CONCERNING NON-DENSE PLANE CONTINUA

BY

J. H. ROBERTS

It has been shown by Menger\(^{\dagger}\) that a necessary and sufficient condition that a plane continuum \(M\) contains no domain is that for each point \(P\) of \(M\) and each positive number \(\varepsilon\) there exists a simple closed curve \(J\) of diameter less than \(\varepsilon\) which encloses \(P\), and such that \(M \cdot J\) is totally disconnected.

In the present paper it is shown that if \(M\) is a continuum which contains no domain then there exists a set \(G\) of simple closed curves filling the whole plane and indeed topologically equivalent to the set of all polygons, such that the common part of \(M\) and any curve of the set \(G\) is vacuous or totally disconnected. Additional results are obtained for the special case where \(M\) is a continuous curve.

I wish to acknowledge my indebtedness to Professor R. L. Moore, and to thank him. Credit is due him for the suggestion of most of the theorems of this paper, and for many helpful criticisms of the proofs.

PART I

In this paper I make frequent use of the notion of a **double ruling**.\(^{\ddagger}\) If, on the simple closed curve \(ABCD\), \(X_1, X_2, \ldots, X_n, Y_1, Y_2, \ldots, Y_m, X'_1, X'_{n-1}, \ldots, X'_1, Y'_1, Y'_{m-1}, \ldots, Y'_1\) are points in the order \(AX_1X_2\ldots X_nBY_1Y_2\ldots Y_mCX'_1X'_{n-1}\ldots X'_1DY'_1Y'_{m-1}\ldots Y'_1A\), and \(X_1X'_1, X_2X'_2, \ldots, X_nX'_n\), and \(Y_1Y'_1, Y_2Y'_2, \ldots, Y_mY'_m\) are arcs which, except for their end points, lie entirely within \(ABCD\), and finally, for every \(i, j\) (where \(1 \leq i \leq n, 1 \leq j \leq m\)) \(X_iX'_i\) has just one point in common with \(Y_jY'_j\) and no point in common with \(X_jX'_j\) (unless \(i = j\)), then these two sets of arcs are said to constitute a **double ruling** of the interior of \(ABCD\) (or merely a **double ruling** of \(ABCD\)). The arcs \(X_1X'_1, X_2X'_2, \ldots, X_nX'_n\) are said to be **parallel** to \(BC\) and to \(AD\), and the arcs \(Y_1Y'_1, Y_2Y'_2, \ldots, Y_mY'_m\) are said **...**

\(^{\dagger}\) Über die Dimensionalität von Punktmengen, Monatshfte für Mathematik und Physik, vol. 33 (1923), pp. 148–160. In his paper Sur les multiplicités Cantoriennes, Fundamenta Mathematicae, vol. 7 (1925), pp. 30–137, Urysohn obtained the slightly weaker result that for each point \(P\) of \(M\) and each positive number \(\varepsilon\) there exists a totally disconnected closed subset \(T\) of \(M\) such that \(M - T\) is the sum of two mutually separated sets \(M_1\) and \(M_2\), such that \(M_1\) contains \(P\) and is of diameter less than \(\varepsilon\).

to be parallel to $AB$ and $CD$. Each of these two sets of arcs is a single ruling of $ABCD$. Two sets of arcs $H$ and $K$ are said to constitute a complete double ruling of $ABCD$ if (1) every finite subset of the set of arcs $H+K$ is a single or double ruling of $ABCD$, (2) through each point within $ABCD$ there is an arc of $H$ and an arc of $K$, and (3) each point of $ABCD$ (except $A$, $B$, $C$ and $D$) is an end point of some arc of $H$ or of some arc of $K$. If $P$ is a point with both coordinates rational, then $P$ is said to be a rational point. By a rational line is meant a line whose equation is $x=r$ or $y=r$ where $r$ is some rational number.

As an obvious corollary of a theorem due to Fréchet* I state the following

**Theorem I.** If $M$ is a point set containing no domain then there exists a continuous transformation $T$ of the plane $S$ into itself such that if $P$ is any point of $M$ then $T(P)$ is not a rational point.

Schoenflies has proved the following theorem:"\[\text{If } T_1 \text{ is a continuous one-to-one correspondence between the points of two simple closed curves } J_1 \text{ and } J_2 \text{ such that } T_1(J_1)=J_2, \text{ then there exists a continuous one-to-one correspondence } T_2 \text{ between } J_1 \text{ plus its interior and } J_2 \text{ plus its interior such that } T_2(P)=T_1(P) \text{ for every point } P \text{ on } J_1.\]

Therefore for every simple closed curve $J$ there exists a continuous transformation $T$ which throws $J$ plus its interior into a square plus its interior. The truth of the following lemma is apparent in view of this fact and Theorem I.

**Lemma I.** Suppose $M$ is a point set containing no rational point, and $A$, $B$, $C$ and $D$ are rational points which are the vertices of a rectangle, and $AB$ is a horizontal interval. Suppose $\alpha$ is a double ruling of $ABCD$ with arcs $h_1, h_2, \ldots, h_n$ parallel to $BA$ and arcs $v_1, v_2, \ldots, v_m$ parallel to $BC$ such that (1) for each $i \leq n, j \leq m$ the end points of $h_i[v_j]$ have the same rational $y$-coordinate [$x$-coordinate], (2) for every $i$ and $j$ ($i \leq n, j \leq m$) the point common to $h_i$ and $v_j$ is rational, and (3) the rational points are everywhere dense on $h_i$ and on $v_j$ ($i \leq n, j \leq m$). Then there exists a continuous transformation $T$ throwing $ABCD$ plus its interior into itself and such that (1) $T$ reduces to the identity transformation on the rectangle $ABCD$, (2) $T(P)$ is not rational for any point $P$ of $M$, and (3) for every $i[j]$ ($i \leq n, j \leq m$) the arc $T(h_i)$ [$T(v_j)$] is an interval of some horizontal [vertical] rational line.

---


Theorem II. If $M$ is the sum of a countable number of closed point sets containing no domain and lying in a euclidean plane $S$, then there exists a continuous transformation $T$ of $S$ into itself such that if $L$ denotes any straight line of $S$ the point set $L \cdot T(M)$ is either totally disconnected or vacuous, and no point of $T(M)$ is rational.

I will assume that $M$ contains no rational point. In view of Theorem I this is no essential restriction. Let $A$, $B$, $C$, $D$ and $E$ be points with coördinates $(0,1)$, $(1,1)$, $(1,0)$, $(0,0)$ and $(1/2, 1/2)$ respectively, and let $R$ denote the interior of the square $ABCD$. Suppose $M = M_1 + M_2 + M_3 + \cdots$, where for every $n$ the set $M_n$ is closed. There exist five circles with centers $A$, $B$, $C$, $D$ and $E$ respectively, such that their interiors $I_A$, $I_B$, $I_C$, $I_D$ and $I_E$ are mutually exclusive, and no one of them contains a point of the closed set $M_1$.

There exists a double ruling $\alpha_1$ (see Fig. 1) of $R$ consisting of four arcs
$h_1, \cdots, h_4$ parallel to $AB$ and four arcs $v_1, \cdots, v_4$ parallel to $BC$ such that

(1) the end points of the arcs $h_1, \cdots, h_4, v_1, \cdots, v_4$ divide each side of the square $ABCD$ into five equal parts, (2) for each $i (i \leq 4)$ the arc $h_i$ contains three points, $x_i, y_i$ and $z_i$, in the order $x_iy_iz_i$ from $AD$ to $BC$ which belong to the domains $I_D, I_E$, and $I_C$ respectively, for $i = 1, 2$, and to the domains $I_A, I_B$, and $I_C$ respectively for $i = 3, 4$, (3) the points $x_1, x_2, x_3, x_4, z_1, z_2, z_3, z_4$ are the points $h_1 \cdot v_1, h_2 \cdot v_2, h_3 \cdot v_3, h_4 \cdot v_4, h_1 \cdot v_4, h_2 \cdot v_3, h_3 \cdot v_2, h_4 \cdot v_1$, respectively, (4) for each $i (i \leq 4)$ $v_i$ contains a point of $I_E$ between the arcs $h_2$ and $h_3$, and (5) for each $i$ the rational points are dense on $h_i$ and on $v_i$ and every point $h_i \cdot v_i (i \leq 4, j \leq 4)$ is rational. Since $M$ does not contain any rational point it is easily seen that the hypotheses of Lemma I are satisfied by $M$ and the double ruling $\alpha_1$. Let $T_1$ denote a transformation satisfying the conclusion of Lemma I. It is easily seen that every straight line which contains two points of the square $ABCD$ contains a point not belonging to $T_1(M_1)$. (See Fig. 2.)

