SURFACE TRANSFORMATIONS*

BY

F. R. BAMFORTH†

In a paper written by Birkhoff‡ on the Problem of Three Bodies, he reduced the study of a certain phase of this celebrated problem to the study of a surface transformation which was the product of two surface transformations each of which represented the reflection, in a curve lying in a surface, of the surface into itself. It is the purpose of this paper to make a start on the study of such surface transformations in general.

In the first section are studied surface transformations of period two, and it is shown that when a suitable coördinate system has been chosen, any such transformation may be represented by one and but one of the following systems of equations:

$$(1)\qquad \begin{aligned} u_1 &= u, \\ v_1 &= v; \end{aligned} \qquad \begin{aligned} u_1 &= -u, \\ v_1 &= -v; \end{aligned} \qquad \begin{aligned} u_1 &= u, \\ v_1 &= -v; \end{aligned}$$

in which the point (u_1, v_1) denotes the transform of the point (u, v).

In the remaining sections, surface transformations are discussed which are the product of two transformations, each of which, when a suitable coördinate system has been selected, can be represented by equations of the third type in (1).

I. Transformations of period two

This section will be concerned with real, one-to-one, analytic transformations of a surface Z into itself, of such a character that after the same transformation has been performed twice in succession all the points of the surface are in the same positions with respect to fixed axes as they were before. These transformations are called transformations of period two, and we shall study the movement under such transformations of the points of the surface Z in a neighborhood of a point which is invariant.

In a neighborhood of such an invariant point the equations representing the transformation may be written in the form

$$(2)\qquad u_1 = au + bv + \cdots, \qquad v_1 = cu + dv + \cdots, \qquad ad - bc \neq 0,$$

* Presented to the Society, December 27, 1929; received by the editors February 28, 1930.

† This paper was partially prepared while the author was a National Research Fellow.

‡ G. D. Birkhoff, Rendiconti del Circolo Matematico di Palermo, vol. 39.

after a proper choice of coördinate system has been made. For this system of coördinates the origin is the invariant point and the series on the right hand sides of the equations (2) are convergent for u and v sufficiently small in absolute value. By means of a real, linear change of variables, the equations (2) of the transformation are equivalent to equations which will have one of the following forms:

$$u_1 = \alpha u + \cdots, \qquad v_1 = \alpha v + \cdots, \ \alpha \neq 0; \tag{3}$$

$$u_1 = v + \cdots, \qquad v_1 = \beta u + \delta v + \cdots, \ \beta \neq 0, \tag{4}$$

where the dots indicate terms of degree greater than one.

Let us now discuss transformations which may be represented by equations of the form (3). If we denote the transform of (u_1, v_1) by (u_2, v_2), from the fact that the transformation is of period two, it follows that

$$u = u_2 = \alpha u_1 + \cdots = \alpha^2 u + \cdots, \qquad v = v_2 = \alpha v_1 + \cdots = \alpha^2 v + \cdots,$$

whence $\alpha = \pm 1$. Suppose first that $\alpha = 1$. The transformation may be represented by

$$u_1 = u + a_{20}u^2 + a_{11}uv + a_{02}v^2 + \cdots,$$
$$v_1 = v + b_{20}u^2 + b_{11}uv + b_{02}v^2 + \cdots.$$

Since the transformation is of period two

$$\begin{aligned} u = u_2 &= u_1 + a_{20}u_1^2 + a_{11}u_1v_1 + a_{02}v_1^2 + \cdots \\ &= u + 2a_{20}u^2 + 2a_{11}uv + 2a_{02}v^2 + \cdots. \end{aligned}$$

From this identity it follows that all the a_{ij} are zero, and from the corresponding one in v it follows that all the b_{ij} are also zero. Hence, when a transformation of period two can be represented by equations of the form (3) with $\alpha = 1$, it is the identity transformation.

Transformations represented by equations of the form (3) with $\alpha = -1$ will be called "rotation transformations" on account of the fact that the simplest example of equations of this form, i.e., $u_1 = -u$, $v_1 = -v$, represents a rotation through 180° when the coördinate system is rectangular. We shall now reduce the equations (3) of a rotation transformation to a simpler form. To this end let us write these equations in the form

$$u_1 = -u + U(u,v), \qquad V_1 = -v + V(u,v), \tag{5}$$

where $U(u, v)$ and $V(u, v)$ have no terms of first degree in u and v. On account of the periodicity of the transformation,

$$u = u_2 = -u_1 + U(u_1,v_1) = u - U(u,v) + U(u_1,v_1),$$
$$v = v_2 = -v_1 + V(u_1,v_1) = v - V(u,v) + V(u_1,v_1),$$

whence

$$U(u,v) = U(u_1,v_1), \qquad V(u,v) = V(u_1,v_1).$$

Hence the equations (5) are equivalent to the equations

$$u_1 - \tfrac{1}{2}U(u_1,v_1) = -u + \tfrac{1}{2}U(u,v),$$
$$v_1 - \tfrac{1}{2}V(u_1,v_1) = -v + \tfrac{1}{2}V(u,v).$$

Under the change of variables

$$u^* = u - \tfrac{1}{2}U(u,v), \qquad v^* = v - \tfrac{1}{2}V(u,v),$$

these equations are evidently transformed into

$$u_1^* = -u^*, \quad v_1^* = -v^*,$$

from which form the structure of the transformation is evident.

If a transformation of period two is represented by equations of the form (4),

$$u = u_2 = v_1 + \cdots = \beta u + \delta v + \cdots,$$
$$v = v_2 = \beta u_1 + \delta v_1 + \cdots = \beta v + \delta\beta u + \delta^2 v + \cdots,$$

whence $\beta = 1$ and $\delta = 0$. The equations representing such a transformation, by means of a suitable change of coördinate system, may be of either of the two forms

$$u_1 = v + \cdots, \quad v_1 = u + \cdots, \tag{6}$$

or

$$u_1 = u + \cdots, \quad v_1 = -v + \cdots. \tag{7}$$

The equivalence is readily seen if the coördinates are considered as rectangular coördinates in the plane and a rotation of axes through an angle of 45° is performed. Since the simplest example of equations of the form (7) representing such a transformation of period two, i.e., $u_1 = u$, $v_1 = -v$, represents a reflection in the u axis, we shall call any transformation of period two which can be represented by equations of the form (6) or (7), a "reflection transformation."

Let us write the equations (7) in the form

$$u_1 = u + U(u,v), \qquad v_1 = -v + V(u,v). \tag{8}$$

Here, on account of the periodicity of the transformation,

$$U(u,v) = -U(u_1,v_1), \qquad V(u,v) = V(u_1,v_1).$$

Hence the equations (8) are equivalent to the equations

$$u_1 + \tfrac{1}{2}U(u_1,v_1) = u + \tfrac{1}{2}U(u,v),$$
$$v_1 - \tfrac{1}{2}V(u_1,v_1) = -v + \tfrac{1}{2}V(u,v),$$

from which it follows immediately that this transformation may be represented by the equations

$$u_1^* = u^*, \quad v_1^* = -v^*, \tag{9}$$

where

$$u^* = u + \tfrac{1}{2}U(u,v), \qquad v^* = v - \tfrac{1}{2}V(u,v).$$

Here, as in the case of rotation transformations, the form (9) tells us everything that we wish to know about the transformation.

When a transformation of the reflection type is represented in a neighborhood of an invariant point by equations of the form (9), we shall say that the u^*v^* system of coördinates is a "normal system" for the transformation. Now it is evident that there are infinitely many normal coördinate systems for any one such transformation. In fact, if we make a reversible change of coördinates defined by the equations

$$u^* = f(w,z), \qquad v^* = g(w,z), \tag{10}$$

a sufficient condition that the new system of coördinates be a normal system for this transformation is that

$$f(w,-z) = f(w,z), \quad g(w,-z) = -g(w,z). \tag{11}$$

This is a consequence of the fact that if (11) be true, $w_1=w$, $z_1=-z$ is a solution of the pair of equations

$$f(w_1,z_1) = f(w,z), \qquad g(w_1,z_1) = -g(w,z).$$

By way of summarizing this first section, we may say that *every transformation of period two, in a neighborhood of an invariant point, after a suitable choice of coördinate system has been made, may be represented by one and but one of the following systems of equations*:

$$\begin{aligned} u_1 &= u, & u_1 &= -u, & u_1 &= u, \\ v_1 &= v; & v_1 &= -v; & v_1 &= -v. \end{aligned}$$

II. Product transformations

In Birkhoff's paper on the Problem of Three Bodies, a transformation of a surface into itself arose which was the product of two transformations each of which was of the reflection type. The object of this and the succeeding sections of this paper is to study such transformations in the neighborhood of a point which is invariant under each of the component transformations of period two.

