidently be continued until we arrive at the integral \( \int_0^a d\phi_1(u)/u^k \), whatever value \( k \) may have.

4. We next prove the theorem mentioned at the end of §1, namely

**Theorem 3.** (a) If there exist a number \( a > 0 \) such that

\[
\int_0^a d\phi_1(u) = \int_a^\infty d\phi_1(u),
\]

then (1) admits a minimal \( k \)th extension for all values of \( k \).

(b) The continued fraction (7) converges to the limit

\[
F_1(z) = \int_0^{1/a} - \frac{ud\phi_1(1/u)}{z + u}
\]

which is the limit of the even convergents of (3).

(c) The series (6) converges for all \( z \) for which \( |z| > 1/a \), and represents \( F_1(z) \) in that region.

(d) In case \( \sum a_i \) converges, \( a \) may be chosen arbitrarily in the open interval \((0, \lambda_1)\), and \( c_p = \sum_{i=1}^p \mu_i / \lambda_i \), \( p = 1, 2, 3, \ldots \).

For by (27)

\[
\int_0^a d\phi_1(u)/u^k = \int_a^\infty d\phi_1(u)/u^k,
\]

and this integral is readily seen to be convergent. Hence, by Theorem 2, (1) admits a minimal \( k \)th extension. Consider now the integral (28). We have

\[
F_1(z) = \int_0^{1/a} - \frac{ud\phi_1(1/u)}{z + u} = \int_0^{1/a} - u \left[ \frac{1}{z} - \frac{u}{z^2} + \frac{u^2}{z^3} - \cdots \right] d\phi_1(1/u).
\]

Since the series within the brackets converges uniformly over \((0, 1/a)\) if \( |z| > \delta > 1/a \), it may be integrated term by term. Therefore

\[
F_1(z) = \frac{-\int_0^{1/a} u d\phi_1(1/u)}{z} + \frac{\int_0^{1/a} u^2 d\phi_1(1/u)}{z^2} - \cdots
\]

\[
= -\frac{\int_0^{\infty} d\phi_1(u)/u}{z} - \frac{\int_0^{\infty} d\phi_1(u)/u^2}{z^2} + \cdots
\]

\[
= -\frac{c_{-1}}{z} + \frac{c_{-2}}{z^2} + \cdots,
\]
convergent if $|z| > 1/a$. It follows* that the continued fraction (7) converges and is equal to $F_1(z)$. When $\sum a_i$ converges the integrals $\int_0^\infty d\phi_t(u)/u^k$ evidently reduce to the sums $\sum_{i=1}^\infty \mu_i/\lambda_i^k$ by (17).

* Cf. Stieltjes, loc. cit., §10, in which it is shown that when a Stieltjes series converges, the numbers $1/(\alpha \alpha_{i+1})$, $i = 1, 2, 3, \ldots$, must increase to a finite limit, and consequently $\sum \alpha_i$ must diverge, thus implying the convergence of the continued fraction.

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