Let $I_1$ be the set of all arcs $k$ such that $k = T_1(g)$, where $g$ is an arc of the
double ruling $\alpha_1$. Obviously $t_i$ is a double ruling of $R$, the arcs of which are horizontal and vertical rational intervals. There exists a positive number $\epsilon_i (\epsilon_i < 1/2)$ such that if $P$ and $Q$ are points of $\overline{R}$ and $\delta(P, Q) \geq 1/2$, then $\delta(T_i(P), T_i(Q)) > \epsilon_i$. Let $P_1, P_2, P_3, \cdots$ denote the rational points on the arcs $AB$ and $BC$. Let $\beta_i$ denote a double ruling of $R$ consisting of a finite set of rational horizontal and vertical intervals such that (1) $\beta_i$ contains an interval through the point $P_1$, and contains $t_i$ and (2) every component of $\overline{R}$ minus the sum of the arcs of the ruling $\beta_i$ is of diameter less than $\epsilon_i$.

Let $F_1$ denote the collection of all components of $R$ minus the sum of the arcs of the ruling $\beta_i$ and let $G$ denote any element of $F_1$. The domain $G$ is the interior of a rectangle $A_0B_0C_0D_0$, where $A_0B_0$ is an interval of a horizontal rational line, and all of the points $A_0, B_0, C_0$, and $D_0$ are rational. Let $E_G$ denote the center of the rectangle and let $I_{A_0}, I_{B_0}, I_{C_0}, I_{D_0}$, and $I_{E_G}$ be mutually exclusive circular domains which contain no point of the closed set $T_i(M_1 + M_2)$ and whose centers are the points $A_0, B_0, C_0, D_0$, and $E_G$. Clearly there exists a double ruling $\alpha_2$ of $G$ having with respect to $\overline{G}$, $T_i(M_1 + M_2), T_i(M)$ and the above mentioned circular domains the same properties which the double ruling $\alpha_1$ has with respect to $\overline{R}, M_1, M$ and the domains $I_A, I_B, I_C, I_D$ and $I_E$. Let $N_1$ be the sum of all arcs $k$ where $k$ belongs to some double ruling $\alpha_2$ for some domain $G$ of $F_1$. The point set $N_1$ is the sum of a finite number of arcs which constitute a double ruling $\gamma_1$ of $R$. Let $\alpha_2$ be the double ruling consisting of all the arcs of the rulings $\beta_i$ and $\gamma_1$. Since $T_i(M)$ contains no rational point the hypotheses of Lemma I are satisfied by $T_i(M)$ and the double ruling $\alpha_2$. There exists a transformation $T_2$ satisfying the conclusion of Lemma I and reducing to the identity transformation on the square $ABCD$ and on the arcs of the ruling $\beta_i$. Let $t_2$ denote the double ruling consisting of all arcs $k$ such that $k = T_2(g)$, where $g$ is an arc of the ruling $\alpha_2$.

Let $r$ denote an interval of length $2\epsilon_1$, which is a subset of $R$. Clearly if $r$ is a subset of some arc of the ruling $\beta_i$ then it contains a point not belonging to $T_2T_i(M_1 + M_2)$. If $r$ is not a subarc of any arc of $\beta_i$ then there exists a rectangle which is a subset of the sum of the arcs of $\beta_i$ and which contains exactly two points of $r$. With the assistance of Fig. 2 it can readily be seen that $r$ contains a point not belonging to the set $T_2T_i(M_1 + M_2)$. Hence if $T$ is any continuous transformation of $\overline{R}$ into itself such that $T(P) = P$ for every point $P$ on $ABCD$ or on some arc of the ruling $t_2$, then every interval in $\overline{R}$

\[ \text{License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use} \]
which is of length greater than or equal to $2\epsilon_1$ contains a point not belonging to the set $TT_2T_1(M_1+M_2)$. There exists a positive number $\epsilon_2$ ($\epsilon_2<1/3$) such that if $\delta(U, V)\geq 1/3$ and $U$ and $V$ belong to $\overline{R}$, then $\delta(T_2T_1(U), T_2T_1(V))>\epsilon_2$. Let $\beta_2$ denote a double ruling of $R$ consisting of rational horizontal and vertical intervals such that (1) some interval of $\beta_2$ contains $P_2$ (the second rational point on $ABCD$), (2) $\beta_2$ contains $k$, (3) every component of $R$ minus the sum of the arcs of the ruling $\beta_2$ is of diameter less than $\epsilon_2$.

Let $F_2$ denote the collection of all components of $R$ minus the sum of the arcs of the ruling $\beta_2$ and let $G$ denote any element of $F_2$. The domain $G$ is the interior of a rectangle composed of subsets of rational lines. There exists a double ruling $\alpha_3g$ of $G$ having with respect to $G$, $T_2T_1(M_1+M_2+M_3)$ and $T_2T_1(M)$ the properties which the double ruling $\alpha_1$ has with respect to $\overline{R}$, $M_1$ and $M$. Let $N_2$ be the sum of all arcs $k$, where $k$ belongs to some double ruling $\alpha_3g$ for some domain $G$ of $F_2$. The point set $N_2$ is the sum of a finite number of arcs which constitute a double ruling $\gamma_2$ of $R$. Let $\alpha_3$ be the double ruling consisting of all arcs of the rulings $\beta_2$ and $\gamma_2$. Since $T_2T_1(M)$ contains no rational point the hypotheses of Lemma I are satisfied by $T_2T_1(M)$ and the double ruling $\alpha_3$. Hence there exists a transformation $T_3$ satisfying the conclusion of Lemma I and reducing to the identity transformation on $ABCD$ and on the arcs of the double ruling $\beta_2$. Let $t_4$ denote the double ruling consisting of all arcs $k$ such that $k=T_3(g)$, where $g$ is an arc of the ruling $\alpha_3$. Each component of $\overline{R}$ minus the sum of the arcs of the ruling $\beta_2$ is of diameter less than $\epsilon_2$. Hence it follows that if $T$ is a continuous transformation of $\overline{R}$ into itself which reduces to the identity transformation on $ABCD$ and on every arc of the ruling $t_4$ then every interval which is a subset of $\overline{R}$ and is of diameter greater than or equal to $2\epsilon_2$ contains a point not belonging to the set $TT_2T_1(M_1+M_2+M_3)$.

Proceeding in this way one can see that there exists a countable infinity of double rulings $t_1, t_2, \ldots$, and a countable infinity of continuous transformations $T_1, T_2, T_3, \ldots$ of $\overline{R}$ into itself such that for every positive integer $n$ the following properties obtain: (1) $t_{n+1}$ contains $t_n$, (2) $t_n$ is composed of horizontal and vertical rational intervals and every component of $\overline{R}$ minus the sum of the arcs of the ruling $t_{n+1}$ is of diameter less than $1/n$, (3) if $U^n$ denotes the point $T_nT_{m-1} \cdots T_1(U)$ and $U$ and $V$ are points of $\overline{R}$ such that $\delta(U, V)\geq 1/n$, then for every integer $m(m>n)$ the points $U^m$ and $V^m$ are separated in $\overline{R}$ by some arc of the ruling $t_n$, and, for every $n$, $\delta(U^{n+1}, U^{n+k})<2/n$ ($k=1, 2, \ldots$), (4) the transformation $T_{n+1}$ reduces to the identity transformation on $ABCD$ and on all arcs of the ruling $t_n$, (5) some arc of $t_n$ contains $P_n$ (the $n$th rational point on $ABCD$), (6) no point belonging to two arcs of $t_n$ belongs to the set $T_nT_{n-1} \cdots T_1(M)$, and (7) if $T$ is any con-
tinuous transformation of \( \mathcal{R} \) into itself which reduces to the identity transformation on \( ABCD \) and on the arcs of the ruling \( t_{n+1} \) then every interval which is a subset of \( \mathcal{R} \) and is of length greater than or equal to \( 2/n \) contains a point not belonging to the set \( TT_{n+1}T_n \cdots T_1(M_1+M_2+\cdots+M_{n+1}) \).

From property (3) it clearly follows that for every point \( U \) of \( \mathcal{R} \) the sequence \( U, U^1, U^2, \cdots \) has a sequential limit point. Let \( T \) be the transformation which carries \( U \) into this sequential limit point. In particular, if for some integer \( n \) the point \( U^n \) belongs to some arc of \( t_n \) or to \( ABCD \), then \( U^n = U^{n+k} \) \( (k = 1, 2, \cdots) \) and \( T(U) \) is merely \( U^n \). I shall now show that \( T \) is a continuous one-to-one transformation of \( \mathcal{R} \) into itself which satisfies the conclusion of Theorem II with respect to \( \mathcal{R} \).

First, \( T \) is a one-to-one transformation of \( \mathcal{R} \) into itself which reduces to the identity transformation on \( ABCD \). Let \( U \) and \( V \) denote distinct points of \( \mathcal{R} \). We see at once from property (3) that there exist two mutually exclusive intervals \( g_1 \) and \( g_2 \) which belong to some double ruling \( t_n \) and each of which separates \( U^m \) from \( V^m \) in \( \mathcal{R} \) for every integer \( m (m > n) \). Then the sequences \( U^1, U^2, U^3, \cdots \) and \( V^1, V^2, V^3, \cdots \) have distinct sequential limit points. Hence for each point \( X \) of \( \mathcal{R} \) there is not more than one point \( U_x \) such that \( X = T(U_x) \). That for each \( X \) there is at least one point \( U_x \) follows from properties (2) and (3).

