

NOTE ON INTEGRO- q -DIFFERENCE EQUATIONS*

BY

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Introduction. We concern ourselves here with a functional equation in which the unknown function $f(x)$ is subjected simultaneously to the q -difference operator and to an integral operator of Volterra type; it may be written

$$(A) \quad \sum_{i=0}^n a_i(x) f(q^{n-i}x) = b(x) + \int_0^x K(x, \xi) f(\xi) d\xi \quad (n \geq 1),$$

and n will be spoken of as the *order* of the equation. Certain facts about the solutions of the equation of first order when $|q| \leq 1$ are already known, thanks to the work of Picone, Popovici, Nalli, and Tamarkin.† It is not unnatural that these writers should have confined their studies to the real domain.

Our purpose in this paper is to investigate the existence of solutions both in the complex and real domains for the equation of order n , including the case of $|q| < 1$ and the essentially different case of $|q| > 1$. There being major differences in the situation when $b(x)$ is not identically zero and when it is, we reserve the symbol (A) for the case of $b(x) \not\equiv 0$ and employ the designation (B) for the equation when $b(x) \equiv 0$. The associated homogeneous q -difference equation, to which (B) reduces when $K(x, \xi)$ vanishes identically, we denote by (C).

COMPLEX DOMAIN

1. In this part of our work we assume that the functions $a_i(x)$ and $b(x)$ are analytic‡ at $x=0$ and that $K(x, \xi)$ is analytic at the place $(0, 0)$; thus we have

$$(1) \quad \begin{aligned} a_i(x) &= a_{i0} + a_{i1}x + a_{i2}x^2 + \cdots & (i = 0, 1, \cdots, n), \\ b(x) &= b_0 + b_1x + b_2x^2 + \cdots, \\ K(x, \xi) &= k_{00} + k_{10}x + k_{01}\xi + \cdots \end{aligned}$$

* Presented to the Society, March 29, 1929; received by the editor of the Bulletin in April, 1929, and transferred to the Transactions.

† Tamarkin, *On Volterra's integro-functional equation*, these Transactions, vol. 28 (1926), pp. 426-431; references to the other authors cited may be found here. This paper will be referred to hereafter as [T]. To Tamarkin's references may be added the recent note by Colombo, *Su una classe de equazioni integro-funzionali a limiti variabili*, Bollettino della Unione Matematica Italiana, vol. 7 (1920), pp. 80-85.

‡ Clearly it is no more general to allow the functions $a_i(x)$ and $b(x)$ to have poles at the origin.

for $|x| < R$, $|\xi| < R'$. The constant q is understood to be $\neq 0$ and of modulus $\neq 1$.

The characteristic equation* of (C) is then

$$(2) \quad \sum_{i=0}^n a_{i0} \rho^{n-i} = 0.$$

Corresponding to any root ρ_i which is finite, not zero, and not equal to a positive integral power of q times another root, there exists a solution of (C),

$$(3) \quad (x) = x^{r_i} g(x),$$

where

$$(4) \quad r_i = \frac{\log \rho_i}{\log q}$$

and $g(x)$ is analytic at the origin. It is natural therefore to seek solutions of (B) of the same nature.

Since r_i is not in general an integer, the function (3) is in general multiple-valued and so a single-valued function of position only on a Riemann surface with branch point at the origin and a suitable number, finite or infinite, of sheets. The integral of such a function from 0 to x , if r_i is not an integer, is thus independent of the path only if we specify the character of the path to the extent of giving the number of times it makes a circuit of the branch point. Such a specification having been made, *and we assume this to have been done*, the right-hand member of (B) is a single-valued function of position on this surface when $f(x)$ is the kind of function in question.

Under these conditions and if the real part of r is > -1 , we readily find by substitution that to the root ρ_i of (2) there corresponds a formal solution of (B),

$$(5) \quad s(x) = x^{r_i}(s_0 + s_1x + s_2x^2 + \dots).$$

Now r_i is uniquely defined only when determinations for $\arg \rho_i$ and $\arg q$ are chosen. For the equation (C) it is immaterial what determinations be selected; *for the equation (B), however, it is essential that determinations be selected which will make the real part of $r > -1$* . That this is always possible is easily verified; we shall assume such a choice to have been made.

