ON THE MAXIMUM ABSOLUTE VALUE OF THE DERIVATIVE OF \( e^{-x^2}P_n(x) \)

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A remarkable theorem due to S. Bernstein\(^\dagger\) asserts that if \( L \) is the maximum absolute value of an arbitrary polynomial \( P_n(x) \) of degree \( n \) in the interval \((a, b)\) then the maximum absolute value of the derivative \( P_n'(x) \) does not exceed \( nL[(b-x)(x-a)]^{-1/2} \) on \((a, b)\). A related theorem for trigonometric sums states that if \( L \) is the maximum of the absolute value of a trigonometric sum of order \( n \), then the maximum absolute value of its derivative does not exceed \( nL \).\(^\ddagger\)

A similar theorem is here given for the function \( e^{-x^2}P_n(x) \), where \( P_n(x) \) is an arbitrary polynomial of degree \( n \).

**Theorem.** If \( L \) is the maximum absolute value of \( e^{-x^2}P_n(x) \) in the interval \(-\infty < x < \infty\), then the maximum absolute value of the derivative is less than \( n^{1/2}L[1.09514+O(n^{-1})] \) in the infinite interval.

It is convenient to establish the corresponding result for functions of the form \( f_n(x) = e^{-x^2/4}P_n(x) \) and then to obtain the stated theorem by the change of variable \( x = 2x' \). The proof follows the line of attack adopted by de la Vallée Poussin, and is accomplished with the aid of the following propositions.

I. If \( f_n'(x) \) attains its maximum absolute value at \( x_0 \), then

\[
x_0^2 < 2k(n+1),
\]

where \( k \) is a constant which may be taken as 3.69264.

II. There exists an analytic function \( \psi_m(x) \), where \( 4m+2 > 2k(n+1) \), such that

(a) \( \psi_m'(x) \) has an extremum equal to \( f_n'(x_0) \) at \( x = x_0 \);

(b) \( \psi_m(x) \) becomes infinite at \(-\infty\), at \( +\infty\), and has \( m+1 \) extrema, with alternating signs at these \( m+3 \) points (counting \( \pm \infty \));

(c) the least extremum of \( \psi_m(x) \) is greater in absolute value than

\[
|f_n'(x_0)| \left[ m + \frac{1}{2} - x_0^2/4 \right]^{1/4} \left[ m + \frac{1}{2} \right]^{-3/4} \left[ 1 + O(m^{-1}) \right].
\]

\(^\dagger\) Presented to the Society, June 20, 1930; received by the editors March 24, 1930.

\(^\ddagger\) S. Bernstein, *Sur l'ordre de la meilleure approximation des fonctions continues par des polynomes de degre donne*, Mémoire Couronné, Brussels, 1912.

III. If
\[ R(x) = \psi_m(x) - f_n(x), \]
then

(a) \( R'(x) \) has a double root at \( x=x_0; \)

(b) \( R'(x) \) has not more than \( m+1 \) roots in the interval \( -\infty < x < \infty. \)

Before demonstrating these propositions let us show how they establish
the theorem. Let the maximum of \( |f_n(x)| \) and of \( |f_n'(x)| \) be \( L \) and \( M \)
respectively, and suppose if possible that the least maximum of \( |\psi_m(x)| \) is
greater than \( L. \) Then \( R(x) \) has the sign of \( \psi_m(x) \) at \( -\infty, \) at \( +\infty \) and at \( m+1 \)
intermediate points. Because of the alternation of signs at these \( m+3 \)
points, \( R(x) \) has at least \( m+2 \) distinct roots and \( R'(x) \) has at least \( m+1 \)
distinct roots. Therefore, by III(a), \( R'(x) \) has at least \( m+2 \) roots. But this
contradicts III(b), and hence \( L \) cannot be less than the least maximum of
\( |\psi_m(x)|. \) Consequently, by III(c),
\[ L > M \left[ m + \frac{1}{2} - x_0^2/4 \right]^{1/4} \left[ m + \frac{1}{2} \right]^{-3/4} \left[ 1 + O(m^{-1}) \right]. \]

Now, \( x_0^2 < 2k(n+1), \) and we shall choose \( m \) as an integer in such a manner that
\[ m = \frac{3}{2}kn \left[ 1 + O(n^{-1}) \right]. \]

With this value of \( m \) and the value of \( k \) given in I the inequality
\[ M < n^{1/4}L \left[ 2.19018 + O(n^{-1}) \right] \]
follows, from which the inequality of the theorem is derived by the change of
variable \( x = 2x'. \)

We turn now to the proof of I. If \( L \) denotes the maximum of \( |e^{-x^2/4}P_n(x)|, \)
then in the interval where \( x^2 \leq 2n \)
\[ |P_n(x)| \leq L \left| 2ex^2/n \right|^{n/2}, \]
and consequently
\[ |e^{-x^2/4}P_n(x)| \leq L(2e/n)^{n/2} \left| x \right|^{n}e^{-x^2/4}, \]
for \( x^2 > 2n. \) When \( x > (2n)^{1/2} \) the function \( e^{-x^2/4}x^n \) is decreasing, so that when
\( x^2 > 2kn \) we merely strengthen the inequality in replacing \( x^2 \) on the right

hand side by \(2kn\). We then find upon calculating the value of the right hand expression with the given value of \(k\) that

\[
| e^{-x^2/4}P_n(x) | < L
\]

when \(x^2 > 2kn\). Since the derivative of \(e^{-x^2/4}P_n(x)\) is \(e^{-x^2/4}\) times a polynomial of degree \(n+1\), the proof of I is completed.