Second, the transformation \( T \) is continuous. Let \( U \) be a point not belonging to a point set \( W \). If \( U \) is not a limit point of \( W \) then there exists an integer \( m \) such that any point of \( T_nT_{n-1}\cdots T_1(W) \) is separated from \( T(U) \) by some arc of the ruling \( t_n \) for every \( n (n > m) \). Hence \( T(U) \) is not a limit point of \( T(W) \). If however \( U \) is a limit point of the set \( W \) then for every \( \epsilon \) there is a point \( V_\epsilon \) of \( W \) and an integer \( n_\epsilon \) such that \( \delta(U^m, V_\epsilon^m) < \epsilon \) for every integer \( m \) greater than \( n_\epsilon \). Hence \( T(U) \) is a limit point of \( T(W) \).

It follows from properties (2), (5) and (6) that if \( U \) is a point such that \( T(U) \) is rational, then \( U \) does not belong to \( M \).

Let \( L \) denote any interval which is a subset of \( \mathcal{R} \), and suppose that \( L \cdot T(M) \) contains a nondegenerate\( ^\dagger \) connected set. Since no continuum is the sum of a countable number of totally disconnected closed sets it follows that there exists an integer \( n \) such that \( L \) contains a nondegenerate connected subset of \( T(M_n) \). Let \( r \) denote some interval which is a subset of \( L \cdot T(M_n) \) and let \( \epsilon \) be its length. Let \( n_1 \) be an integer such that \( 2/n_1 < \epsilon \) and \( n_1 > n \). Now the transformation \( T \) reduces to the transformation \( T_{n+1}T_{n_1} \cdots T_1 \) on \( ABCD \) and on every arc of \( t_n \) (property 4). Hence if \( T^* \)

\( ^\dagger \) A point set containing but a single point is said to be degenerate.
denotes the transformation such that $T(U) = T^{*}T_{n_1+1}T_{n_1} \cdots T_1(U)$ then $T^{*}$ reduces to the identity transformation on $ABCD$ and on every arc of $t_{n_1}$. Hence by property (7) the interval $r$ contains a point not belonging to the set $T^{*}T_{n_1+1}T_{n_1} \cdots T_1(M_1+\cdots+M_{n_1+1})$. But this set is exactly $T(M_1+\cdots+M_{n_1+1})$. Hence we have a contradiction which means that no interval which is a subset of $R$ contains a nondegenerate connected subset of $T(M)$.

I have now shown that $T$ is a continuous one-to-one transformation of $R$ into itself such that if $L$ is any straight line then the point set $L \cdot T(M) \cdot R$ is either vacuous or totally disconnected, and no point which belongs to $T(M)$ is rational. Hence $T$ satisfies the conclusion of Theorem II with respect to the rectangle $ABCD$ plus its interior. The extension to the whole plane is obvious.

**Part II**

In his paper † *Grundzüge einer Theorie der Kurven*, Karl Menger proves that if $M$ is a bounded continuum containing no domain then a necessary and sufficient ‡ condition that $M$ be a continuous curve [regular curve.§] is that, for each positive number $\varepsilon$, $M$ is the sum of a finite number of continua all of diameter less than $\varepsilon$, such that the common part of any two of these continua is vacuous or totally disconnected [finite]. The necessity of this condition follows from Theorem III for the case where $M$ is a continuous curve, and from Theorem VI for the case where $M$ is a regular curve.

**Theorem III.** If $M$ is any continuous curve which contains no domain then there exists a continuous transformation $T$ of the plane $S$ into itself such that (1) no straight line contains a nondegenerate connected subset of $T(M)$ and no point of $T(M)$ is rational, and (2) if $R$ is the interior of a rectangle composed of intervals of lines with equations of the form $x = r$ and $y = r$, where $r$ is a rational number, then the point set $R \cdot T(M)$ is the sum of a finite number of connected sets and every point of $M$ on the boundary of $R$ is a limit point of $R \cdot T(M)$.

To help establish Theorem III I will first prove the following lemma.

---

‡ For the case of a continuous curve the sufficiency of the given condition was proved by W. Sierpinski, *Sur une condition pour qu'un continu soit une courbe jordaniennne*, Fundamenta Mathematicae, vol. 1 (1920), pp. 44–60.
§ A continuum $M$ is said to be a *regular curve* if for each point $P$ of $M$ and each positive number $\varepsilon$ there exists a domain of diameter less than $\varepsilon$ which contains $P$ and whose boundary contains only a finite number of points of $M$. See K. Menger, loc. cit.
Lemma II. If $M$ is a continuous curve containing no domain and $J$ is a simple closed curve $EKFGLHE$ such that the arcs $EKF$ and $GLH$ of $J$ contain no point of $M$, then there exists a simple continuous arc $KL$ lying, except for $K$ and $L$, within $J$, and such that (1) the arc $KL$ contains no nondegenerate connected subset of $M$, (2) every point of $M$ on the arc $KL$ is a limit point of $M$ from both sides of $KL$, and (3) the point set $M \setminus M \cdot KL$ is the sum of a finite number of connected sets.

Proof of Lemma II. Let $I$ denote the interior of $J$ and let $N$ be the point set $M \cdot I + EGH + FG$. If $N$ is not connected there exists an arc $KL$ which contains no point of $M$. I will therefore suppose that $N$ is connected. Then $N$ is a continuous curve. The set of junction points of $M$ is countable.‡ With the help of this fact and Theorem I it is easily seen that there exists an arc $KZL$ which except for $K$ and $L$ is within $J$ and which contains no junction point of $M$ and no nondegenerate connected subset of $M$. Let $S_E$ and $S_F$ denote the components of $N \setminus N \cdot KZL$ containing $E$ and $F$ respectively. Let $S_E^*$ denote the component of $N \setminus S_E$ which contains $E$. Let $S_F^*$ denote $N \setminus S_F$.

Since $N$ is a continuous curve, $S_E^*$ and $S_F^*$ are mutually separated sets. The common part of the two sets $S_E^*$ and $S_F^*$ is a subset of $N \cdot KL$ and is therefore totally disconnected. Now $S_E^*$ is connected, by definition. I will show that $S_F^*$ is also connected.

Suppose that $Q$ is a point of $S_F$ and $P$ is any point of $S_E^* \setminus S_F^* \cdot S_F$. There exists§ in $N$ a simple continuous arc $PQ$. Let $P_1$ be the first point on this arc from $P$ to $Q$ which belongs to the set $S_E^* + S_F^*$. The point $P_1$ belongs to $S_F$, for otherwise the subarc $PP_1$ of $PQ$ belongs to $S_E^*$, which is contrary to the supposition that $P$ is in $S_F^*$. If $P_1$ is not in $S_E^*$ then it is in $S_F$ and the arc $PP_1$ is connected to $Q$ by an arc in $S_F^*$.

Suppose then that $P_1$ belongs to both of the sets $S_E^*$ and $S_F$. Then $P_1$ is a

† If $M$ is a point set and $P$ is an interior point of an arc $AB$ then $P$ is said to be a limit point of $M$ from both sides of $AB$ if there exists a simple closed curve $J$ containing $AB$ such that, $J$ denoting the interior of $J$, $P$ is a limit point of $M \cdot I_J$ and of $M \cdot (S-I_J)$.

‡ Cf. R. L. Moore, Concerning triods in the plane and the junction points of plane continua, Proceedings of the National Academy of Sciences, vol. 14 (1928). If $P$ is a point of a continuous curve $N$, and $K$ is a domain containing $P$ such that $P$ is a cut point of the component of $N \cdot K$ which contains $P$, and furthermore there exist three arcs $PA_1$, $PA_2$, and $PA_3$ which lie in $N$ and have only the point $P$ in common, then $P$ is said to be a junction point of $N$. The continuum $PA_1 + PA_2 + PA_3$ is called a triod and the point $P$ is its emanation point.

