A PROOF OF THE GENERALIZED SECOND-LIMIT THEOREM IN THE THEORY OF PROBABILITY*

BY

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Introduction. A function \( F(x) \), defined for all real \( x \), will be called a "law of probability," if the following conditions are satisfied:

(i) \( F(x) \) is monotone non-decreasing in \((-\infty, \infty)\) and continuous to the left,

(ii) \( F(-\infty) = 0, \ F(\infty) = 1. \)

A particular case is represented by \( dF(x) = f(x)dx \), where \( f(x) \), summable and \( \geq 0 \), is the "probability density" or "law of distribution" for \( x \).

The expression \( \int_{-\infty}^{\infty} x^s dF(x) \) is called the "\( s \)th moment" of the distribution, \( s \) taking values \( 0, 1, 2, \ldots \).

The Second Limit-Theorem, which was the starting point of this paper, can be stated, with A. Markoff,‡ as follows:

If a sequence of laws of probability \( F_k(x) (k = 1, 2, \cdots) \) is such that they admit moments of all orders, and if

\[
\lim_{k \to \infty} \int_{-\infty}^{\infty} x^s dF_k(x) = \pi^{-1/2} \int_{-\infty}^{\infty} x^s e^{-x^2} dx \quad (s = 0, 1, \cdots),
\]

then, for all \( x \),

\[
\lim_{k \to \infty} \int_{-\infty}^{x} dF_k(x) = \pi^{-1/2} \int_{-\infty}^{x} e^{-x^2} dx.
\]

Markoff's proof is rather complicated, being based on the distribution of roots and other properties of Hermite polynomials, also on the so-called Tchebycheff inequalities in the theory of algebraic continued fractions. He points out that the theorem still holds if we replace the law of probability \( \pi^{-1/2} \int_{-\infty}^{\infty} e^{-x^2} dx \) by a more general one: \( \int_{-\infty}^{\infty} f(x)dx \) (in which case, however, his considerations need many supplements).§

* Presented to the Society, April 18, 1930; received by the editors August 22, 1930.

† In fact, if \( X \) is a fortuitous variable (finite, not necessarily bounded), and if \( F(x) \) is the probability that \( X < x \), then \( F(x) \) will satisfy these conditions, provided we assume that the principle of total probabilities still holds for a countable infinity of inconsistent events.


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The same theorem has recently attracted the attention of many investigators: R. von Mises,* G. Pólya,† Paul Lévy,‡ Cantelli,§ Jacob|| and others.

The object of this paper is (a) to establish a general limit-theorem, removing many restrictions imposed otherwise on the functions involved and their moments, so that the above statement dealing with the law of probability \( \pi^{-1/2} \int_{-\infty}^{\infty} e^{-x^2} \, dx \) (we shall call it hereafter the “classical case”) is therein included as a very special case; (b) to give an elementary proof, which does not use either characteristic functions or algebraic continued fractions, being based on a well known Montel-Helly theorem concerning sequences of monotonic functions.

A brief account will first be given of the “moments-problem” to which the theorem in question is closely related.

1. The moments-problem. Given a certain interval \( (a, b) \), finite or infinite, and an infinite sequence of real constants \( c_0, c_1, \ldots \), find a function \( \psi(x) \), non-decreasing in \( (a, b) \), such that

\[
\int_a^b x^s \psi(x) \, dx = c_s \quad (s = 0, 1, \ldots).
\]

We call this the moments-problem corresponding to the data \( \{c_s\} \). We may assume, without loss of generality, \( \psi(a) = 0 \).

It is known that if \( (a, b) \) be finite, then the moments-problem cannot have more than one solution (if it has any), if we generally agree to consider as

‡ Paul Lévy, Calcul des Probabilités, Paris, 1925, Chapter IV.
|| Jacob, De l'application des intégrales généralisées de Fourier au calcul des probabilités, Comptes Rendus, vol. 188 (1929), pp. 541-43, 754-56.

† The case of \( \psi(x) \) having but a finite number of points of increase in \( (a, b) \) is trivial.