In the notation we shall use, the script letters $\mathcal{R}$ and $\mathcal{S}$ will stand for transformations of the reflection type of a surface Z into itself, and the script letter $\mathcal{T}$ will stand for the product transformation $\mathcal{T}=\mathcal{R}\mathcal{S}$. In a neighborhood of a point which is invariant under the transformation $\mathcal{S}$, by choosing a normal system of coördinates for this transformation, it may be represented by the equations

$$S_0:\quad y_1 = y, \quad x_1 = -x. \tag{12}$$

Here, S_0 is interpreted as symbolical of the equations (12). If we make a reversible change of coördinates defined by

$$V:\quad y = f(w,z), \quad x = g(w,z), \tag{13}$$

the equations (12) are equivalent to the equations

$$f(w_1,z_1) = f(w,z), \quad g(w_1,z_1) = -g(w,z). \tag{14}$$

Since we have denoted the equations (13) symbolically by V, let us denote the equations obtained by solving them for w and z in terms of y and x by V^{-1}, i.e.,

$$V^{-1}:\quad w = f_0(y,x), \quad z = g_0(y,x).$$

On solving (14) for w_1 and z_1 we obtain

$$\begin{aligned} w_1 &= f_0[f(w,z),\ -g(w,z)], \\ z_1 &= g_0[f(w,z),\ -g(w,z)]. \end{aligned}$$

These equations may be represented symbolically by $V^{-1}S_0V$ when this symbol is interpreted according to the law which will now be stated. If A, B and C denote the systems of equations

$$A:\ \begin{aligned} \alpha &= f_1(\gamma,\delta) \\ \beta &= g_1(\gamma,\delta) \end{aligned},\qquad B:\ \begin{aligned} \gamma &= f_2(\xi,\eta) \\ \delta &= g_2(\xi,\eta) \end{aligned},\qquad C:\ \begin{aligned} \xi &= f_3(\zeta,\theta) \\ \eta &= g_3(\zeta,\theta) \end{aligned},$$

then AB and ABC denote, respectively, the systems of equations

$$AB:\ \begin{aligned} \alpha &= f_1[f_2(\xi,\eta),\ g_2(\xi,\eta)], \\ \beta &= g_1[f_2(\xi,\eta),\ g_2(\xi,\eta)]; \end{aligned}$$

and

$$ABC:\ \begin{aligned} \alpha &= f_1\{f_2[f_3(\zeta,\theta),\ g_3(\zeta,\theta)],\ g_2[f_3(\zeta,\theta),\ g_3(\zeta,\theta)]\}, \\ \beta &= g_1\{f_2[f_3(\zeta,\theta),\ g_3(\zeta,\theta)],\ g_2[f_3(\zeta,\theta),\ g_3(\zeta,\theta)]\}. \end{aligned}$$

Thus we have the result that if a transformation $\mathcal{S}$ of a surface Z into itself be denoted by the equations S_0 of (12), and if a change of coördinate systems denoted by the equations V of (13) takes place, the transformation $\mathfrak{S}$ is denoted in the new system of coördinates by the equations $V^{-1}S_0V$.

Let us suppose now that the transformation $\mathfrak{R}$ is represented by the system of equations

$$(15) \qquad R:\ u_1 = u, \quad v_1 = -v,$$

and let the yx and the uv systems of coördinates be related by means of the equations

$$(16) \qquad U:\ y = F(u,v), \qquad x = G(u,v),$$

where F and G are analytic functions of their arguments in a neighborhood of the point $(u, v) = (0, 0)$, the jacobian

$$(17) \qquad \left| \begin{matrix} \partial F/\partial u & \partial F/\partial v \\ \partial G/\partial u & \partial G/\partial v \end{matrix} \right|_{(u,v)=(0,0)}$$

is greater than zero, and $F(0, 0) = G(0, 0) = 0$. The condition that the jacobian (17) be greater than zero is not so strong as one might imagine. If we have a correspondence between the two systems of coördinates which fulfills the other conditions named above but which is such that the jacobian (17) is negative, we may change our yx coördinate system to another by means of the equations

$$y = w, \quad x = -z.$$

The wz system of coördinates will be normal for $\mathfrak{S}$ and the analogue of the negative jacobian will be positive.

Since the yx normal coördinate system can be replaced by another by means of the equations (10), where f and g satisfy the relations (11), it it evident that we may assume that the determinant of the coefficients of the linear terms of the functions F and G in (16) has the value 1, and that, if the linear term in u (or the linear term in v) is lacking either in F or in G, the coefficient of the linear term in v (or the linear term in u) is unity in absolute value. Thus there exist what may be termed certain canonical forms for the equations (16) which are characterized by the linear terms of F and G. These may be listed as:

$$(18) \qquad \mathrm{I(A)}:\ \begin{aligned} y &= u + \cdots, \\ x &= v + \cdots; \end{aligned}$$

$$\mathrm{I(B)}:\ \begin{aligned} y &= u + av + \cdots, \quad a \neq 0, \\ x &= v + \cdots; \end{aligned}$$

$$\text{II(A)}: \quad \begin{aligned} y &= v + \cdots, \\ x &= -u + \cdots; \end{aligned}$$

$$\text{II(B)}: \quad \begin{aligned} y &= au + v + \cdots, \quad a \neq 0, \\ x &= -u + \cdots; \end{aligned}$$

$$\text{III}: \quad \begin{aligned} y &= u + \cdots, \\ x &= au + v + \cdots, \quad a \neq 0; \end{aligned}$$

$$\text{IV}: \quad \begin{aligned} y &= v + \cdots, \\ x &= -u - av + \cdots, \quad a \neq 0; \end{aligned}$$

$$\text{V}: \quad \begin{aligned} y &= a_{10}u + a_{01}v + \cdots, \\ x &= b_{10}u + b_{01}v + \cdots, \quad a_{10}a_{01}b_{10}b_{01} > 0; \end{aligned}$$

$$\text{VI}: \quad \begin{aligned} y &= a_{10}u + a_{01}v + \cdots, \\ x &= b_{10}u + b_{01}v + \cdots, \quad a_{10}a_{01}b_{10}b_{01} < 0. \end{aligned}$$

The equations I(B), II(B), III and IV may be assumed to be in a still simpler form. Consider, for example, the equations I(B), and make use of the change of coördinates

$$\begin{aligned} y &= y^*, & u &= u^*, \\ x &= x^*/(2a); & v &= v^*/(2a). \end{aligned}$$

Evidently the transformed equations of I(B) have the form

$$\begin{aligned} y^* &= u^* + \tfrac{1}{2}v^* + \cdots, \\ x^* &= v^* + \cdots. \end{aligned}$$

Since analogous transformations may be employed in the other cases, it is evident that *we may assume that wherever "a" appears in the equations* (18) *we may write* $\frac{1}{2}$. This choice of constant "a" gives a particularly simple form to the canonical forms which will be given later of the equations representing transformations $\mathfrak{T}$.

Let us now consider the transformation $\mathfrak{T}$ of the surface Z into itself which is the product of the transformations $\mathfrak{R}$ and $\mathfrak{S}$, i.e., $\mathfrak{T}=\mathfrak{R}\mathfrak{S}$, which is to mean that the transformation $\mathfrak{S}$ is followed by the transformation $\mathfrak{R}$. We shall first discuss one of the properties of this transformation $\mathfrak{T}$ which can be readily proved out of the fact that each of the transformations $\mathfrak{R}$ and $\mathfrak{S}$ is of period two. Let the symbol S represent the equations of the transformation $\mathfrak{S}$ in the uv system of coördinates which is normal for the transformation $\mathfrak{R}$ represented by the equations (15). On account of the equations (16) we see that symbolically $S=U^{-1}S_0U$, where S_0 is given in (12). If T

represents the equations of the transformation $\mathfrak{T}$ in the uv system of coördinates, evidently $T=RS$. If we denote the identity transformation by I, we have symbolically

$$SS = I,\ S = S^{-1},\ R = R^{-1},\ T = RS,\ TSR = RSSR = I,$$
$$T^{-1} = SR = SRSS^{-1} = RRSR = STS^{-1} = R^{-1}TR.$$

It is this last equation, $T^{-1}=R^{-1}TR$, which we wish to examine more closely. Since a change of coördinate systems denoted by V changes the equations T into $V^{-1}TV$ we see that the inverse transformation of $\mathfrak{T}$ is represented in the uv system of coördinates by the same equations as the transformation $\mathfrak{T}$ after the change of coördinate systems denoted by R has taken place. Thus, if the equations T are

$$(19)\qquad T:\quad \begin{aligned} u_1 &= a_1u + a_2v + \cdots, \\ v_1 &= b_1u + b_2v + \cdots, \end{aligned}$$

and if we perform a change of coördinates denoted by

$$(20)\qquad R:\quad u = s,\ v = -t,$$

obtaining thereby

$$(21)\qquad R^{-1}TR:\quad \begin{aligned} s_1 &= a_1s - a_2t + \cdots, \\ t_1 &= -b_1s + b_2t + \cdots, \end{aligned}$$

in the uv system of coördinates $\mathfrak{T}^{-1}$ is given by

$$(22)\qquad T^{-1}:\quad \begin{aligned} u &= a_1u_1 - a_2v_1 + \cdots, \\ v &= -b_1u_1 + b_2v_1 + \cdots. \end{aligned}$$

Thus, observing the movement of the points of the surface Z under the transformation $\mathfrak{T}^{-1}$ is equivalent to observing the movement of the points of the reflection of the surface Z under $\mathfrak{T}$ in a mirror which is plane for the uv system of coördinates. We shall make use of this fact later on in the discussion.

Let us now return to a further consideration of the normal forms (18) of the equations of the change of coördinate systems. The transformation $\mathfrak{S}$ in the uv system is given by

$$(23)\qquad F(u_1,v_1) = F(u,v), \qquad G(u_1,v_1) = -G(u,v).$$

Hence the transformation $\mathfrak{T}$ is given by

$$(24)\qquad F(u_1,-v_1) = F(u,v), \qquad G(u_1,-v_1) = -G(u,v).$$

The normal forms for the equations T which correspond to the normal forms

(18) are the following:

$$
(25)\qquad
\begin{array}{rl}
\mathrm{I(A)}: & \begin{aligned} u_1 &= u + \cdots, \\ v_1 &= v + \cdots; \end{aligned} \\
\mathrm{I(B)}: & \begin{aligned} u_1 &= u + v + \cdots, \\ v_1 &= v + \cdots; \end{aligned} \\
\mathrm{II(A)}: & \begin{aligned} u_1 &= - u + \cdots, \\ v_1 &= - v + \cdots; \end{aligned} \\
\mathrm{II(B)}: & \begin{aligned} u_1 &= - u + \cdots, \\ v_1 &= - u - v + \cdots; \end{aligned} \\
\mathrm{III}: & \begin{aligned} u_1 &= u + \cdots, \\ v_1 &= u + v + \cdots; \end{aligned} \\
\mathrm{IV}: & \begin{aligned} u_1 &= - u - v + \cdots, \\ v_1 &= - v + \cdots; \end{aligned} \\
\mathrm{V}: & \begin{aligned} u_1 &= \alpha u + \beta v + \cdots, \\ v_1 &= \gamma u + \alpha v + \cdots, \ \beta\gamma > 0, \ \alpha \neq 0; \end{aligned} \\
\mathrm{VI}: & \begin{aligned} u_1 &= \alpha u + \beta v + \cdots, \\ v_1 &= \gamma u + \alpha v + \cdots, \ \beta\gamma < 0; \end{aligned}
\end{array}
$$

where $\alpha = a_{10}b_{01} + a_{01}b_{10}$, $\beta = 2a_{01}b_{01}$, $\gamma = 2a_{10}b_{10}$, $\alpha^2 - \beta\gamma = 1$. Evidently, each of the type sets of equation in (25) is but an example of

$$
(26)\qquad T: \quad \begin{aligned} u_1 &= \alpha u + \beta v + \cdots, \\ v_1 &= \gamma u + \alpha v + \cdots, \end{aligned}
$$

where α, β or γ may have the value zero.