limit point of each of the mutually separated connected sets $S_E^*$ and $S_F$. Therefore there exists an arc $R_1P_1$ in $S_E^* + P_1$, and an arc $T_1P_1$ in $S_F + P_1$, where $R_1$ and $T_1$ are on the arcs $EH$ and $FG$, respectively. (See Fig. 3.) Since the point $P_1$ is an emanation point of the triod of $N$ composed of the three arcs $R_1P_1$, $PP_1$, and $T_1P_1$, and is not a junction point of $M$, it is not a cut point of $N$. Therefore there exists an arc from $P$ to $Q$ which does not contain the point $P_1$. Let $P_2$ denote the first point of this arc in the order from $P$ to $Q$ which belongs to the set $S_E^* + S_F$. Then as before either $P$ and $Q$ can be connected by an arc in $S_F^*$ or $P_2$ belongs to both $S_E^*$ and $S_F$. Suppose the latter is true. Then $P_2$ does not belong to any of the arcs $PP_1$, $R_1P_1$ or $T_1P_1$. In $S_F^* + P_2$ there exists an arc $P_2T_2$, where $T_2$ is either on the arc $FG$ or on the arc $P_1T_1$, and no other point of the arc $P_2T_2$ belongs to either $FG$ or $P_1T_1$. In $S_E^* + P_2$ there exists an arc $P_2R_2$, where $R_2$ is on $EH$ or on $R_1P_1$, and no other point of $P_2R_2$ is on $EH$ or $R_1P_1$. Suppose for definiteness that $T_2$ and $R_2$ are on the arcs $FG$ and $EH$, respectively. Let $X$ and $Y$ be points

on $FG$ and $EH$ between $T_1$ and $T_2$, and $R_1$ and $R_2$, respectively. Then the simple closed curve $h$ ($h = T_1X_TP_3R_2YR_1P_1T_1$) encloses the segments $PP_1$ and $PP_2$. Let $R$ be a domain which contains the arc $PP_1$ but does not contain the point $P_2$. The component of $N\cdot R$ which contains $P_1$ contains subarcs of $PP_1$, $R_1P_1$, and $T_1P_1$. Since $P_1$ is not a junction point of $N$ it is not a cut point of this component. Hence there exists within $R\cdot (N-P_1)$ a simple continuous arc $PW$, where $W$ belongs to one of the segments $R_1P_1$ and $T_1P_1$. Let $P_3$ be the first point in the order from $P$ to $W$ which the arc $PW$ has in common with the set $S_{S^*}^*+S_{F^*}$. Assume that $P_3$ belongs to both $S_{S^*}^*$ and $S_{F^*}$. Clearly $P_3$ is within the simple closed curve $h$. There exists a segment of an arc within $h$ with $P_1$ and $P_2$ as end points which contains no point of $S_{S^*}^*+S_{F^*}$. This segment $P_3P_2$ divides the interior of $h$ into two connected domains, one of which contains $P_3$. Now there exist arcs from $P_3$ to $T_1$ and from $P_2$ to $R_1$ which lie in the sets $S_T^*+P_3$ and $S_{F^*}^*+P_3$, respectively. Since the simple closed curve $P_3R_1YR_2P_3P_1$ contains no point of $S_{F^*}^*+P_3$, and the simple closed curve $P_3T_1X_TP_2P_1$ contains no point of $S_{S^*}^*+P_3$ it is clear that we have reached a contradiction. Hence every point $P$ of $S_{S^*}^*S_{F^*}^*$ lies in the component of $S_{F^*}^*$ which contains $S_F$, which means that $S_{F^*}^*$ is connected.

We now have $N=S_{S^*}^*+S_{F^*}^*$ where $S_{S^*}^*$ and $S_{F^*}^*$ are connected and have no point in common, and $S_{S^*}^*+S_{F^*}^*$ is totally disconnected. Let $X_S$ and $X_F$ be the two continua obtained by adding to $S_{S^*}^*$ and $S_{F^*}^*$, respectively, all of their bounded complementary domains. Now no point is in a bounded complementary domain of both $S_{S^*}^*$ and $S_{F^*}^*$. Therefore the point set $X_S\cdot X_F$ is the same as the set $S_{S^*}^*-S_{F^*}^*$. Call this set $T$. Then $X_S-T$ is connected, and neither $X_S$ nor $X_F$ separates the plane. As a result of a theorem of R. L. Moore† it follows that there exists a simple closed curve $k$ enclosing $X_S-T$, containing $T$, and not containing or enclosing any point of $X_F-T$. Clearly $k$ contains an arc whose end points lie on the segments $EF$ and $GH$, respectively, but which otherwise lies within $J$. This arc can be modified so as to have $K$ and $L$ for end points and retain the property of separating $X_S-T$ and $X_F-T$ as above.‡ I will show that this arc satisfies the conclusion of Lemma II.

(1) Clearly the arc $KL$ contains no nondegenerate connected subset of $M$, for it contains no point of $M$ not belonging to the totally disconnected set $T$. (2) Every point of $M$ on the arc $KL$ is a limit point of each of the sets $S_{S^*}^*$ and $S_{F^*}^*$ and is therefore a limit point of $M$ from both sides of $KL$.

† Concerning the separation of point sets by curves, Proceedings of the National Academy of Sciences, vol. 11 (1925), pp. 469-476.
‡ This follows readily from the fact that the segments $EKF$ and $GLH$ do not contain any point of $N$. 

License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use
(3) Every component of \( M - M \cdot KL \) contains a point either on the arc \( EH \) or on the arc \( FG \) because of the fact that \( (M - T) + EH + FG \) is the sum of two or less connected sets. Since not infinitely many components of \( M - T \) have a point on \( EH \) or \( FG \) and also have a limit point on \( KL \) it follows that only a finite number of such components can have a limit point on \( KL \). But every component of \( M - T \) has a limit point on \( KL \). Hence the number of such components is finite. This completes the proof of Lemma II.

**Proof of Theorem III.** As in Theorem II there is no loss of generality in assuming that no point with both coordinates rational belongs to \( M \). Clearly the plane \( S \) can be regarded as the sum of a countable infinity of rectangles plus their interiors, the rectangles being subsets of *irrational* horizontal and vertical lines. Let \( R_1, R_2, R_3, \ldots \) denote the mutually exclusive interiors of such a set of rectangles.

If \( \epsilon \) is any positive number and if we have any finite double ruling \( \alpha \) of \( R_i \) (\( i \) being any positive integer) such that no arc of \( \alpha \) contains a nondegenerate connected subset of \( M \), then we can obtain a finite double ruling \( \beta \) which contains the arcs of \( \alpha \) and is such that every component of \( R_i \) minus the sum of the arcs of \( \beta \) is of diameter less than \( \epsilon \), and every arc of \( \beta \) which does not belong to \( \alpha \) has properties (1), (2), and (3) of Lemma II, and such that no point common to two such arcs belongs to \( M \). Now the arcs of the double ruling \( \beta_n \) defined in the proof of Theorem II were taken to be *rational* lines so that no nondegenerate connected subset of \( T_n T_{n-1} \cdots T_1 (M) \) would be a subset of an arc of \( \beta_n \). In view of Theorem II, however, it can be seen that transformations \( T_1, T_2, T_3, \ldots \) can now be chosen so that no nondegenerate connected subset of \( T_n T_{n-1} \cdots T_1 (M) \) is a subset of *any* straight line. Hence some of the arcs of the rulings \( \beta_n \) can be taken as *irrational* horizontal and vertical intervals. By obvious modification of the argument given in the proof of Theorem II it can be seen that there exists a countable infinity of double rulings \( t_1, t_2, t_3, \ldots \) of \( R_i \), and a countable infinity of continuous transformations \( T_1, T_2, T_3, \cdots \) of \( R_i \) into itself which, except for (6) and a modification of (2) to allow \( t_n \) to contain intervals of *irrational* horizontal and vertical lines, have properties (1)-(7) as stated in the proof of Theorem II, and the additional property that the double ruling \( t_n \) contains a double ruling \( r_n \) every arc of which has properties (1), (2), and (3) of Lemma II and \( P_n \) (the \( n \)th rational point on the boundary of \( R_i \)) is an end point of some arc of the ruling \( r_n \). Let \( W_i \) be the transformation corresponding to the transformation \( T \) as defined in the proof of Theorem II. Let \( T \) be the transformation of the plane \( S \) into itself which for every \( i \) reduces to \( W_i \) over \( R_i \).

Clearly then no straight line contains a nondegenerate connected sub-
set of $T(M)$, and no point of $T(M)$ is rational. The second conclusion of Theorem III follows readily from the fact that every rational horizontal and vertical interval with rational end points has properties (1), (2), and (3) of Lemma II with respect to the continuous curve $T(M)$. Hence Theorem III is established.

The following is an example of a regular curve which contains an uncountable set $H$ of points such that no arc containing a point of $H$ has property (3) of Lemma II. It therefore follows that Theorem III would be false if the stipulation that the boundary of $R$ is composed of intervals of rational horizontal and vertical lines were omitted, or if the word rational were replaced by the word irrational.