If none of the numbers q^i ($i=0, 1, 2, \dots$) is a root of (2), the equation (A) is satisfied formally by a power series

$$(6) \quad s(x) = s_0 + s_1x + s_2x^2 + \dots.$$

* Cf., e.g., Adams, *On the linear ordinary q -difference equation*, *Annals of Mathematics*, (2), vol. 30 (1929), pp. 195-205. This will be referred to as [A].

2. The case of $|q| > 1$. When $|q|$ is > 1 the convergence of the formal solution (5) can be established immediately by the methods employed by the author in [A].* Only slight and natural modification of the details of that work is necessary. In constructing the convergence proof it is desirable at first to restrict the radii of convergence of $a_i(x)$ ($i=0, 1, \dots, n$) and the associated radii of convergence of $K(x, \xi)$ all to exceed 1; that this hypothesis is not essential may be shown as in [A]. We thus obtain the following

THEOREM 1. *When $|q|$ is > 1 and the equation (B) is satisfied formally by a series of the type (5), that series converges in the neighborhood of $x=0$ and represents there an analytic solution of (B). The region of analyticity of the solution may be extended by repeated use of the equation (B) itself out to the nearest zero of $a_0(x)$ or nearest singularity of the functions $a_i(x)$ ($i=0, 1, \dots, n$) and $K(x, \xi)$.*

As in [A] we may also establish

THEOREM 2. *When $|q|$ is > 1 and the equation (A) is satisfied by a series of the type (6), that series converges at and in the neighborhood of $x=0$ and represents there an analytic solution of (A). The region of analyticity of the solution may be extended by repeated use of the equation (A) itself out to the nearest zero of $a_0(x)$ or nearest singularity of the functions $a_i(x)$ ($i=0, 1, \dots, n$), $b(x)$, and $K(x, \xi)$.*

We shall not attempt to say what is the most general solution of either (B) or (A). It may be noted, however, (i) that *any linear combination, with constant coefficients, of particular solutions of (B) is a solution of (B)*; and (ii) that *the sum of a particular solution of (A) and any such linear combination of solutions of (B) is a solution of (A)*.

3. The case of $|q| < 1$. Although Picone and Tamarkin confined themselves to the real domain, some of their results carry over immediately to the complex domain if the given functions in the equations studied are assumed analytic in the neighborhood of the origin. The following theorem may be inferred from the work of [T] and a simple generalization thereof;† it also follows in part from Picone's results.

* Cf. the third and seventh paragraphs of §2.

† [T] treats the equation

$$u(x) = f(x) + s(x)u[\theta_1(x)] + \int_{-\infty}^x K(x, \xi)u[\theta_2(\xi)]d\xi,$$

where $u(x)$ stands for the function to be determined. The results of §1 of [T] follow in the same manner if the lower limit of integration is 0. If we make this change and set $\theta_1(x) = qx$ and $\theta_2(x) = x$, the equation clearly reduces to an equation of first order of our type (A).

THEOREM 3. *If for $|q| < 1$ the sum $\sum_{i=0}^{n-1} |a_{i0}/a_{n0}|$ is < 1 , the equation (B) has no solution bounded in the neighborhood of $x=0$ that does not vanish there identically and the equation (A) has a unique solution analytic in that neighborhood.*

Results for the equation (A) which are in certain respects more inclusive may be obtained by the method of successive approximations. We employ the following set of equations:

$$(7) \quad \begin{aligned} \sum_{i=0}^n a_i(x) f_1(q^{n-i}x) &= b(x), \\ \sum_{i=0}^n a_i(x) f_m(q^{n-i}x) &= b(x) + \int_0^x K(x, \xi) f_{m-1}(\xi) d\xi \quad (m = 2, 3, \dots). \end{aligned}$$

It is now assumed that the equation (C) has a complete set of solutions,

$$(8) \quad F_i(x) = x^{r_i}(s_{i0} + s_{i1}x + s_{i2}x^2 + \dots) \text{ for } |x| < R \quad (i = 1, 2, \dots, n)$$

with

$$D = \begin{vmatrix} F_1(qx) & F_2(qx) & \dots & F_n(qx) \\ F_1(q^2x) & F_2(q^2x) & \dots & F_n(q^2x) \\ \dots & \dots & \dots & \dots \\ F_1(q^nx) & F_2(q^nx) & \dots & F_n(q^nx) \end{vmatrix}$$

$$= x^{r_1+r_2+\dots+r_n}(d_0 + d_1x + d_2x^2 + \dots) \quad (d_0 \neq 0).$$

Such a set of solutions will exist if the roots of (2) are all finite and different from zero and if no one of them is equal to q^m times another, where m is a positive integer or zero; they may exist even if these conditions are not all fulfilled.* We assume further that $a_0(0)$ is $\neq 0$, in which case there is no loss of generality in taking $a_0(x) \equiv 1$.