Thus we have obtained the result that *every transformation which is the product of two transformations of the reflection type, in a neighborhood of a point which is invariant under each of the component transformations, can be represented in one and only one way by equations which are of one of the types displayed in* (25).

III. Invariant directions and simple examples

Let us now inquire into the possibility of there being directions through the origin which are either invariant under T or are rotated through 180°. From (26) when $\alpha\beta\gamma \neq 0$ we obtain at $(u, v) = (0, 0)$ for du/dv finite and different from $-\alpha/\gamma$

$$\frac{du_1}{dv_1} - \frac{du}{dv} = \frac{\alpha du + \beta dv}{\gamma du + \alpha dv} - \frac{du}{dv} = \frac{\beta - \gamma\left(\dfrac{du}{dv}\right)^2}{\gamma\dfrac{du}{dv} + \alpha}.$$

If $du/dv = -\alpha/\gamma$ or is infinite, then $dv/du \neq -\alpha/\beta$, since $\alpha^2 - \beta\gamma = 1$, and

$$\frac{dv_1}{du_1} - \frac{dv}{du} = \frac{\gamma - \beta\left(\dfrac{dv}{du}\right)^2}{\alpha + \beta\dfrac{dv}{du}}.$$

Hence du/dv remains unchanged by T if and only if

(27) $$\beta(dv)^2 - \gamma(du)^2 = 0.$$

Now consider the case where $\alpha = 0$. Then $\beta\gamma \neq 0$, and

$$\frac{du_1}{dv_1} = \frac{\beta}{\gamma}\frac{dv}{du}$$

when dv/du is finite and

$$\frac{dv_1}{du_1} = \frac{\gamma}{\beta}\frac{du}{dv}$$

when du/dv is finite. Thus the invariant values of du/dv are given by (27). In a similar manner it can readily be shown that the invariant values of du/dv for $\beta = 0$ or $\gamma = 0$ are also given by (27).

Thus we find that *for all transformations of the types* I(A) *and* II(A), *all slopes through the origin are invariant*; *for all transformations of the types* I(B), II(B), III *and* IV *there are two coincident invariant slopes which are the only ones*; *for all transformations of the type* V *there are two distinct invariant slopes which are the only ones*; *and for transformations of the type* VI *there is no invariant slope.*

It would, perhaps, be useful in clarifying our ideas to give at this point a short discussion of transformations of the different fundamental types in which only linear terms appear on the right hand sides of the equations (25). Each of these transformations is area preserving since the jacobian of the functions on the right hand sides of each pair of these equations is 1.

The transformation $u_1 = u$, $v_1 = v$, which is of type I(A), is evidently the identity transformation under which every point is an invariant point.

For the transformation

(28) $$u_1 = u + v, \quad v_1 = v,$$

of type I(B), the u and y axes are coincident while the equation of the x-axis in the uv-system of coördinates is given by $u+\frac{1}{2}v=0$. The only invariant curve for this transformation which passes through the origin is the u-axis, and since v is an invariant function, the points move parallel to the u-axis under the transformation $\mathfrak{T}$. Moreover, this invariant curve is made up of invariant points. It is easily verified that this transformation has the origin as a hyperbolic, unstable,† invariant point, since for $v>0$, $u_1-u>0$, and for $v<0$, $u_1-u<0$. In fact, if any neighborhood of the origin be taken, it is evident that the only points which remain in the neighborhood under the indefinite iteration of T and of T^{-1} are those lying on the u-axis which are invariant points.

The transformation $u_1=-u$, $v_1=-v$, of type II(A), is a rotation through 180°. Thus every straight line through the origin is invariant and the points on it are reflected in the origin, which is an isolated, hyperbolic, stable, invariant point for $\mathfrak{T}$. The x and the u axes are coincident, as are also the y and the v.

For the transformation $u_1=-u$, $v_1=-u-v$, of type II(B), the y and v axes are coincident and the equation of the x-axis in the uv-system of coördinates is given by $\frac{1}{2}u+v=0$. The only invariant curve through the origin is the v-axis on which all the points are reflected in the origin by $\mathfrak{T}$. The origin is a hyperbolic, unstable, invariant point.

For the transformation $u_1=u$, $v_1=u+v$, of type III, the x and v axes are coincident and the equation of the y axis is $\frac{1}{2}u+v=0$. The origin is a hyperbolic, unstable, invariant point, and the only invariant curve through it is the v axis. The discussion of the simple transformation of type IV is obviously similar to that of type III.

For the transformation

$$u_1 = \alpha u + \beta v, \quad v_1 = \gamma u + \alpha v, \quad \alpha > 0, \quad \beta > 0, \quad \gamma > 0, \tag{29}$$

of type V, $\beta v^2-\gamma u^2$ is an invariant function which is zero on the two invariant lines $\beta v^2-\gamma u^2=0$ through the origin, which is therefore a hyperbolic, invariant point. The u axis bisects the angle between these two invariant lines. Let us examine the movement under $\mathfrak{T}$ of the points on these invariant lines. Let (u, v) be any point on the invariant line $u=v(\beta/\gamma)^{1/2}$. Then

$$u_1 = \alpha u + \beta v = (\alpha + (\beta\gamma)^{1/2})u,$$
$$v_1 = \gamma u + \alpha v = ((\beta\gamma)^{1/2} + \alpha)v.$$

† Definitions of certain technical terms used in the discussion of this paper will be found in a paper by G. D. Birkhoff, Acta Mathematica, vol. 43: conservative transformation, p. 2; formal conservative transformation, p.22; quasi-invariant function, p.2; stable and unstable invariant points, p. 5; formal invariant curve through origin, p. 26; hypercontinuous invariant curve, p. 67; elliptic and hyperbolic invariant points, p. 26. This paper will be referred to hereafter as B.

Now $(\alpha+(\beta\gamma)^{1/2})$ $(\alpha-(\beta\gamma)^{1/2})=1$. Hence $\alpha+(\beta\gamma)^{1/2}$ is positive and either is greater than 1 or less than 1. If $\alpha+(\beta\gamma)^{1/2}>1$, the points on this invariant line will recede from the origin on iteration of $\mathfrak{T}$ and if $\alpha+(\beta\gamma)^{1/2}<1$ the points will approach the origin on iteration of $\mathfrak{T}$. But if $\alpha+(\beta\gamma)^{1/2}<1$, $\alpha-(\beta\gamma)^{1/2}>1$, and hence the points on one of the invariant lines recede from the origin on iteration of $\mathfrak{T}$ and the points on the other invariant line approach the origin on iteration of $\mathfrak{T}$. This shows that the origin is an unstable invariant point.

Suppose that for the transformation represented by (29) we make a change of coördinates which takes these invariant curves into the axes. Then the equations (29) become $\bar{u}_1=b\bar{u}$, $\bar{v}_1=c\bar{v}$, $b>1$, $0<c<1$, which shows that no other invariant curves exist for the transformation (29) than those already mentioned.

The transformation which is the same as (29) with the exception that $\alpha<0$, can be discussed in a manner similar to that in which we have discussed (29). In this transformation the points on one of the invariant lines recede from the origin on iteration of $\mathfrak{T}$ but oscillate about the origin in so doing. By this is meant that the origin always separates a point on this invariant line from its image under $\mathfrak{T}$. The points on the other invariant line approach the origin under iteration of $\mathfrak{T}$ but also oscillate with respect to it.

For the transformation $u_1=\alpha u+\beta v$, $v_1=\gamma u+\alpha v$, $\beta>0$, $\gamma<0$, which is of the type VI, as has already been proved there is no invariant slope through the origin. Hence there is no invariant curve through it and the origin is an elliptic, invariant point. In this case the function $\beta v^2-\gamma u^2$ is an invariant function which shows that the origin is a stable invariant point.

Since all the simple transformations which we have just been discussing preserve areas and hence have invariant integrals, it would seem that it would be advantageous to link that classification which we have given for transformations $\mathfrak{T}=\mathfrak{R}\mathfrak{S}$ with that given by Birkhoff in his paper in the Acta Mathematica. It will be assumed that the reader has this paper before him; the classification is on page 4. We shall designate the types of problems which we are studying by I(A), I(B), etc., and those which Birkhoff studied by I′, I″, etc., as he did. With this notation in mind, it is evident that we have the following correspondence based upon the coefficients of the linear terms in the equations of the transformations: I(A)$\sim$II″, I(B)$\sim$III′, II(A)$\rightarrow$II‴, II(B)$\sim$III″, III$\sim$III′, IV$\sim$III″, V$(\alpha>0)\sim$I′, V$(\alpha<0)\sim$I″, VI$(\alpha=0)\sim$II‴, VI$(\alpha\neq 0)\sim$II′ or II‴. Reference will be made to this correspondence from time to time. When interpreting this correspondence we should remember the properties of the u and v axes of coördinates as used in this paper, made clear by the equations (15).