Example 1. (See Fig. 4.) Let $H$ denote a nondense perfect point set on the interval $0 \leq x \leq 1$, and let $K$ denote any acyclic† continuous curve such that $H$ is the set of end points† of $K$. Let $G_1, G_2, G_3, \ldots$ denote a

---

† A continuous curve is said to be acyclic if it contains no simple closed curve. See H. M. Gehman, *Concerning acyclic continuous curves*, these Transactions, vol. 29 (1927), pp. 553–568. An end point of an acyclic continuous curve is a point which is not an interior point of any arc of that curve. See R. L. Wilder, loc. cit., or H. M. Gehman, loc. cit.
contracting sequence† of mutually exclusive simple closed curves such that every point of \( H \) is enclosed by infinitely many curves of the sequence \( G_1, G_2, G_3, \ldots \), and for every \( n \) the curve \( G_n \) has just one point in common with \( K \), and this point does not belong to \( H \). Let \( M \) be the continuous curve \( K+G_1+G_2+G_3+\cdots \). Let \( P \) denote a point of \( K-H \). Then if \( Q \) is any point of \( H \) there is one and only one arc \( PQ \) from \( P \) to \( Q \) in \( M \). Let \( J \) denote a simple closed curve containing \( Q \) but not containing a non-degenerate connected subset of \( M \). Either (1) \( PQ-PQ\cdot J \) is the sum of infinitely many maximal connected subsets, in which case \( M-M\cdot J \) is not the sum of a finite number of connected sets, or (2) there exists a point \( X \) on the arc \( PQ \) distinct from \( Q \) such that the arc \( XQ \) has only the point \( Q \) on \( J \). Suppose for definiteness that \( X \) is within \( J \). There exist infinitely many simple closed curves of \( M \) enclosing \( Q \), having points within \( J \) and points without \( J \), and having only one point on the arc \( XQ \) of \( PQ \). Since no maximal connected subset of \( M-M\cdot J \) which lies in the exterior of \( J \) can contain a point of the arc \( XQ \) it follows that the number of components of \( M-M\cdot J \) is infinite.

In his paper‡ Concerning irreducible cuttings of continua, G. T. Whyburn raises the question as to whether or not every open§ subset of a plane continuous curve \( M \) contains an irreducible cutting of \( M \). This question is answered by the following theorem which is an application of Theorem III.

**Theorem IV.** Every open subset of a plane continuous curve \( M \) contains an irreducible cutting of \( M \).

Let \( G \) denote an open subset of a continuous curve \( M \). Clearly if \( G \) contains a domain then it contains a circle which is an irreducible cutting of \( M \). If \( G \) contains no domain let \( R \) denote the interior of a circle such that \( R \) contains a point of \( G \) but \( R \) does not contain a point of \( M-G \). Let \( M_1 \) denote any maximal connected subset of \( M\cdot R \) which contains more than one point. Then|| \( M_1 \) is a continuous curve which contains no domain. From Theorem III it readily follows that there exists a simple closed curve \( J \) which encloses some point of \( M_1 \) but does not contain or enclose any point

† If \( H \) is a sequence of point sets and for each positive number \( \varepsilon \) only a finite number of point sets of the set \( H \) are of diameter greater than \( \varepsilon \) then \( H \) is said to be a contracting sequence of point sets. See R. L. Moore, Concerning upper semi-continuous collections, Monatshefte für Mathematik und Physik, vol. 36 (1929), pp. 81–88.


§ An open subset of a continuum \( M \) is a set such that its complement with respect to \( M \) is closed. An irreducible cutting of a continuum \( M \) is a point set \( K \) of \( M \) such that \( M-K \) is not connected, but such that if \( G \) is any proper subset of \( K \) then \( M-G \) is connected. See G. T. Whyburn, ibid.

|| H. M. Gehman, Concerning the subsets of a plane continuous curve, Annals of Mathematics, vol. 27, p. 34.
of $M - M_1$, and such that the point set $M \cdot J$ is totally disconnected and separates $M_1$ into a finite number (greater than 1) of connected sets. Clearly then $M \cdot J$ has the same properties with respect to $M$. Then the set $M \cdot J$ contains a subset which is an irreducible cutting of $M$.

**Theorem V.** A necessary and sufficient condition that a continuum $M$ (not the whole plane) be a regular curve is that if $R$ is a connected domain containing two distinct points $A$ and $B$ not belonging to $M$ then in $R$ there exists a simple continuous arc from $A$ to $B$ which contains only a finite number of points of $M$.

The condition is necessary. Suppose $M$ is a regular curve and $R$ is a connected domain containing two points $A$ and $B$ not belonging to $M$. Let $AB$ denote any simple continuous arc from $A$ to $B$ which lies in $R$, and let $A'$ and $B'$ be points in the order $AA'B'B$ such that no point of $M$ is on the arc $AA'$ or the arc $BB'$ of $AB$. Enclosing each point of the arc $A'B'$ there exists a simple closed curve containing only a finite number of points of $M$ and not containing or enclosing $A$ or $B$ or any point not in the domain $R$. There exists a finite set of such curves whose interiors cover the arc $A'B'$. Call the curves of such a set $J_1, J_2, \ldots, J_n$. If $H$ denotes the continuous curve $AA' + BB' + J_1 + J_2 + \cdots + J_n$ then $H$ contains only a finite number of points of $M$. Let $AXB$ denote an arc from $A$ to $B$ which is a subset of $H$. Obviously this arc contains only a finite number of points of $M$.

The condition is sufficient. Clearly $M$ cannot contain a domain. Suppose $P$ is any point of $M$ and $\epsilon$ is any positive number. Let $J_1$ and $J_2$ denote two circles with $P$ as center and radii $\epsilon/2$ and $\epsilon/3$, respectively. Let $P_1$ and $P_2$ denote the extremities of a diameter of $J_1$. Let $D$ denote the domain bounded by $J_1 + J_2$, and let $A$ and $B$ denote two points not belonging to $M$ and lying in $D$ on different sides of the diameter $PP_1$. Let $D_1$ and $D_2$ be the connected domains $D - D \cdot PP_1$ and $D - D \cdot PP_2$, respectively. Let $AXB_1$ and $AXB_2$ denote arcs lying in $D_1$ and $D_2$, respectively, and containing only a finite number of points of $M$. The continuous curve $AXB_1 + AXB_2$ contains a simple closed curve which encloses $P$, contains only a finite number of points of $M$, and is of diameter less than $\epsilon$. Hence the point $P$ is a regular point and $M$ is a regular curve.

**Theorem VI.** If $M$ is a regular curve (not necessarily bounded) in a euclidean plane $S$, then there exists a continuous transformation $T$ of $S$ into itself such that (1) no straight line contains a nondegenerate connected subset of $T(M)$ and no point of $T(M)$ is rational, and (2) each rational horizontal or vertical line has in common with $T(M)$ a point set which has no limit point.

† See G. T. Whyburn, loc. cit.
The only essential difference between the proof of this theorem and that of Theorem III is that here we require that the arcs of the double ruling \( r_n \) shall have only a finite number of points in common with \( M \), instead of requiring that they have properties (1), (2) and (3) of Lemma II. In case \( M \) is a \textit{bounded} regular curve the second conclusion of Theorem VI is equivalent to the statement that no rational line contains more than a finite number of points of \( M \).

It follows that if \( R \) is the interior of a rectangle whose sides are intervals of rational horizontal and vertical lines, then the point set \( T(M) \cdot (R - R) \) contains only a finite number of points of \( M \). However the following example shows that it does not follow that the set \( R \cdot T(M) \) is the sum of a finite number of connected sets.

\textbf{Example 2.} (See Fig. 5.) For each pair of positive integers \( n \) and \( k (k \leq 2^n) \) let \( I_{kn} \) denote the interval with end points \( [(k-1)/2^n, 0] \) and \( [k/2^n, 0] \), and let \( C_{kn} \) denote the semicircle above the \( x \)-axis with \( I_{kn} \) as diameter. Let \( H \) denote the continuum which is the sum of the interval \( I(0 \leq x \leq 1) \) and all semicircles \( I_{kn}(k \leq 2^n, n = 1, 2, 3, \ldots) \). Let \( P_1, P_2, P_3, \ldots \) denote the points of the \( x \)-axis which are extremities of diameters of semicircles belonging to \( H \), and for each \( n \) let \( a_1n, a_2n, a_3n, \ldots \) denote a contracting sequence of arcs all of diameter less than \( 1/n \), such that for each \( m \) the arc \( a_{mn} \) contains the point \( P_n \) but no other point of \( H \) and no other point of the arc \( a_{kn} \) \((k \neq m)\).

Let \( M \) be the continuum \( H + \sum_{i=1}^{\infty} \sum_{n=1}^{\infty} a_{in} \). Then \( M \) is a regular curve. Now any arc which lies between the lines \( x = 0 \) and \( x = 1 \) and has a point above and a point below the \( x \)-axis either contains the point \( P \) for some integer \( n \) or it contains infinitely many points of \( M \). In the first case it cuts \( M \) into infinitely many components.

\textbf{Theorem VII.} If every point of a bounded regular curve \( M \) in a euclidean plane \( S \) is of finite order\(^\dagger\), then there exists a continuous transformation \( T \) of \( S \) into itself such that (1) no straight line contains a nondegenerate connected subset

\(^\dagger\) If \( P \) is a point of a regular curve \( M \) and there exists an integer \( n \) such that for every positive number \( \epsilon \) there is a domain of diameter less than \( \epsilon \) which contains \( P \) and whose boundary has not more than \( n \) points in common with \( M \) then the point \( P \) is said to be of finite order. Cf. K. Menger, loc. cit.
of $T(M)$ and no point of $T(M)$ is rational, (2) the rational lines contain only a finite number of points of $T(M)$, and (3) if $R$ is the interior of a rectangle composed of intervals of rational lines, then $R \cdot T(M)$ is the sum of a finite number of connected sets.