To establish II we let \(w_1\) and \(w_2\) be two solutions of

(1) \[ d^2w/dx^2 + [m + \frac{1}{2} - x^2/4]w = 0, \]

with the initial values

\[
w_1 = (m + \frac{1}{2})^{-1/4}, \quad w_2 = 0, \\
w_1' = 0, \quad w_2' = (m + \frac{1}{2})^{1/4},
\]

at \(x=0\), and express the solution of (1) in the form

(2) \[ w_m(x) = [w_1^2 + w_2^2]^{1/2} \cos [\psi(x) - \theta], \]

where

\[
\psi(x) = \int_{-\infty}^{x} [w_1^2 + w_2^2]^{-1}dx.
\]

As \(\theta\) increases the extrema of \(w_m(x)\) move continuously to the right\(^*\) in the interval \(- (4m+2)^{1/2} < x < (4m+2)^{1/2}\), so that if we select \(m\) as an integer such that

(3) \[ 4m + 2 > 2k(n + 1) \]

it is clearly possible to choose \(\theta\) so that an extremum of \(w_m(x)\) occurs at \(x = x_0\), since, by I, \(|x_0| < (2kn)^{1/2}\).†

Corresponding to a given integral value of \(m\) there is a single critical value of \(\theta\), \(0 \leq \theta < \pi\), for which \(w_m(x)\) vanishes at \(\pm \infty\), while for all other values of \(\theta\), \(w_m(x)\) becomes infinite at \(\pm \infty\).† We desire to construct a function that will always become infinite at \(\pm \infty\) and therefore, if the \(\theta\) chosen above should prove to be critical, we shall take a new \(m\) equal to the original \(m\) increased by unity. Since the critical function is

\[ Ce^{-x^2/4}H_m(x), \]

where \(H_m(x)\) is an Hermitian polynomial, we see from the known properties of \(H_m(x)\) and \(H_{m+1}(x)\) that if \(\theta\) is critical for \(m\) it will not be so for \(m+1\). By this arrangement we are sure that \(w_m(x)\) will always become infinite at \(\pm \infty\).

\(^*\) The proof is similar to that for the behavior of the roots. See W. E. Milne, these Transactions, vol. 30 (1928), pp. 797–802, especially p. 800, formula (16).

The function \( \psi_m(x) \) is now defined as follows:

\[
\psi_m(x) = \left[ f'(x_0)/w_m(x_0) \right] \int_a^x w_m(s) \, ds,
\]

in which \( a \) denotes the abscissa of the extremum of \( w_m(x) \) nearest the origin (or one of the two nearest). It is clear from (4) that II(a) and III(a) are verified.

Next consider the roots of \( w_m(x) \). When \( \theta \) is critical \( w_m(x) \) becomes \( Ce^{-x^{1/4}H_m(x)} \) and is known to have exactly \( m \) real distinct roots. As \( \theta \) increases each root moves continuously to the right, no root is gained or lost in the finite interval, but a new root appears at \( -\infty \). Hence, for non-critical values of \( \theta \), \( w_m(x) \) has exactly \( m+1 \) real distinct roots. Therefore \( \psi_m(x) \) has \( m+1 \) distinct extrema, and obviously becomes infinite at \( \pm \infty \).

Finally it is known that the amplitudes of the oscillations and the intervals between the roots of \( w_m(x) \) increase as \( x \) recedes from the origin, so that the areas bounded by the successive arches increase. This assures us of the alternation in sign of \( \psi_m(x) \) at the extrema and at \( \pm \infty \), and completes the proof of II(b).

The proof of II(c) is easily effected with the aid of (2) and (4) and certain inequalities previously established.*

Finally, to prove III(b) we note that

\[
e^{x^{1/4}}R'(x) = \psi_m(x) + P_n'(x) - xP_n(x)/2,
\]

where \( \psi_m(x) \) is a solution of the differential equation

\[
v'' - xv' + mv = 0.
\]

Differentiating this equation \( m \) times we get

\[v^{(m+2)} - xv^{(m+1)} = 0,\]

whence

\[v^{(m+1)} = Ce^{x^{1/2}}.\]

The value \( C = 0 \) gives the critical solution, hence \( C \neq 0 \). Therefore in view of the fact that \( m > n + 1 \) because of (3)

\[
(d^{m+1}/dx^{m+1})(e^{x^{1/4}}R'(x)) = Ce^{x^{1/2}} \neq 0,
\]

which shows that \( R'(x) \) has not more than \( m + 1 \) roots.

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* W. E. Milne, these Transactions, vol. 31, pp. 907–918. See pp. 909–910, formulas (8) to (15).

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