IV. Formal invariant series

The question of the existence of real, formal series which are invariant under the equations representing a transformation $\mathfrak{T}$ is important for at least two reasons. If there is such a series which is convergent, by setting it equal to a constant we obtain a curve which is invariant under $\mathfrak{T}$. Hence we can study the transformation by studying the totality of such invariant curves. On the other hand, if there does not exist any series which is convergent, but there does exist one that is divergent for all sets of values of the variables different from $(0, 0)$, it can be used in many cases to reduce the equations representing the transformation to a simpler form and to prove the existence or non-existence of real invariant curves through the origin which is assumed to be an invariant point of the transformation. Hence we shall search for series which are formally invariant under $T=RS$.

Let us now recall that $T^{-1}=R^{-1}TR=S^{-1}TS$, which means that the equations giving the inverse transformation $\mathfrak{T}^{-1}$ are the same in form as those for the transformation $\mathfrak{T}$ itself after a change of variables has taken place in which the equations of the change of variables are those of the transformation $\mathfrak{R}$ or of the transformation $\mathfrak{S}$. In this connection, let us return to the consideration of the equations (19), (20), (21) and (22), and the discussion given concerning them. Thus we see that, if $H(u, v)$ is a formal invariant series under (19), $H(s,-t)$ is invariant under (21) and hence $H(u,-v)$ is invariant under (22). Thus, if the series $H(u, v)$ is invariant under T, the series $H(u,-v)$, $H(u, v)+H(u,-v)$ and $H(u, v)-H(u,-v)$ are too; and every series which is invariant under T is the sum of an invariant series which is even in v and an invariant series which is odd in v. This follows immediately from the fact that if a series is invariant under T it is also invariant under T^{-1}.

Furthermore, if the transformation $\mathfrak{T}$ is written in the variables y and x by means of the equations (16) it is readily seen that every series that is formally invariant is the sum of an invariant series which is even in x and an invariant series which is odd in x.

Now let us consider the equations (24) for the transformation $\mathfrak{T}$. These equations may be written in the form

$$
(30)\qquad
\begin{aligned}
F(u,v) &= a_0(u) + a_1(u)v + a_2(u)v^2 + a_3(u)v^3 + \cdots \\
&= a_0(u_1) - a_1(u_1)v_1 + a_2(u_1)v_1^2 - a_3(u_1)v_1^3 + \cdots, \\
G(u,v) &= b_0(u) + b_1(u)v + b_2(u)v^2 + b_3(u)v^3 + \cdots \\
&= -\, b_0(u_1) + b_1(u_1)v_1 - b_2(u_1)v_1^2 + b_3(u_1)v_1^3 - \cdots .
\end{aligned}
$$

From the reasoning given in the preceding paragraph it is evident that if there is a formal series which is invariant under T there is one, $K(y, x)$, in the variables y and x which is invariant under T and is even in x and consequently in $G(u, v)$ when the variables y and x are replaced by the functions $F(u, v)$ and $G(u, v)$ of (16). Hence from (24),

$$K[F(u_1,v_1), G(u_1,v_1)] = K[F(u,v), G(u,v)] = K[F(u_1, -v_1), -G(u_1, -v_1)].$$

But K is an even function of G by hypothesis. Hence

$$K[F(u_1,v_1), G(u_1,v_1)] = K[F(u_1, -v_1), G(u_1, -v_1)],$$

whence K is even in v_1, and hence, when it is expressed in terms of u and v, even in v. In an analogous manner it can easily be proved that any invariant series which is odd in x in the yx-system of coördinates is odd in v when written in the uv-system of coördinates. Since the square of an invariant series which is odd in v is an invariant series which is even in v we see that there exists a series which is formally invariant under T if and only if there exists such a series which is even in v.

We shall now examine transformations $\mathfrak{T}$ to find out whether formally invariant series can exist, and we shall limit the discussion at first to the cases in which the transformations are of the type V, or VI with $\alpha \neq 0$. Since we have shown that there exists a formally invariant series for such a transformation if and only if there exists one which is even in x, and hence, as has been shown, even in v, we shall try to show that, for the cases under consideration, there exists a formally invariant series which is even in x and, when expressed in terms of u and v, is even in v.

It is readily provable that if there exists an invariant series it can not have any terms of the first degree in u or in v. Hence we shall start with second degree terms. Let us write the equations (30) in the form

$$(31)\qquad \begin{aligned} F(u,v) &= a_{10}u + a_{01}v + a_{20}u^2 + a_{11}uv + a_{02}v^2 + \cdots \\ &= a_{10}u_1 - a_{01}v_1 + a_{20}u_1^2 - a_{11}u_1v_1 + a_{02}v_1^2 + \cdots, \\ G(u,v) &= b_{10}u + b_{01}v + b_{20}u^2 + b_{11}uv + b_{02}v^2 + \cdots \\ &= -b_{10}u_1 + b_{01}v_1 - b_{20}u_1^2 + b_{11}u_1v_1 - b_{02}v_1^2 + \cdots. \end{aligned}$$

Now from the way in which the series $F(u, v)$ and $G(u, v)$ are transformed by T we see that our invariant series which is to be even in x and consequently in G must be such that when F and G are expanded in terms of u and v no term of odd degree in v appears. In so far as the second degree terms are concerned, except for a constant factor, there is only one combination of F and G which is even in v and invariant under T up to terms of degree three, and this is

$$(32)\qquad b_{10}b_{01}F^2 - a_{01}a_{10}G^2.$$

This expression transforms under T in the same manner as do F and G^2, and has no term of the second degree in u and v which is odd in v. The method which we shall use to build up our formally invariant series consists in adding to the function (32) a function of the third degree in F and G which is even in G and which is such that the resulting sum contains no term of the third degree in u and v which is odd in v; and then adding to this sum a function of the fourth degree in F and G which is even in G and which is such that this last sum will contain no term of the fourth degree in u and v which is odd in v; this process being continued so long as any terms of odd degree in v remain in the sum. Now the series (32) is invariant up to its terms of the third order; the series obtained from (32), after the first addition has been made to it, will be invariant up to terms of the fourth order; and so on. Thus, if we can prove that at no stage does the above outlined process of adding functions break down, we shall have established the fact that there exists an invariant series for transformations $\mathfrak{T}$ of the types V and VI with $\alpha\neq 0$.

Let us now investigate what we need to prove in order to show that this process does not break down at any stage. Suppose we wish to eliminate terms of odd order in v in the group of terms of the rth degree in u and v. To do this we may employ a sum of the form

$$\alpha_0 G^r + \beta_0 G^{r-2}F^2 + \cdots + \eta_0 F^r \tag{33}$$

if r is even, and of the form

$$\alpha_0 G^{r-1}F + \beta_0 G^{r-3}F^3 + \cdots + \eta_0 F^r \tag{34}$$

if r is odd.

Let us first consider the case when r is odd. When a sum of the form (34) is employed, we see that to determine the $(r+1)/2$ multipliers $\alpha_0, \beta_0, \cdots, \eta_0$, we shall have $(r+1)/2$ linear algebraic equations. Hence, if we are to show that in every case we shall be able to solve these equations we shall have to show that the determinant of the coefficients of $\alpha_0, \cdots, \eta_0$ in these equations is different from zero. Proving this is equivalent to proving that if the function (34) is even in v up to terms of the $(r+1)$th order, $\alpha_0=\beta_0= \cdots =\eta_0=0$. Evidently, in this proof only the linear terms of F and G will be involved. Hence let us define

$$\begin{aligned} F_0(u,v) &= a_{10}u + a_{01}v, \\ G_0(u,v) &= b_{10}u + b_{01}v, \end{aligned} \tag{35}$$

where F_0 and G_0 are thus the linear terms of F and G, respectively. Hence $a_{10}b_{01}-a_{01}b_{10}$ may be assumed to be 1. Consequently the transformation $\mathfrak{T}_0$, which is defined by

$$(36)\qquad T_0:\quad \begin{aligned} u_1 &= \alpha u + \beta v, \quad \alpha = a_{10}b_{01} + a_{01}b_{10}, \quad \beta = 2a_{01}b_{01}, \\ v_1 &= \gamma u + \alpha v, \quad \gamma = 2a_{10}b_{10}, \end{aligned}$$

and hence is the transformation obtained by using only the first degree terms in the equations T of (26), is the transformation defined by

$$(37)\qquad \begin{aligned} F_0(u_1, -v_1) &= F_0(u,v), \\ G_0(u_1, -v_1) &= -G_0(u,v). \end{aligned}$$

Since we are considering only transformations $\mathfrak{T}$ of the types V and VI in which $\alpha\beta\gamma\neq 0$, by means of a linear change of variables,

$$w = \beta^{1/2}v - \gamma^{1/2}u, \quad z = \beta^{1/2}v + \gamma^{1/2}u,$$

the equations of (36) may be written as

$$(38)\qquad w_1 = \rho w, \quad z_1 = \frac{1}{\rho} z,$$

where now imaginary quantities may occur. For the argument to follow, it is necessary that θ, where $\rho = e^{i\theta}$, be not a rational multiple of 2π.

Any series which is invariant under the transformation (38) must be of the form

$$\sum_r c_r w^r z^r.$$

Hence, any series which is invariant under T_0 must be of the form

$$\sum c_r(\beta v^2 - \gamma u^2)^r,$$

which is the same as

$$(39)\qquad \sum c_r 2^r(b_{10}b_{01}F_0^2 - a_{10}a_{01}G_0^2)^r.$$

This shows first of all that any series that is invariant under T_0 has only terms of even order in u and v; and secondly, it shows that the terms of any degree form an integral power of $(b_{10}b_{01}F_0^2 - a_{10}a_{01}G_0^2)$. Thus any linear combination of terms of even degree in F_0 and G_0 such as

$$(40)\qquad \alpha_0 G_0^{r-1}F_0 + \beta_0 G_0^{r-3}F_0^3 + \cdots + \eta_0 F_0^r,$$

r being an odd integer, can not be invariant under T_0 unless $\alpha_0 = \beta_0 = \cdots = \eta_0 = 0$, which, on account of the definition of T_0 by the equations (36), means that if the function (40) is even in v, $\alpha_0 = \cdots = \eta_0 = 0$, since if it is even in v it is invariant under T_0. Hence the process described above relative to adding functions of F and G to the function (32) to form an invariant series will never fail when we wish to eliminate terms of odd degree in u and v which are odd in v.