Let $T$ be a transformation satisfying the conclusion of Theorem VI. Since each point of $M$ on a rational line $L$ is of finite order it is not a limit point of infinitely many components of $T(M) - L$. In view of this, and the additional fact that the set of points of $M$ on any rational line is finite, it is clear that the transformation $T$ satisfies the conclusion of Theorem VII.

**Theorem VIII.** If $M$ is a bounded regular curve which contains only a finite number of simple closed curves then there exists a transformation $T$ satisfying the conclusion of Theorem VII.

To help prove Theorem VIII I will establish the following lemma.

**Lemma III.** If $M$ is a bounded regular curve which contains only a finite number of simple closed curves, and $R$ is a connected domain containing two points $A$ and $B$ not belonging to $M$, then there exists a simple continuous arc $AB$ which lies within $R$, contains only a finite number of points of $M$, and is such that $M - M \cdot AB$ is the sum of a finite number of connected sets.

**Proof of Lemma III.** Let $H$ denote the set of junction points of $M$. Let $J$ be a simple closed curve enclosing $A$ and $B$ and lying in $R$, and let $g$ denote a simple continuous arc from $A$ to $B$ which lies within $J$, contains no non-degenerate connected subset of $M$, and no point of $H$. But the outer boundary of every bounded complementary domain of a continuous curve is† a simple closed curve and‡ no two bounded complementary domains of a continuous curve have the same outer boundary. Hence since $M$ contains only a finite number of simple closed curves it follows that only a finite number of complementary domains of $M + J$ have boundary points on the arc $g$. If one of these domains contains both $A$ and $B$ the lemma is obviously established. If not let $P_1$ be the last point of $g$ in the order from $A$ to $B$ which is on the boundary of that complementary domain of $M + J$ which contains $A$. Then§ there exists an arc $AP_1$ which lies wholly in this domain except for the point $P_1$. Since $P_1$ is a limit point of the points of $S - M$ on the

Theorem IX. If $A$ and $B$ are distinct points of a continuous curve $M$ then $M$ contains a simple continuous arc from $A$ to $B$ every subarc of which contains a subarc which either lies on the boundary of some complementary domain of $M$ or lies in some domain which belongs to $M$.

(1) Suppose $M$ contains no domain. Let $AXB$ denote a simple continuous arc from $A$ to $B$ such that the common part of $AXB$ and $M$ is totally disconnected. Let $T$ denote the set $M \cdot AXB$. Let $D_1, D_2, \ldots$ denote the complementary domains of $M$ which contain limit points on the arc $AXB$ and for each $i$ let $J_i$ denote the boundary of $D_i$. Let $K$ be the point set $T + J_1 + J_2 + J_3 + \cdots$. Since $J_1, J_2, J_3, \ldots$ is a contracting sequence of continuous curves all containing points on the arc $AXB$ it is readily seen that the set $K$ is closed. If $P$ is an interior point of the arc $AXB$ which does not belong to $K$ then there exists a connected subset of $K$ containing the last point of $K$ which precedes $P$ on the arc $AXB$ and the first point of $K$ which follows $P$ on this arc. Therefore $K$ is connected. With the use of the fact that the boundary of every complementary domain of a continuous curve is itself a continuous curve it readily follows that $K$ is connected im kleinen. Hence $K$ is a continuous curve. Let $AB$ denote any arc which lies in $K$ and let $EF$ denote any subarc of $AB$. Since $T$ is totally disconnected the arc $EF$ contains a subarc $E'F'$ which contains no point of $T$. The arc $E'F'$ is a subset of $J_1 + J_2 + J_3 + \cdots$, and hence is equal to $J_1 \cdot E'F' + J_2 \cdot E'F' + J_3 \cdot E'F' + \cdots$. But the sum of a countable number of totally dis-


‡ See third footnote on p. 14.
connected closed point sets is not connected. Hence there exists at least one integer \( i \) such that the set \( J_i'E'F' \) contains an arc. Thus every subarc of \( AB \) contains an arc belonging to the boundary of some complementary domain of \( M \).

(2) If \( A \) and \( B \) are distinct points of a continuous curve \( M \) which contains a domain let \( M_1 \) be a continuous curve containing \( A \) and \( B \) which is obtained by taking from \( M \) the interiors \( I_1, I_2, I_3, \ldots \) of a contracting sequence of circles such that (1) for every domain \( D \) which is a subset of \( M \) there is an integer \( n \) such that \( I_n \) contains at least one point of \( D \), (2) for every \( n \), \( I_n \) is a subset of some domain belonging to \( M \) and (3) \( I_kI_n = 0(k \neq n) \). Let \( AB \) denote an arc satisfying the conclusion of the theorem with respect to \( M_1 \). Since the boundary of a complementary domain of \( M_1 \) which is not a complementary domain of \( M \) belongs in a domain lying in \( M \) it is obvious that \( AB \) satisfies the conclusion of the theorem with respect to \( M \).

**Theorem X.** If \( M \) is a bounded continuous curve which contains no domain then there exists a continuous transformation \( T \) of the plane \( S \) into itself such that (1) if \( AB \) is an arc such that \( T(AB) \) is a subset of a rational line then \( AB \cdot M = c_1 + c_2 + \cdots + c_n \) where for each \( i (i \leq n) \) \( c_i \) is an arc or a point and if \( c_i \) is an arc then every subarc of \( c_i \) contains a subarc lying on the boundary of some complementary domain of \( M \) and (2) if \( AB \) is an arc such that \( T(AB) \) is a subset of an irrational horizontal or vertical line then \( AB \cdot M \) is vacuous or totally disconnected.

To help establish Theorem X, I will prove several lemmas. To avoid repetition I will say that an arc \( AB \) has property \( c \) with respect to \( M \), or merely that it has property \( c \), if the common part of \( M \) and \( AB \) is the sum of a finite number of connected sets such that each of these sets which is an arc is of diameter less than \( \epsilon \) and has the property that every subarc of it contains a subarc lying on the boundary of some complementary domain of \( M \).

**Lemma IV.** If \( I \) is the interior of a simple closed curve and \( M \) is a bounded continuous curve containing no domain and \( A \) and \( B \) are distinct points lying in \( I \) and \( \epsilon \) is any positive number, then there exists a simple continuous arc from \( A \) to \( B \) which is a subset of \( I \) and which has property \( c \).

With the help of Theorem II it can readily be seen that there exists a simple closed curve \( J_1 \) lying in \( I \), enclosing \( A \) and \( B \), and such that (1) \( J_1 \cdot M \) is totally disconnected, and (2) if \( I_1 \) denotes the interior of \( J_1 \) then no component of \( M \cdot I_1 \) is of diameter greater than or equal to \( \epsilon \). Let \( AXB \) denote any simple continuous arc from \( A \) to \( B \) which lies in \( I_1 \) and let \( s_1, s_2, \ldots, s_n \) denote the components of \( M \cdot I \), which have points on \( AXB \). For each \( i \)
(i ≤ n) s_i is† a continuous curve. In view of this fact and Theorem IX it follows that there exists an arc \( AB \) in \( I_i \) such that \( M \cdot AB \) is a subset of \( s_1 + s_2 + \cdots + s_n \) and is the sum of \( n \) or less connected sets such that each of these sets which is an arc has the properties of the arc of Theorem IX with respect to that one of the continuous curves \( s_1, s_2, \ldots, s_n \) to which it belongs. Let \( EF \) denote an arc belonging to \( M \cdot AB \). Since \( EF \) contains a subarc lying wholly within \( J \) it can easily be shown that \( EF \) has the properties stated in Theorem IX with respect to \( M \). Since in addition \( EF \) is of diameter less than \( \epsilon \) the lemma is proved.

**Lemma V.** If \( J \) is a simple closed curve and \( KL \) is a simple continuous arc which lies within \( J \) except that \( K \) and \( L \) are on \( J \) and \( KL \) is on the boundary of a complementary domain \( D \) of \( M \), then there exists a simple continuous arc \( AB \) which lies within \( J \) such that (1) the common part of \( AB \) and \( KL \) is a single point, (2) \( KL \) separates \( A \) from \( B \) within \( J \), and (3) \( AB \cdot M \) is either an arc or a point, and if it is an arc it has property \( c_\alpha \).

Let \( C \) denote the interior of a circle which lies within \( J \) and encloses a point of \( KL \). There exists a point \( A \) in \( C \cdot D \) and a subarc \( E'F' \) of \( EF \) such that for every point \( P \) of \( E'F' \) there exists an arc \( AP \) which lies in \( C \cdot D \) except for the point \( P \). Let \( O \) denote some interior point of \( E'F' \) and let \( C_1 \) denote the interior of a circle \( J_1 \) which lies in \( C \) such that \( C_1 \) contains \( O \) but contains no point of \( KL - E'F' \). Let \( A' \) and \( B' \) denote points in \( C_1 \) lying respectively on the \( A \) side and the non \( A \) side of \( KL \) and let \( B'A' \) denote an arc having property \( c_\alpha \) and lying in \( C_1 \). Let \( Q \) denote the first point of \( B'A' \) on \( KL \) in the order from \( B' \) to \( A' \). Let \( AQ \) denote an arc lying in \( C \cdot D \) except for the point \( Q \). Let \( QB \) denote a subarc of \( QB' \) such that \( QBM \) is connected. The sum of the arcs \( AQ \) and \( QB \) gives an arc \( AB \) which satisfies the conclusion of the lemma.