The first equation of the set (7) will then be satisfied by

$$f_1(x) = \sum_{i=1}^n l_i(x) F_i(x)$$

The work of §1 of [T] may be extended at once, without modification of the method of proof, to cover the equation

$$u(x) = f(x) + \sum_{i=1}^n s_i(x) u[\theta_i(x)] + \int_{-x}^x K(x, \xi) u[\theta_{n+1}(\xi)] d\xi.$$

Tamarkin's σ is then replaced by the sum of the bounds of the absolute values of the functions $s_i(x)$ ($i=1, 2, \dots, n$) on the interval $-X \leq x \leq X$. From this generalization we may draw the conclusions indicated in Theorem 3 when the order of (A) is > 1 .

* Cf. [A].

if the functions $l_i(x) (i = 1, 2, \dots, n)$ are determined by the relations

$$l_i(x) = \Sigma (D_i/D),$$

where D_i represents the determinant obtained from D by replacing $F_j(q^jx) (j = 1, 2, \dots, n-1)$ by zero and $F_i(q^n x)$ by $b(x)$, and the operator Σ is defined by the equation*

$$\Sigma g(qx) - \Sigma g(x) = g(x).$$

But we have

$$(D_i/D) = x^{-ri}(\theta_0 + \theta_1x + \theta_2x^2 + \dots),$$

and if $|q^r|$ is > 1 , we may clearly employ for Σ the evaluation

$$\Sigma g(x) = -g(x) - g(qx) - g(q^2x) - \dots$$

and obtain

$$\Sigma (D_i/D) = x^{-ri} \phi_i(x),$$

where $\phi_i(x)$ is analytic for $|x| < R$. Thus if each root of (2) is in absolute value > 1 , we obtain a solution $f_1(x)$ which is analytic for $|x| < R$.

Defining $f_0(\xi)$ as zero, we have from (7)

$$\sum_{i=0}^n a_i(x) [f_m(q^{n-i}x) - f_{m-1}(q^{n-i}x)] = \int_0^x K(x, \xi) [f_{m-1}(\xi) - f_{m-2}(\xi)] d\xi \quad (m = 2, 3, \dots).$$

For $m = 2$ we have

$$f_{m-1}(\xi) - f_{m-2}(\xi) = f_1(\xi),$$

which is analytic for $|\xi| < R$. Now let R_1 denote any positive number less than the smaller of the two numbers R, R' ; we then have

$$|f_1(\xi)| \leq \sum_{i=1}^n |l_i(\xi)F_i(\xi)| < M \quad \text{for } |\xi| \leq R_1,$$

where M is a suitably chosen positive constant. Let us assume in general that $f_{m-1}(\xi) - f_{m-2}(\xi)$ is analytic and

$$|f_{m-1}(\xi) - f_{m-2}(\xi)| < M_1 \quad \text{for } |\xi| \leq R_1,$$

and let us examine $|f_m(x) - f_{m-1}(x)|$. As above we have

* Cf. Adams, *Note on the existence of analytic solutions of non-homogeneous linear q -difference equations, ordinary and partial*, *Annals of Mathematics*, (2), vol. 27 (1925), pp. 73-83.