Now consider the case when the terms to be eliminated are of even degree in u and v and a series of the form (33) is employed. We see that to determine the $\alpha_0, \cdots, \eta_0$, we shall have to solve a system of $r/2$ linear algebraic equations in the $(r+2)/2$ unknowns $\alpha_0, \cdots, \eta_0$. If we choose $\alpha_0=0$ we shall have $r/2$ equations to solve for $r/2$ unknowns and it is evident from the reasoning for the case when r is odd that the determinant of the coefficients of the unknowns will be always different from zero and that the equations can always be solved. This completes the proof that the process of building up a series which is formally invariant under T when T is of the type V or VI with $\alpha\neq 0$, and the condition following (38) satisfied, never fails.

Thus we may assert that when the transformation $\mathfrak{T}$ is of the type V or of the type VI with $\alpha\neq 0$ and the condition following (38) satisfied, there exists a series, $H^*(u, v)$, whose initial terms are $\beta v^2-\gamma u^2$, which is formally invariant under T.

Let us now examine transformations $\mathfrak{T}$ of the type I(B) for invariant series, and let us use the same general plan that we used when $\mathfrak{T}$ was of the type V or VI. In this case we may choose F and G so that

$$F(u,v) = u + \tfrac{1}{2}v + \cdots,$$
$$G(u,v) = v + \cdots,$$

as has already been shown. Evidently the only quadratic function which is invariant under T up to terms of the third order is a multiple of G^2.

Define $F_0=u+\frac{1}{2}v$, $G_0=v$, and the corresponding transformation $\mathfrak{T}_0$ has as its equations

$$T_0:\quad \begin{aligned} u_1 &= u + v,\\ v_1 &= v,\end{aligned}$$

and the only series which is formally invariant under T_0 is evidently a series in v alone, and hence a series in G_0 alone. Thus, an argument similar to that given when T was of the type V or VI shows that any series of the form

$$\alpha_0 G_0^{r-1}F_0 + \beta_0 G_0^{r-3}F^3 + \cdots + \eta_0 F_0^r, \tag{40}$$

where r is an odd positive integer, is an even function of v only if $\alpha_0= \cdots =\eta_0=0$; and any series of the form

$$\beta_0 G_0^{r-2}F_0^2 + \cdots + \eta_0 F_0^r, \tag{41}$$

where r is an even positive integer, is an even function of v only if $\beta_0= \cdots =\eta_0=0$. Hence, from reasoning similar to that given when $\mathfrak{T}$ was of the type V or VI, it follows that every transformation of the type I(B) possesses a formally invariant series whose initial term is v^2.

Let us now consider transformations of the type II(B). The only series invariant under the corresponding T_0 are series in $-u$, i.e., G_0. Thus it is evident, from the type of reasoning given above, that for any transformation of this type there exists an invariant series whose first and second degree terms are all zero except the one involving u^2, which is not zero.

Now consider the possibility of invariant series for the cases where $\mathfrak{T}$ can be represented by equations of the form I(A), II(A), III or IV. Since series invariant under the corresponding equations T_0 may be series in F_0 only, we see that our method of proving the existence of invariant series fails since a series of the form (40) or (41) may be an even function of v and still not have $\alpha_0 = \cdots = \eta_0 = 0$. However, on making use of the fact that our method does fail, we can actually set up transformations of these types which possess no formally invariant series. This is due to the fact that, for these cases, the determinant of the coefficients in the equations which determine the $\alpha_0, \cdots, \eta_0$ of (33) and (34) may now be zero, and by a proper choice of the coefficients of the F and the G these equations will be inconsistent. Hence we can only say that transformations of the types I(A), II(A), III and IV may or may not have formal invariant series.

We may say by way of summary that *for equations T of the types* I(B), II(B), V *and* VI ($\alpha \neq 0$, *and θ satisfying the condition mentioned after* (38)), *there always exist formal invariant series*; *but for equations of the other types such series need not exist.*

V. Totality of invariant series

For certain types of transformations $\mathfrak{T}$ we have proved the existence of formal invariant series and have given rules for finding particular ones. We now wish to prove that *for any given transformation every formal invariant series is a formal series in powers of a particular one.*

Let us first consider those transformations of types V and VI for which we proved the existence of invariant series. As we have seen, there is no invariant series with any terms of degree less than two. Hence, in every invariant series, the terms of lowest degree are of at least the second degree. Now suppose that the degree of the lowest degree terms of a formal invariant series is $r \geqq 2$, and suppose that these terms do not form a multiple of some power of

$$b_{10}b_{01}F_0^2 - a_{01}a_{10}G_0^2$$

of (32). Since the invariancy of these terms is due entirely to their properties as functions of F_0 and G_0 it follows from the discussion relative to the equations (37), (38) and (39) that we have a contradiction. Hence every formal series invariant for a transformation of the type V, or of the type VI in which $\alpha \neq 0$ and θ satisfies the condition mentioned after the equation (38), is a formal series in the one whose existence was proved in the last section. This is seen as follows. Let us denote the invariant series whose existence was proved in the last section by F^*, and denote any other invariant series by F. Let F_m^* and F_m be the terms of minimum degree in F^* and in F, respectively. Then we have that $F_m = c(F_m^*)^r$ for some positive integer r and some constant c. Then is $F - c(F^*)^r$ an invariant series, and its terms of minimum degree have a degree greater than r and form a multiple of a power of F_m^*. This series may now take the place of F in the argument just given and that argument may be repeated.

Now consider transformations of the type I(B). We have shown that there exists a formal invariant series whose only term of degree two or less is v^2. On the other hand, if there exists a formal invariant series which has terms odd in v, it follows from the argument of the last section that there exists a formal invariant series which contains v as a factor. In this case the line $v=0$ is an invariant curve, which, on account of the fact that the transformation $\mathfrak{T}$ is of the form $\mathfrak{R}\mathfrak{S}$, is a line of invariant points.

Conversely, if $v=0$ is a line of invariant points, the u and the y axes coincide, which implies that the function $G(u, v)$ of (16) has the form

$$G(u,v) = v[1 + G_1(u,v)], \tag{42}$$

where G_1 has no constant term. We wish now to show that in this case every invariant series has v as a factor. An invariant series contains v as a factor if and only if it contains G as a factor since G contains v as a factor, so we shall have proved what we wish to prove if we can show that every invariant series has G as a factor.

We shall prove first of all that there exists an invariant series which is odd in v by using a method similar to that used in proving the existence of invariant series which are even in v. The function $G(u, v)$ is invariant under T up to terms of the second degree and every term contains v as a factor. Now under T, $G(u, v)$ and $F(u, v)$ are transformed according to the equations (31) where $b_{i0}=0$, $i=1, 2, \cdots$. Furthermore, every series in F and G which is odd in G transforms under T as G itself does and contains v as a factor. Hence we wish to add to G a multiple of FG so that the resulting sum up to terms of the third degree is invariant under T, and odd in v; then add a multiple of F^2G so that the result will be invariant under T up to terms of the fourth degree, and will be odd in v; and so on. At no stage do we introduce terms in u alone. When we add terms of the rth degree they are of the form

$$\alpha_0 F^{r-1}G + \beta_0 F^{r-3}G^3 + \cdots + \eta_0 F^{r-r^*}G^{r^*} \tag{43}$$

where r^* is $r-1$ if r is even, and is $r-2$ if r is odd. Hence there are in (43) $r/2$ coefficients to determine if r is even, and $(r-1)/2$ if r is odd. But these coefficients are to be determined so that there will be no terms of degree r in the series which are of even degree in v. Hence there will be $r/2$ linear algebraic equations to determine these coefficients if r is even, and $(r-1)/2$ if r is odd. From the reasoning given in the preceding section, it follows that the determinant of the coefficients of the $\alpha_0, \cdots, \eta_0$ in these equations is always different from zero so that they can be solved. Hence the above process of building up an invariant series having a linear term in v and having v as a factor fails at no stage. Hence there exists such a series which we shall denote by S^*.

Now consider any other series which is invariant under T. There is only one term of minimum degree and that is a multiple of a power of v. This follows at once from the form of the linear terms in the equations representing the transformation, T. Hence this formal invariant series is a formal series in S^*, for if it were not, we could add to it a formal series in S^* and obtain a formal invariant series which has a term of minimum degree containing u only, which is impossible. Hence, *if* $v=0$ *is a curve of invariant points for a transformation of the type* I(B), *every formal invariant series contains* v *as a factor as many times as the degree of its terms of lowest degree.*

Thus we have shown that for transformations of type I(B), if there exists an invariant series which is odd in v, there exists one which is odd in v and has a linear term in v, and has the further property that every other invariant series is a formal power series in this one. On the other hand, if there exists no invariant series which is odd in v, we still have existing one which is even in v and has as initial term v^2. In this case, also, it can be shown that every other invariant series is a power series in this one.

In a similar manner it can be shown that for every transformation of the type II(B) every invariant series is a formal series in the one whose initial term is u^2.