**Lemma VI.** If \( M \) is a continuous curve containing no domain and lying within a simple closed curve \( J \) whose interior is \( R \), \( \epsilon_1 \) and \( \epsilon_2 \) are any positive numbers and \( \alpha \) is a double ruling of \( R \) such that every arc of \( \alpha \) has property \( c_\epsilon_1 \), then there exists a double ruling \( \beta \) of \( R \) such that every arc of \( \alpha \) is also an arc of \( \beta \), every arc of \( \beta \) which is not an arc of \( \alpha \) has property \( c_\epsilon_2 \), and every component of \( \overline{R} \) minus the sum of the arcs of the ruling \( \beta \) is of diameter less than \( \epsilon_2 \).

With the help of a theorem of Schoenflies‡ it is easily seen that there exists a continuous transformation \( T_1 \) of the plane into itself which throws \( J \) into a square \( ABCD \) and the arcs of \( \alpha \) into horizontal and vertical inter-

---

† See H. M. Gehman, loc. cit.
‡ Loc. cit.
vals. Let \( \alpha_i \) be the double ruling of \( T(R) \) which is composed of all arcs \( T(g) \) where \( g \) is an arc of \( \alpha \). Since \( T_1 \) is continuous it obviously follows that there exist two finite sets of rectangles \( h_1, h_2, \ldots, h_n, v_1, v_2, \ldots, v_n \) such that (1) for each \( i \) \((i \leq n)\), \( h_i[v_i] \) has no point in common with any arc of \( \alpha \) which is parallel to \( AB \) [BC] and no point in common with \( h_i[v_i] \) \((j \leq n, j \neq i)\) but contains at least one point of \( AD \) and one point of \( BC \) [AB and CD], and (2) if \( \alpha_2 \) is any double ruling of \( R \) which contains all of the arcs of \( \alpha \) and in addition contains arcs \( a_i \) and \( b_i \) such that \( T(a_i) \) lies in \( h_i \) plus its interior and \( T(b_i) \) lies in \( v_i \) plus its interior \((i = 1, 2, \ldots, n)\) then every component of \( \bar{R} \) minus the sum of the arcs of \( \alpha_2 \) is of diameter less than \( \epsilon_2 \). It is easily shown with the help of Lemmas IV and V that a particular such ruling \( \beta \) can be obtained such that the arcs of \( \beta \) which do not belong to \( \alpha \) have property \( c_{e_1} \).

**Lemma VII.** Suppose \( M \) is a continuous curve which contains no domain and lies in the interior \( R \) of a square \( ABCD \). Let \( \alpha \) be any double ruling of \( R \) and let \( P \) be a point of \( R \) not belonging to any arc of \( \alpha \). Then there exists an integer \( k \) such that if \( n > k \) and \( \beta \) is any double ruling of \( R \) such that (1) \( \beta \) contains \( \alpha \), (2) no arc of \( \beta \) contains \( P \), (3) every component of \( \bar{R} \) minus the arcs of \( \beta \) is of diameter less than \( 1/n \), and (4) every arc of \( \beta \) which does not belong to \( \alpha \) has property \( c_{1/n} \), then if \( E \) denotes the component, containing \( P \), of \( \bar{R} \) minus the arcs of \( \beta \) which are parallel to \( AB \) [BC], there exists in \( E \) an arc \( a_k \) with property \( c_{(n+1)/n} \) which together with the arcs of \( \beta \) forms a double ruling of \( R \) and such that no component of \( M \cdot (E - a_k) \) contains points in more than two components of \( \bar{R} \) minus the arcs of \( \alpha \). (See Fig. 6.)

Let \( \epsilon_1 \) be a positive number such that if \( a_1 \) and \( b_1 \) denote any arcs of \( \alpha \) which have no point in common then the distance from any point of \( a_1 \) to any point of \( b_1 \) is greater than \( \epsilon_1 \). Let \( \epsilon_2 \) be a positive number such that a circle with \( P \) as center and \( \epsilon_2 \) as radius neither contains nor encloses any point of any arc of \( \alpha \) or of \( ABCD \). Let \( k \) be any integer greater than both \( 1/\epsilon_1 \) and \( 1/\epsilon_2 \). Suppose \( \beta \) is a double ruling of \( R \) with properties (1), (2), (3), and (4) as given above. Let \( E \) denote the component, containing \( P \), of \( \bar{R} \) minus the arcs of \( \beta \) which are parallel to \( AB \) (for example). In view of property (3) and the additional fact that \( 1/n < \epsilon_2 \) it is obvious that \( a_1 \) and \( a_2 \), the arcs of \( \beta \) on the boundary of \( E \), do not belong to \( \alpha \). From (4) it follows that both \( a_1 \) and \( a_2 \) have property \( c_{1/n} \). Since \( 1/n < \epsilon_1 \) it follows that between each two distinct arcs of \( \beta \) parallel to \( BC \) the arc \( a_i \) \((i = 1, 2)\) contains a point not belonging to \( M \). Hence if \( m + 1 \) denotes the number of arcs of \( \beta \) which are parallel to \( BC \) it is easily seen that there exist \( 2m \) circles lying in \( R \) with interiors \( C_{11}, C_{12}, \ldots, C_{1m}, C_{21}, C_{22}, \ldots, C_{2m} \) such that (1) \( C_{1h}(i = 1, 2; k \leq m) \)
contains no point of \( M \) and no point of any arc of \( \alpha \), and (2) for each two adjacent arcs \( b_1 \) and \( b_2 \) of \( \alpha \) which are parallel to \( BC \) there exist integers \( i \) and \( j \) such that \( C_{1i} \) and \( C_{2j} \) contain points of \( a_1 \) and \( a_2 \), respectively, which lie between \( b_1 \) and \( b_2 \). With the help of Lemmas IV and V it is seen that there exists in \( E \) an arc \( a_{i\ell} \) with property \( e_{1/(m+1)} \) which together with the arcs of \( \beta \) forms a double ruling of \( R \), and in addition contains a point in \( C_{ik}(i=1, 2; k=1, 2, \ldots, m) \). Obviously no component of \( M \cdot (E - a_{i\ell}) \) contains points in more than two components of \( \mathcal{R} \) minus the arcs of \( \alpha \).

**Proof of Theorem X.** Suppose \( M \) lies in the interior \( R \) of a square \( ABCD \). With the help of Lemmas IV, V, and VI one can readily see that there exists a double ruling \( \beta_1 \) of \( ABCD \) such that (1) every component of \( \mathcal{R} \) minus the arcs of \( \beta_1 \) is of diameter less than 1, (2) every arc of \( \beta_1 \) has property \( c_1 \), and (3) between each two adjacent arcs of \( \beta_1 \) parallel to \( BC \) there exists on each arc of \( \beta_1 \) parallel to \( AB \) a point not belonging to \( M \). Let \( a_1 \) and \( a_2 \) denote any two adjacent arcs of \( \beta_1 \) parallel to \( AB \) (or \( a_1 \) or \( a_2 \) may be \( AB \) or \( CD \)) and let \( E \) denote the set of all points of \( \mathcal{R} \) which lie between \( a_1 \) and \( a_2 \). If \( m+1 \) denotes the number of arcs of \( \beta_1 \) which are parallel to \( BC \) it is clear that

![Figure 6](image-url)
there exist $2m$ circles lying in $R$ with interiors $C_{11}$, $C_{12}$, $\cdots$, $C_{1m}$, $C_{21}$, $C_{22}$, $\cdots$, $C_{2m}$ such that (1) $C_{1k}(i=1, 2; k \leq m)$ contains no point of $M$ and no point of any arc of $\beta_1$ which is parallel to $BC$, and (2) for each two adjacent arcs $b_1$ and $b_2$ of $\beta_1$ parallel to $BC$ there exist integers $j$ and $k$ such that $C_{1j}$ and $C_{2k}$ contain points of $a_1$ and $a_2$, respectively, which lie between $b_1$ and $b_2$.