$$f_m(x) - f_{m-1}(x) = \sum_{i=1}^n l'_i(x)F_i(x),$$

where

$$l'_i(x) = \Sigma (D'_i/D),$$

D'_i denoting the determinant obtained from D by replacing $F_i(q^i x)$ ($j = 1, 2, \dots, n-1$) by zero and $F_i(q^n x)$ by $\int_0^x K(x, \xi)[f_{m-1}(\xi) - f_{m-2}(\xi)]d\xi$. For $|x| \leq R_1$, $|\xi| \leq R_1$, and K a suitably chosen positive constant, we have

$$|K(x, \xi)| < K,$$

and thus

$$|D'_i/D| < KM_1R_1 |x^{-r_i}(\theta'_0 + \theta'_1 x + \theta'_2 x^2 + \dots)|,$$

a bound in which only M_1 is dependent upon m . Employing for Σ the same evaluation as before, we clearly obtain for $l'_i(x)$ a function analytic for $0 \neq |x| \leq R_1$ and satisfying the inequality

$$|l'_i(x)| < KM_1R_1 |x^{-r_i}\phi'_i(x)| \quad \text{for } |x| \leq R_1.$$

It follows that we have

$$|f_m(x) - f_{m-1}(x)| < KM_1R_1M' \quad \text{for } |x| \leq R_1,$$

where M' denotes a bound for $\sum_{i=1}^n |x^{-r_i}\phi'_i(x)F_i(x)|$.

Therefore the series

$$(9) \quad f_1(x) + [f_2(x) - f_1(x)] + [f_3(x) - f_2(x)] + \dots$$

is dominated by the series

$$M + MKR_1M' + M(KR_1M')^2 + \dots \quad \text{for } |x| \leq R_1.$$

Thus if R_1 be chosen so small that $KR_1M' < 1$, the series (9) converges uniformly for $|x| \leq R_1$ and

$$\lim_{m \rightarrow \infty} f_m(x) \text{ exists uniformly for } |x| \leq R_1.$$

Since $f_m(x)$ is analytic, the limit function, which we denote by $f(x)$, is also analytic for $|x| \leq R_1$; that $f(x)$ satisfies the equation (A) is readily verified.

We have therefore proved

THEOREM 4. *If for $|q| < 1$ and $a_{00} \neq 0$ the equation (C) has a full set of solutions (8) and each root of (2) is in absolute value > 1 , the equation (A) has a solution analytic at and in the neighborhood of $x = 0$.*

It is readily shown that if the hypothesis of Theorem 3,

$$\sum_{i=1}^n \left| \frac{a_{i0}}{a_{n0}} \right| < 1,$$

be satisfied, each root of (2) is > 1 ; obviously, however, the latter hypothesis may be fulfilled without the former.

REAL DOMAIN

4. Naturally in the above discussion the independent variable x may be restricted to the real domain without essential change in the results. It should be pointed out, however, that under less stringent hypotheses for the known functions certain conclusions are immediate.

The case of $|q| < 1$. Let the function $K(x, \xi)/a_n(x)$ be bounded in the region $0 \leq x \leq X$, $0 \leq \xi \leq X$ and if it has discontinuities let them be regularly distributed (cf. [T]); let the functions $b(x)/a_n(x)$ and $a_i(x)/a_n(x)$ ($i=0, 1, \dots, n-1$) be continuous on the interval $0 \leq x \leq X$, with

$$\left| \frac{b(x)}{a_n(x)} \right| \leq b \quad \text{and} \quad \sum_{i=0}^{n-1} \left| \frac{a_i(x)}{a_n(x)} \right| \leq \sigma,$$

b and σ being positive constants. Under these conditions we have, by §1 of [T] and the generalization indicated above,

THEOREM 5. *If σ is < 1 the equation (B) has no solution bounded on the interval $0 \leq x \leq X$ that does not vanish there identically, and the equation (A) has a unique solution continuous on that interval.*

Under the hypotheses (i) that $K(x, \xi)$ is continuous in the vicinity of $x=0, \xi=0$; (ii) that $b(x)$ is continuous in the vicinity of $x=0$ and, if discontinuous at the point, satisfies the condition

$$|b(x)| < B |x|^m,$$

B being a suitably chosen constant; and (iii) that the other known functions in the equation (A) are analytic at $x=0$, the method of §3 yields

THEOREM 6. *If for $a_{00} \neq 0$ the equation (C) has a full set of solutions (8) and each root of (2) is in absolute value $> |q|^m$, the equation (A) has a solution continuous in the neighborhood of $x=0$; if $b(x)$ is continuous at $x=0$, the solution is also.*

5. It may be remarked that all of the theorems of this paper are valid also for the equations obtained from (A) and (B) by replacing $f(\xi)$ by $f(q^j \xi)$, where j has any one of the values $1, 2, \dots, n-1$.