Now let us turn our attention to transformations of the types I(A), II(A), III, IV and VI (cases not included in the argument at the first of the section). As has been noted, a transformation may be of one of these types and not have a formal invariant series. If a transformation is of one of these types and possesses an invariant series, it possesses one whose terms of minimum degree have a degree less than or equal to that of every other invariant series. Then it can be proved by means of methods other than those used above that every formal invariant series is a formal series in one such as has just been described.†

VI. Invariant curves

Let us fix our attention on any particular transformation $\mathfrak{T}=\mathfrak{R}\mathfrak{S}$, and consider any curve C which passes through the origin. It is reflected by $\mathfrak{S}$ in the y axis into a curve C_s and C_s is reflected by $\mathfrak{R}$ in the u axis into a curve $C_{rs}=C_t$. If the curve C is invariant under $\mathfrak{T}$, it is the same as C_t. But C_t and C_s are, in the uv system of coördinates, the reflection images of one another in the u axis. Hence, if $C=C_t$, we see that C and C_s are the reflection images of one another in the u axis in the uv system of coördinates as well as the reflection images of one another in the yx system of coördinates. Hence, if C is invariant under $\mathfrak{T}$, C_s is also, and thus we see that invariant curves occur in pairs, each curve of every pair being the reflection of the other curve of the same pair in the y axis under $\mathfrak{S}$ and in the u axis under $\mathfrak{R}$. We shall call such a pair of invariant curves a "pair of conjugate curves." It may happen, of course, that a pair of conjugate invariant curves consists of only one curve counted twice.

Let us now discuss the possibility of there being curves of invariant points through the origin. If such curves exist, it follows from the equations (24) for the transformation $\mathfrak{T}$ that the coördinates of their points must satisfy the equations

$$F(u,\,-v)=F(u,v),\quad G(u,\,-v)=-G(u,v),$$

which implies that such curves are analytic. The above equations also show that either the origin is an isolated invariant point or there exists a curve of invariant points passing through it.

It is of interest to note that if the points on an invariant curve through the origin are not invariant, they either approach the origin under the transformation $\mathfrak{T}$ or recede from it. If the points on one invariant curve approach the origin under the transformation, the points on its conjugate recede from the origin. But a proof of this latter fact will be omitted since it is not used in any of the discussion to follow.

We now wish to discuss the relationships existing among formal invariant series, formal invariant curves through the origin, and actual invariant curves through the origin. In case there exists a formal invariant series,‡

† B, p. 25. Reference is to the first part of the theorem only. To prove our statements are correct for transformations of the types II(A) and IV, note that every series formally invariant under T is also invariant under T^2.

‡ B, pp. 18–33.

$F(u, v)$, then every real formal curve which makes a factor of this series vanish is a formal invariant curve, and conversely, every formal invariant curve which is not made up of invariant points makes one of the factors of F vanish. That a formal invariant curve consisting of invariant points does not necessarily make a factor of F vanish will be shown to be the case by means of an example in the next section. It will be shown in this same section that there may exist formal invariant curves and yet no invariant series. This illustrates some of the radical differences between transformations of the type discussed in this paper and those of the type discussed by Birkhoff, for when a transformation is represented by equations whose linear terms are those of the type I(A), it possesses an invariant series if it is conservative, and every formal invariant curve, whether made up of invariant points or not, makes one of the factors of this series vanish.

It has been shown,† in an ingenious manner, that to every real formal invariant curve of a conservative transformation of any one of a number of types there corresponds one and only one actual invariant curve, which, moreover, is hypercontinuous at the origin which is the invariant point under consideration, and has the formal invariant curve as its asymptotic representation.

Furthermore, in any sector of a neighborhood of the origin, sufficiently small, which does not contain any invariant curve which has a formal invariant curve as its asymptotic representation at the origin, and which is bounded by two such invariant curves which are not curves of invariant points, no point remains on indefinite iteration of the transformation or of its inverse.

We wish now to outline proofs of analogous statements concerning general transformations whose representing equations are of the form

$$u_1 = u + \cdots, \qquad v_1 = v + \cdots, \tag{44}$$

where the dots indicate terms of degree greater than one. It will be assumed that the equations (44) possess formal invariant series. The formal invariant series used in the discussion is one whose leading terms are of degree not greater than that of the leading terms of every other formal invariant series. It will be recalled that every formal invariant series is a power series in such a one. A comparatively simple case will be treated first in order to illustrate the method, which is a modification of that used by Birkhoff to prove the theorems just mentioned. We shall not consider the case where the formal invariant curves are curves of invariant points since such curves are analytic.

† B, Chapter II.

We shall suppose that the real formal invariant curve under consideration is regular at the origin, i.e., can be represented in the form u = a power series in v, or in the form v = a power series in u. Let us suppose that our curve is given in the latter form and that its equation is

$$v = p(u) = a_1 u + a_2 u^2 + \cdots . \tag{45}$$

If we perform the formal change of coördinates

$$u^* = u, \qquad v^* = v - p(u), \tag{46}$$

the formal curve (45) is taken as the u^* axis and the equations of the transformation are reduced to the form

$$\begin{aligned} u_1^* &= u^* + \cdots , \\ v_1^* &= v^*(1 + \cdots), \end{aligned} \tag{47}$$

and the transform of the formal invariant series used has v^* as a factor, i.e.,

$$F(u,v) \sim v^*(\cdots). \tag{48}$$

In the series on the right hand side of the first equation of (47) let $au^{*\nu}$ be the first term, other than u^*, which does not contain v^* as a factor. There is such a term, for otherwise $v^* = 0$ is a formal curve of invariant points which is contrary to supposition. For definiteness in argument we shall suppose that $a > 0$.

There are two distinct cases to consider, the first being where the formal invariant series of (48) in u^* and v^* contains a term containing u^*, and the other the case where the formal invariant series contains no such term. We shall now consider the first case.

Let the equations† which represent the kth iterate of this transformation be

$$u_k^* = \sum_{m+n=1}^{\infty} \overset{(k)}{\phi}_{mn} u^{*m} v^{*n},$$

$$v_k^* = \sum_{m+n=1}^{\infty} \overset{(k)}{\psi}_{mn} u^{*m} v^{*n},$$

and define

$$\delta u^* = \left.\frac{\partial u_k^*}{\partial k}\right|_{k=0}, \quad \delta v^* = \left.\frac{\partial v_k^*}{\partial k}\right|_{k=0}, \tag{49}$$

whence δu^* and δv^* are formal series in u^* and v^*. Now the formal series F satisfies the equation

† B, p. 10.

$$\frac{\partial F}{\partial u^*}\delta u^* + \frac{\partial F}{\partial v^*}\delta v^* = 0,$$

and the first term in δu^* which is independent of v^* is au^{*p}.

Let F contain v^{*l} and no higher power of v^* as a factor. Then $\partial F/\partial u^*$ can be factored into the form

$$\frac{\partial F}{\partial u^*} = v^{*l}(u^{*k-1} + A_1 u^{*k-2} + \cdots + A_{k-1})(c_1 + \cdots),$$

where c_1 is a constant different from zero and the A_i are zero for $v^* = 0$. Using analogous notations we have

$$\frac{\partial F}{\partial v^*} = v^{*l-1}(u^{*k} + \cdots)(c_2 + \cdots),$$

$$\delta u^* = (u^{*p} + \cdots)(c_3 + \cdots),$$

$$\delta v^* = v^{*s}(u^{*q} + \cdots)(c_4 + \cdots).$$

From the differential equation displayed after the equations (49) we have immediately that there is a term in δv^* of the form cv^*u^{*p-1} where $ac < 0$, and there is no term in δv^* of lower degree in u^* which does not contain at least v^{*2}. But this implies that the series for v_1^* contains the term cv^*u^{*p-1} and there is no other term of this series of the form dv^*u^{*m} where $m < k-1$.

The formal invariant series contains a term $bv^{*l}u^{*k}$. The change of coördinates (46) is not analytic, but if we replace the series $p(u)$ by $p_n(u)$ where n is arbitrarily large and larger than $l+k+2p$, the $p_n(u)$ being the sum of the first n terms of $p(u)$, the new change of coördinates is analytic and the new equations of the transformation agree with the formal equations (47) out to terms of the nth degree. Assuming such a transformation has been made, dropping the asterisks for simplicity in notation, we may write the equations of our transformation, and the formal invariant series, as

$$\begin{aligned} u_1 &= [u + \cdots + au^p + \cdots] + \cdots, \\ v_1 &= v[1 + \cdots + cu^{p-1} + \cdots] + \cdots, \end{aligned} \tag{50}$$

$$F = v^l[\cdots + bu^k + \cdots] + \cdots, \tag{51}$$

where the expression inside the brackets in the first equation of (50) is a polynomial of degree n at most, those inside the other brackets in (50) and (51) are polynomials of degree $n-1$ at most, and all the terms outside the brackets are of degree at least $n+1$.

We have now the hypotheses necessary for the application of the results of Birkhoff. We shall quote them without going into the details of the proofs.

There exists a positive integer n_0 such that for $n>n_0$ there exists a constant d and a closed set Σ of curves defined for u sufficiently small and positive, analytic for $u>0$, which is invariant under $\mathfrak{T}$ and lies within the region defined by

$$0 < u < d, \qquad |v| < Eu^{n-n_0},$$

where E is a positive constant depending on n and d. Define $m=n-n_0$, which is greater than p due to the choice of n. These curves have contact of order at least m at the origin with the u axis as chosen in (50). Now consider the region which is bounded by these curves of Σ and which contains all of them. The upper and lower boundaries of this region are invariant curves for the transformation under consideration and have an order of contact of at least m at the origin with the u axis. We wish now to show that these are one and the same curve, and hence, that Σ consists of only one curve.

Since the distance of any point (u, v) on either of these boundary curves from the u axis is of the order $|u|^m$, by making a transformation which takes one of them and its analogue for $u<0$ into the new u axis, we may write the transform of (50) as

$$\begin{aligned} u_1 &= [u + au^p] + \phi_1(u,v), \\ v_1 &= v\{[1 + cu^{p-1}] + \phi_2(u,v)\}, \end{aligned} \tag{52}$$

where $\phi_1(u, v)=O(u^q)$ and $\phi_2(u, v)=O(u^{q-1})$, for some positive integer $q>p$ if $|v|<c_0|u^p|$, where c_0 is an arbitrary constant. From these facts and the equations (52) it follows that for u small enough in absolute value, and $|v|<c_0|u^p|$, the ratio $(v_1-v)/(u_1-u)$ is negative if u and v are positive, positive if u is positive and v is negative, etc., since $ac<0$. This shows that there can not be another invariant curve passing through the origin in this restricted region, besides that which we have chosen as our new u axis. Hence Σ consists of exactly one curve and is analytic except possibly at the origin.