With the help of Lemmas IV and V it is seen that there exists in $E$ an arc $a_3^B$ which together with the arcs of $\beta_1$ forms a double ruling of $R$ and such that (1) $a_3^B$ contains points in the set $C_{ik}(i=1, 2; k=1, 2, \cdots, m)$, and (2) $a_3^B$ has property $c_{1/2}$. Obviously no component of $M \cdot (E-a_3^B)$ contains points in more than two components of $\overline{R}$ minus the arcs of the ruling $\beta_1$. Let $\alpha_1$ be the double ruling obtained by adding to $\beta_1$ the arc $a_3^B$ for every component $E$ of $\overline{R}$ minus the arcs of $\beta_1$ which are parallel to $AB$. Let $\beta_2$ be a double ruling which contains $\alpha_1$ and is such that (1) every component of $\overline{R}$ minus the arcs of $\beta_2$ is of diameter less than $1/2$ and (2) the arcs of $\beta_2$ which do not belong to $\beta_1$ have property $c_{1/2}$.

Now let $E$ denote the set of all points of $\overline{R}$ which lie between two adjacent arcs of $\beta_2$ which are parallel to $BC$. If there exists in $E$ an arc $a_3^*$ which together with the arcs of $\beta_2$ forms a double ruling of $R$ and such that (1) no component of $M \cdot (E-a_3^*)$ has points in more than two components of $\overline{R}$ minus the arcs of $\beta_2$, and (2) the arc $a_3^*$ has property $c_{1/2}$, then let $a_3^B$ be such an arc $a_3^*$. If no such arc exists, but there does exist an arc $a_3^{**}$ having the above properties except that in (1) the symbol $\beta_1$ replaces the symbol $\beta_2$, then let $a_3^B$ denote such an arc $a_3^{**}$. If neither $a_3^*$ nor $a_3^{**}$ exists let $a_3^B$ be any arc in $E$ which together with the arcs of $\beta_2$ forms a double ruling of $R$ and which has property $c_{1/2}$. Let $\alpha_2$ be the double ruling obtained by adding to $\beta_2$ the arc $a_3^B$ for every component $E$ of $\overline{R}$ minus the arcs of $\beta_2$ which are parallel to $BC$. Let $\beta_3$ be a double ruling which contains $\alpha_2$ and is such that (1) every component of $\overline{R}$ minus the arcs of $\beta_3$ is of diameter less than $1/3$, and (2) the arcs of $\beta_3$ which do not belong to $\beta_2$ have property $c_{1/3}$.

Proceeding in this way one can show that there exists an infinite sequence of double rulings $\beta_1$, $\beta_2$, $\beta_3$, $\cdots$, of $R$ such that for every $n$ the following properties obtain: (1) $\beta_{n+1}$ contains $\beta_n$, (2) every component of $\overline{R}$ minus the arcs of $\beta_n$ is of diameter less than $1/n$, (3) every arc of $\beta_{n+1}$ which does not belong to $\beta_n$ has property $c_{1/(n+1)}$, and (4) if $a_{3n}$ and $a_{2n}$ are adjacent arcs of $\beta_n$ which are parallel to $AB$ for $n$ odd and parallel to $BC$ for $n$ even, and $E$ is the set of all points of $\overline{R}$ which lie between $a_{3n}$ and $a_{2n}$, then if there exists a positive integer $j(j \leq n-1)$ and an arc $a_{3E}$ which lies in $E$ such that (a) the arc $a_{3E}$ together with the arcs of $\beta_n$ forms a double ruling of $R$, and (b) no component of $M \cdot (E-a_{3E})$ contains points in more than two components of $\overline{R}$ minus the arcs of $\beta_j$, then, $k_{3E}$ denoting the largest such integer $j$, $\beta_{n+1}$
contains an arc having the properties stated above for the arc $a_3b$ with $j$ replaced by $k_E$.

Let $P$ denote a point of $R$ not belonging to any arc of any of the double rulings $\beta_1, \beta_2, \beta_3, \ldots$. For each positive integer $n$ let $E_n,P$ denote the component, containing $P$, or $\bar{R}$ minus the arcs of $\beta_n$ which are parallel to $AB$ (for example). In view of properties (1), (2), and (3), and Lemma VII, it can be seen that the integer $n$ can be taken large enough so that the integer $j$ as qualified in property (4) does exist for $E_n,P$, and furthermore $k_{E_n,P}$ increases indefinitely as $n$ increases indefinitely. Hence it follows that for every connected subset $L$ of $M$ there exists an integer $n_L$ such that some arc of $\beta_{n_L}$ parallel to $AB[BC]$ has a point in common with $L$. In view of properties (1) and (2) it follows by methods employed in proving previous theorems that there exists a continuous transformation $T_1$ of $R$ into itself such that (1) for every $n$, $T_1$ throws the arcs of $\beta_n$ into intervals of rational lines, and (2) if $L$ is any rational line then there exists an integer $n$ and an arc $g$ of $\beta_n$ such that $T_1(g)$ is a subset of $L$. Obviously there exists a continuous transformation $T$ of the plane $S$ into itself which reduces to $T_1$ for points of $R$. Such a transformation satisfies the conclusion of the theorem.

Now as shown in the proof of Theorem IX a continuous curve $M$ which contains a domain contains a continuous curve $M_1$ such that $M_1$ contains no domain but does contain every boundary point of $M$, and such that if $D$ is a complementary domain of $M_1$ which is not a complementary domain of $M$ then $D$ lies in a domain of $M$. In view of this fact and the previous theorem the following corollaries may be easily established.

Corollary 1. If $M$ is a bounded continuous curve then there exists a continuous transformation $T$ of the plane into itself such that if $AB$ is an arc and $T(AB)$ is an interval of some rational line then (1) $AB \cdot M$ is the sum of a finite number of connected sets, and (2) every arc which is a subset of $AB \cdot M$ contains a subarc which either lies on the boundary of a complementary domain of $M$ or lies in a domain which belongs to $M$.

Corollary 2. If $M$ is a bounded continuous curve and $P$ is a point of $M$ which is not in a domain belonging to $M$ then there exists a continuous transformation $T$ of $S$ into itself such that if $APB$ is any arc such that $T(APB)$ is a subset of a horizontal line, then the component of $M \cdot APB$ which contains $P$ is $P$, and if $AB$ is an arc such that $T(AB)$ is a subset of a rational horizontal line then the number of components of $AB \cdot M$ is finite.

Let $A$ be any point and for each $n$ let $C_n$ be a circle of radius $1/n$ and center $A$. Let $AB$ be a unit interval and let $M$ be the continuous curve

License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use
This example shows that it is not true that if \( M \) is any continuous curve then there exists a continuous transformation \( T \) of the plane \( S \) into itself such that if \( AB \) is an arc and \( T(AB) \) is a horizontal interval then \( AB \cdot M \) is the sum of a finite number of connected sets.

**Theorem XI.** If \( P \) is a point of a bounded continuous curve \( M \) then there exists an upper semi-continuous collection\( ^\dagger \) \( G \) of subcontinua of \( M \) which fills up \( M \) such that \( P \) is an element of \( G \) and \( G \) is a regular curve with respect to its elements.

Suppose first that \( P \) is a point which does not belong to a domain which belongs to \( M \) and let \( T \) denote a transformation satisfying the conclusion of Corollary 2. For each point \( x \) of \( M \) let \( g_\ast \) be the greatest continuum containing \( x \) such that \( T(g_\ast) \) is a subset of some horizontal line, and let \( G \) denote the collection of continua \( g_\ast \) for all points \( x \) of \( M \). Clearly \( g_\ast = P \), and the collection \( G \) is upper semi-continuous. Now if \( M_1 \) is any continuum such that the common part of any rational horizontal line and \( M_1 \) is a finite point set, and the common part of any horizontal line and \( M_1 \) is totally disconnected, then \( M_1 \) is a regular curve. Hence \( G \) is a regular curve with respect to its elements.

Suppose \( P \) is a point lying in a domain \( D \) of \( M \). There exists a set \( K \) of mutually exclusive simple closed curves lying in \( D \), all enclosing \( P \), no two having a point in common, and such that every point of \( D - P \) belongs to some curve of the set \( K \). Let \( G \) be the upper semi-continuous collection of continua consisting of the curves of the collection \( K \) and the continua \( M - D \) and \( P \). The collection \( G \) is an arc with respect to its elements and one of its elements is \( P \).

\( ^\dagger \) See R. L. Moore, *Concerning upper semi-continuous collections of continua*, these Transactions, vol. 27 (1925), pp. 416-428. A collection \( G \) of continua is said to be an upper semi-continuous collection if for each element \( g \) of the collection \( G \) and each positive number \( e \) there exists a positive number \( d \) such that if \( x \) is any element of \( G \) at a lower distance from \( g \) less than \( d \) then the upper distance of \( x \) from \( g \) is less than \( e \). If \( M \) is a point set and \( P \) is a point, then by \( l(PM) \) is meant the lower bound of the distances from \( P \) to all the different points of \( M \). If \( M \) and \( N \) are two point sets, then by \( l(MN) \) is meant the lower bound of the values \( [l(PN)] \) for all points \( P \) of \( M \), while by \( u(MN) \) is meant the upper bound of these values for all points \( P \) of \( M \). The point set \( M \) is said to be at the upper distance \( u(MN) \) from the point set \( N \) and is said to be at the lower distance \( l(MN) \) from \( N \).

**University of Texas,**
**Austin, Texas**