But this u axis has contact of at least the mth order with the original one used in (50). Hence any curve which has contact of order greater than m with the u axis of (50) has contact of order m at least with the u axis as chosen in the equations (52), and, since $m>p$, must lie in the above mentioned region for u small enough in absolute value, and for some value of c. Hence the invariant curve that we have shown is the only curve of Σ, is analytic for $(u, v)\neq(0, 0)$, and has contact of at least order m with the u axis of the equations (50), is that which will be obtained if we choose higher and higher values for n mentioned in connection with the equations (50). This completes the proof that *if F of* (48) *contains a term having u^* as a factor, to each real formal invariant curve which is regular at the origin there corresponds exactly* *one actual invariant curve which has this formal invariant curve as its asymptotic representation at the origin.*

Let us now consider what can be said about the series for v_1 in case the formal invariant series F in (48) contains no term in u^*. In this case there is only one real formal invariant curve which is not a curve of invariant points, and when that has been transformed formally into the u^* axis the expression for v_1^* is a power series in v^* alone. But if the series F has no term involving u^* its term of minimum degree is a multiple of a power of v^*, and, since the degree of this term is not greater than that of the corresponding term in every other formal invariant series, it is one. But if there is a formal invariant series whose leading term is v^* which contains no term involving u^*, there is one whose only term is v^*. Hence there is a formal invariant series for the transformation represented by the equations (44) of the form $v-p(u)$ where $p(u)$ is a power series in u.

Let us now recall that we intend to apply the discussion of this section to transformations of the type $\mathfrak{T}=\mathfrak{R}\mathfrak{S}$ where $\mathfrak{R}$ and $\mathfrak{S}$ are each of the reflection type, and we shall assume that the coördinate system used is a normal one for the transformation $\mathfrak{R}$. Under these assumptions, if there exists a formal invariant series of the form $v-p(u)$ there exists one of the form $v+p(u)$ which is implied by the discussion of Section IV, and this implies that v is an invariant function which, in turn, implies that the second equation of (44) has the form $v_1=v$. In this case $v=0$ is an invariant curve. Hence, *when a transformation $\mathfrak{T}=\mathfrak{R}\mathfrak{S}$ is represented by equations of the form* (44) *where the uv system of coördinates is normal for $\mathfrak{R}$, and if there is a formal invariant series for this transformation which can be formally reduced to a formal series in v^* alone by means of a change of variables of the form* (46), *$v=0$ is an invariant curve and, on account of the fundamental properties of $\mathfrak{T}$ as the product of $\mathfrak{R}$ and $\mathfrak{S}$, it follows that $v=0$ is a curve of invariant points.*

Now let us consider the transformations whose representative equations have the form (44) but whose formal invariant curves have "cusps" at the origin. By use of transformations of the type

$$u = u^*v^*, \qquad v = v^*, \tag{53}$$

such formal invariant curves are taken into ones of the first type considered in the above argument, i.e., the reduced equations representing the transformation have not a formal invariant series of the form $v^*-p(u^*)$. This is due to the fact that on account of the nature of the change of variables (53) there is one formal invariant series F^* which contains either u^* or v^* as a factor, and to the fact that, according to Section V, every other formal invariant series is either a root or a formal power series in F^*.

Hence, *if there exists a real, formal, invariant series, to every real formal invariant "cusp," there corresponds one and but one real invariant curve which has this formal invariant curve as its asymptotic representation.*

Finally, concerning the stability of the invariant point $(u, v)=(0, 0)$ under a transformation represented by equations of the form (44), it may be said that *in case a formal invariant series exists, in any sector of a sufficiently small neighborhood of the origin, which does not contain any invariant curve which has a formal invariant curve as its asymptotic representation at the origin, and which is bounded by two such invariant curves which are not curves of invariant points, no point remains on indefinite iteration of the transformation or of its inverse.* The proof of this statement is omitted here since it is almost exactly that given in the discussion of conservative transformations which has been mentioned before.

VII. Transformations of type I(A)

One of the chief differences between conservative transformations and those of the type studied in this paper is that for conservative transformations† of type II$''$, which correspond to transformations of the type I(A), there always exist real formal invariant series, while such series need not exist for transformations of the type I(A). Due partially to the fact that formal invariant series need not exist, a wide variety of situations occur which are interesting in comparison with the facts known concerning conservative transformations of the type II$''$. These situations will now be studied in some detail, while the analogous ones which arise for the other types of transformations for which there may not be any formal invariant series will hardly be mentioned since the discussion for type I(A) will be typical of that which could be given for them too.

We shall confine our attention to a neighborhood of the origin of coördinates which is supposed to be an invariant point. Questions which have to be answered are those relative to the existence of invariant curves through the origin, existence of invariant integrals, existence of invariant series, and stability.

Let us first note that in Section III we have already shown that all directions through the origin are invariant under a transformation of the type I(A). We shall now show that *for transformations of type* I(A), (i) *the existence of a formal invariant integral implies the existence of a formal invariant series*; (ii) *the existence of a formal invariant series does not imply the existence of a formal invariant integral*; (iii) *the existence of invariant curves does not imply the existence of formal invariant series*; *and* (iv) *the existence*

† See last of Section II.

of formal invariant series which possess real factors† *implies the existence of real invariant curves through the origin.* The proof of (i) is already in the literature.‡

We shall prove (ii) by means of an example. Consider the transformation defined by the following equations:

$$(54)\qquad R:\ \begin{array}{l} u_1 = u, \\ v_1 = -v; \end{array} \qquad S:\ \begin{array}{l} u_1 = u\dfrac{1-v}{1+v}, \\ v_1 = -v; \end{array}$$

$$T = RS:\ \begin{array}{l} u_1 = u\dfrac{1-v}{1+v}, \\ v_1 = v. \end{array}$$

Each of the transformations defined by the equations R and S is of the reflection type and it is evident that any series in v is formally invariant under T. We need only show that there exists no quasi-invariant series. Evidently

$$\frac{\partial(u_1,v_1)}{\partial(u,v)} = \frac{1-v}{1+v}.$$

Now if $Q(u, v)$ is a quasi-invariant series we have

$$Q(u,v) = Q(u_1,v_1)\frac{\partial(u_1,v_1)}{\partial(u,v)},$$

and we must remember that $Q(0, 0)$ is different from zero. Let

$$Q(u,v) = q_{00} + q_{10}u + q_{01}v + \cdots.$$

Then

$$\begin{aligned} Q(u_1,v_1)\frac{\partial(u_1,v_1)}{\partial(u,v)} &= (q_{00} + q_{10}u_1 + q_{01}v_1 + \cdots)\left(\frac{1-v}{1+v}\right) \\ &= q_{00} + q_{10}u + (q_{01} - 2q_{00})v + \cdots. \end{aligned}$$

Since $q_{00}\neq 0$, this equation shows that no formal quasi-invariant series Q can exist for the transformation represented by (54).

We shall now show by an example that a transformation of the type I(A) may possess an invariant curve through the origin and yet possess no formal invariant series. Consider the transformation for which the equations (16) are

$$(55)\qquad y = u + v^3, \quad x = v + u^3.$$

† B, p. 18.

‡ B, p. 16. In this paper, the assumption is made that the transformation possess an actual invariant integral, but for the proof of (i) no use is made of the fact that the quasi-invariant series Q is convergent.

Evidently

$$x^2 - y^2 = (v^2 - u^2)(1 - 2uv - v^4 - v^2u^2 - u^4).$$

Hence $x^2-y^2=0$ is equivalent to $v^2-u^2=0$. But the curves $x^2-y^2=0$ are the images of one another under $\mathcal{S}$ and the same curves $v^2-u^2=0$ are the images of one another under $\mathcal{R}$. Hence the curves $x^2-y^2=0$, which are the same as the curves $v^2-u^2=0$, are a pair of conjugate invariant curves through the origin for the transformation $\mathcal{T}=\mathcal{R}\mathcal{S}$.

Now turn to the equations (55) and investigate the possibility of the existence of a formal invariant series. We have seen that if such a series exists at all, there exists one that is even in $v+u^3$ and which, when expanded in terms of u and v, is even in v. Furthermore, if there exists such a series, there exists one whose lowest degree terms are of even degree in $(v+u^3)$ and $(u+v^3)$. Let the lowest degree terms of such a series be

$$a_r(v + u^3)^r + a_{r-2}(v + u^3)^{r-2}(u + v^3)^2 + \cdots + a_0(u + v^3)^r, \tag{56}$$

where r is an even positive integer. In this series (56) there exists no term of degree $r+1$ in u and v. Furthermore, in our invariant series, when the terms of degree $r+1$ in $(v+u^3)$ and $(u+v^3)$ are expanded, there will be no terms of degree $r+2$. Hence the terms of (56) which are of degree $r+2$ in u and v but odd in v have to be counterbalanced as in Section IV, by the addition of a sum of the form

$$b_0(v + u^3)^{r+2} + b_2(v + u^3)^r(u + v^3)^2 + \cdots + b_{r+2}(u + v^3)^{r+2}. \tag{57}$$

But there exist no terms in this series of degree $r+2$ which are of odd degree in v, and since the terms of (56) of degree $r+2$ which are of odd degree in v are

$$\begin{array}{lllll}
 & ra_rv^{r-1}u^3 & & & \\
+\ 2a_{r-2}v^{r-1}u & & +\ (r-2)a_{r-2}v^{r-3}u^5 & & \\
 & +\ 4a_{r-4}v^{r-1}u^3 & & +\ (r-4)a_{r-4}v^{r-5}u^7 & \\
 & & +\ 6a_{r-6}v^{r-3}u^5 & & +\ \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots
\end{array}$$

it is evident that $a_{r-2}=a_{r-6}=a_{r-10}= \cdots =0$ and $ra_r+4a_{r-4}=0$, $(r-4)a_{r-4}+8a_{r-8}=0, \cdots$. But since $a_{r-4\alpha}=0$ where 4α is the largest multiple of 4 which is not greater than r, we see that each of this last set of a_i is also zero, and hence there can exist no invariant series, although there exist analytic invariant curves through the origin.

In connection with transformations of the type I(A) it is of interest to know when there are curves of invariant points through the origin. Let the equations (16) be denoted by

$$y = u + f(u,v), \quad x = v + g(u,v),$$

where f and g are convergent series in u and v which have no terms of degree less than two. Then the transformation $\mathfrak{T}$ is represented by the equations

$$u_1 + f(u_1, -v_1) = u + f(u,v),$$
$$-v_1 + g(u_1, -v_1) = -v - g(u,v).$$

The points that are invariant under $\mathfrak{T}$ are given by

$$u + f(u, -v) = u + f(u,v), \qquad -v + g(u, -v) = -v - g(u,v),$$

i.e., $f(u,-v)=f(u, v)$, $g(u,-v) = -g(u, v)$. This means that f is an even function of v and g is an odd function of v for (u, v) an invariant point. Hence, if f and g are such that for curves passing through the origin these equations are satisfied, these curves are curves of invariant points.

We have given examples of transformations under which certain curves passing through the origin are invariant. On the other hand, there exist transformations of the type I(A) under which no curve through the origin is invariant. An example of such a one is the transformation for which the equations (16) are

$$y = u + v^3, \quad x = v - u^3,$$

but we shall not pause to prove this fact.

In the study of conservative transformations the fact was discovered that, for each transformation of type II'', every invariant curve that passes through the origin and is hypercontinuous there has an asymptotic expansion at the origin which formally makes every formal invariant series zero, and formal invariant series always exist. We have already shown that for the analogous transformations of type I(A) there may exist analytic invariant curves through the origin and yet no formal invariant series; now we shall give an example of a transformation of type I(A) which has an invariant series and an analytic invariant curve through the origin along which the series is not formally zero. This invariant curve is necessarily a curve of invariant points.†

Consider again the transformation represented by

$$u_1 = u(1 - v)/(1 + v), \quad v_1 = v.$$

† B, pp. 31–32.

Any formal series in v is a formal invariant series. Yet $u=0$ is a line of invariant points and u is not a factor of any formal invariant series.

Finally, in connection with the questions relative to actual invariant curves corresponding to real factors of a formal invariant series, and questions of stability, we shall merely draw attention to the fact that the discussion of Section VI is applicable in this one.

VIII. Transformations of type I(B)

We have already shown that for such transformations there always exists a formal invariant series which is even in v, whose only term of degree less than three is v^2. Hence this formal invariant series can be factored in the form

$$(v^2 + f(u))(1 + \cdots)$$

where $f(u)$ is a formal series in u whose term of lowest degree we shall denote by au^p in case $f(u)$ is not identically zero. As has already been seen in Section IV, $f(u)$ is identically zero if and only if $v=0$ is a line of invariant points.

On the other hand, if $f(u)$ is not identically zero, $v^2+f(u)=0$ represents a real, formal, invariant curve with a "cusp" at the origin if p is odd, and two real, formal, invariant curves tangent at the origin if $a<0$ and p is even. If $a>0$ and p is even, there is no real, formal, invariant curve through the origin.

A change of coördinates of the type $u=u^*$, $v=u^*v^*$ reduces the equations representing a transformation of the type I(B) to equations of the type (44), and hence the methods of Section VI show that, when there is a formal invariant "cusp," there is a unique actual invariant curve having this formal "cusp" as its asymptotic representation at the origin, and when there are two real formal invariant curves through the origin there are exactly two actual invariant curves tangent to the u axis at the origin and having these formal curves as their asymptotic representations.

Furthermore, when invariant curves through the origin exist, the origin is an unstable hyperbolic point, and the argument at the end of Section VI shows that if a sufficiently small neighborhood of the origin be chosen, points not on the invariant curves in this neighborhood do not remain in it on iteration of the transformation or of its inverse.

With transformations of the type I(B) there are connected formal differential equations of the type (49) which were used in the discussion of equations of the type (44). But for such a transformation the highest common factor of δu and δv is v, and there is a curve of invariant points only if there exists an invariant series having v^2 as a factor. Hence,† transformations of the type I(B) are formally conservative.

† B, p. 19 et seq.

IX. Transformations of types II(A) · · · IV

For all these transformations, except those of type II(B), there may or may not exist formal invariant series. For transformations of the type II(B), a discussion analogous to that given for transformations of type I(B) can be given. It may be noted, however, that in this case there can be no curve of invariant points through the origin. There may, however, exist a curve through the origin which is invariant under the transformation and which consists of points all of which are invariant under $\mathfrak{T}^2$.

As has already been mentioned, a discussion similar to that given for transformations of the type I(A) may be given here for each of the other transformations named in this section. This we shall not do, but shall content ourselves with merely pointing out a few of the main distinctions.

For transformations of the type II(A) the origin is an isolated invariant point, and the points on every invariant curve through the origin transform under $\mathfrak{T}$ into points, each of which is separated on the invariant curve from its image by the origin. This is a direct consequence of the form of the linear terms in the equations for T in (25). The discussion of Section VI is evidently applicable to the second iterate of a transformation of the type II(A).

If a transformation is of the type III, the origin is an isolated invariant point unless there is a single curve of invariant points through it. This is shown as follows. Let the equations (16) be denoted by

$$y = u + f(u,v), \quad x = \tfrac{1}{2}u + v + g(u,v),$$

where all the terms of f and g are of degree higher than two. The invariant points are given by

$$f(u, -v) = f(u,v), \quad -u = g(u, -v) + g(u,v).$$

The form of these equations at once proves that the statement above is correct.

The origin, for transformations represented by equations of the types II(B), III and IV of (25), is an unstable invariant point, and may be either hyperbolic or elliptic.

X. Transformations of type V

If a transformation is of type V, by means of a real linear change of variables, equations representing it may be chosen which have the form

$$(58) \quad u_1 = \rho u + \cdots, \quad v_1 = (1/\rho)\, v + \cdots, \quad \rho = \alpha \pm (\alpha^2 - 1)^{1/2} \neq 0,$$

where now the uv system of coördinates is not normal for $\mathfrak{R}$. Any transformation represented by equations of this form in which the right hand members are analytic possesses† two real analytic invariant curves through the origin whose equations may be written in the form $u=f(v)$, $v=g(u)$. When these curves are taken as axes, it can be readily shown that the points on one of these curves in a neighborhood of the origin move into the origin on iteration of the transformation, while the points on the other curve move away from the origin. In the case where $\rho<0$ the origin always lies between a point and its image when these points lie on one of the invariant curves through the origin.‡ Furthermore, a neighborhood of the origin can be taken so small that any point in it not on one of these invariant curves through the origin moves out of it on iteration of the transformation or of its inverse. Hence, transformations of the type V are hyperbolic and unstable.

We now wish to point out the interesting facts that *there is a formal equivalence existing between transformations of the type* V *and conservative transformations of the types* I′ *and* I″. We shall first prove that every transformation of the type I′ or of the type I″ is formally of the type V and hence is formally the product of two transformations each of the reflection type.§ Now the equations representing a conservative transformation of the type I′ or of the type I″, by a formal change of variables, may be given either the form¶

$$U_1 = \rho U e^{cU^lV^l}, \quad V_1 = \frac{1}{\rho} V e^{-cU^lV^l}, \quad c \neq 0, \tag{59}$$

or the form

$$U_1 = \rho U, \quad V_1 = \frac{1}{\rho} V. \tag{60}$$

But the equations (59) are the product RS of the two systems

$$R\colon \begin{aligned} U_1 &= \rho V e^{cU^lV^l}, \\ V_1 &= \frac{1}{\rho} U e^{-cU^lV^l}; \end{aligned} \qquad S\colon \begin{aligned} U_1 &= V, \\ V_1 &= U, \end{aligned}$$

each of which is of the "reflection" type. Evidently the equations (60) are also the product of two systems each of the "reflection" type.

We shall now prove that, conversely, the equations representing a transformation of type V are, by a formal change of variables, equivalent to equations of the form (59) or (60), and that every transformation of type V is

† Poincaré, Œuvres, vol. I, p. 202.

‡ Cf. Section III.

§ This fact was kindly pointed out to the author by Professor Birkhoff.

¶ B, pp. 34, 55.

formally conservative. Let us assume that the equations representing the transformation of type V under consideration are in the form (58). From our previous discussion we know that there exists a formal invariant series whose only term of degree less than three is cuv where c is a constant different from zero. When $\rho > 0$, we know that there exists† a formal quasi-invariant series and hence that our transformation is formally conservative and that its representative equations may be formally reduced to the form (59) or to the form (60). When $\rho < 0$, the equations for $\mathfrak{T}^2$ are of the same form as (58) with ρ replaced by ρ^2, and hence represent a formal conservative transformation. Hence they can be reduced to the form (59) or to the form (60) and hence‡ the transformed equations of $\mathfrak{T}$ have the same form.

XI. Transformations of type VI

Transformations of type VI where ρ satisfies the condition mentioned after the equations (38) are formally conservative, and transformations which are conservative and of the type II′ are formally the product of two transformations each of the reflection type. We shall not give proofs of these statements since they are wholly similar to those of the analogous statements made in the previous section.

On account of the form of the initial terms in the invariant series which we proved to exist for transformations of this type, it is evident that no formal invariant curves exist through the origin. In fact, the discussion given in Section III shows that there can exist no invariant curve of any kind through the origin since no slope through the origin is invariant. Hence the origin is an elliptic invariant point which raises many important and interesting questions. It will have been observed that the questions answered in the latter part of this paper have been, to a great extent, concerned with hyperbolic invariant points. The author hopes to study the elliptic cases in the near future.

† B, pp. 19–20.
‡ B, p. 55.

Cornell University,
Ithaca, N. Y.