** A simple proof is the following. The existence of two solutions \( \psi_1(x), \psi_2(x) \) implies

\[
\int_a^b x^s \psi_1(x) \, dx = 0 \quad (s = 0, 1, \ldots; F(x) = \psi_1 - \psi_2; F(a) = 0),
\]

\[
F(b) = F(a) \quad (for \ s = 0), \quad \int_a^b x^l \psi(x) \, dx = 0 \quad (l = 0, 1, \ldots; \text{integration by parts}).
\]

The latter relations lead to the required conclusion: \( F(x) = 0 \) at all points of continuity in \( (a, b) \), by the following reasoning due to Stieltjes (Correspondance d'Hermite et Stieltjes, Paris, 1905, pp. 337–338). If such a point \( s \) exists (\( a < s < b \)) where \( F(z) \neq 0 \), then \( 1 - (x-s)^2/M > 0 \) for \( a \leq z \leq b \), for a sufficiently large \( M \); hence, it is easily seen that

\[
I = \int_a^b F(z) \left[ 1 - (x-z)^2/M \right] \, dx,
\]
identical two solutions \( \psi_1(x) \), \( \psi_2(x) \) which coincide at all points of continuity.*

We express this property by saying that the moments-problem for a finite interval is "determined."

On the other hand, the moments-problem for an infinite interval may be "indeterminate," i.e., it may admit infinitely many solutions. In fact, in the formula

\[
\int_0^\infty y^{a-1}e^{-by}dy = \frac{\Gamma(a)}{b^a}
\]

take

\[
b = k + di \quad (k > 0), \quad a = (n + 1)/\lambda, \quad (2n + 1)/\lambda \quad (n = 0, 1, \ldots),
\]

\[
d = \tan \lambda \pi, \quad \tan \frac{\mu \pi}{2} \quad (\lambda, \mu \text{ defined below}), \quad y = x^\lambda,
\]

and we get functions having all moments = 0:†

\[
\int_0^\infty x^ne^{-x^\lambda} \sin (\kappa x^\lambda \tan \lambda \pi)dx = 0 \quad (\kappa > 0, 0 < \lambda < \frac{1}{2}),
\]

\[
\int_{-\infty}^\infty x^ne^{-x^\lambda} \cos \left(\kappa x^\lambda \tan \frac{\mu \pi}{2}\right)dx = 0
\]

\[
(\kappa > 0, 0 < \mu = \frac{2s}{2s + 1} < 1, s \text{ a positive integer}).
\]

Hence we get infinitely many non-decreasing functions

\[
\int_0^x e^{-x^\lambda} \left[1 + h \sin (\kappa x^\lambda \tan \lambda \pi)\right]dx,
\]

(1)

\[
\int_{-\infty}^x e^{-x^\lambda} \left[1 + h \cos \left(\kappa x^\lambda \tan \frac{\mu \pi}{2}\right)\right]dx \quad (-1 \leq h \leq 1),
\]

for \( n \) very large, is different from zero, which is impossible, \( I \) being a linear combination of the moments of \( F(x) \), all of which vanish. (We notice that such \( M \) does not exist for \( (a, b) \) infinite.) Moreover, if \( \psi_i(x) \) is continuous to the left, then \( \psi_i(x) = \psi_i(x) \) everywhere in \( (a, b) \), since \( \psi_i(x-0) = \lim_{x \to X} \psi_i(x) \), where \( X(\leq x) \) converges to \( x \), being always a point of continuity of \( \psi_i(x) \), \( i = 1, 2 \).

* Also at the points \( a, b \), if \( (a, b) \) be finite; this, however, necessarily follows from the relations

\[
\psi_1(a) = 0, \quad \int_a^b \phi(x)dx = \int_a^b \psi(x) = c_0.
\]

† These have been given by Adamoff (Proof of a theorem of Stieltjes, Proceedings of the Kazan Mathematical Society (1911, in Russian)) and by Stekloff (Application de la théorie de fermeture . . . , Mémoires de l'Académie des Sciences, Petrograd, vol. 33 (1915)), but the original statement is due to Stieltjes (loc. cit., p. 230).
solutions of the same moments-problem for $(0, \infty)$ and $(-\infty, \infty)$ respectively. Either of the following conditions ensures the determined character of the moments-problem for an infinite interval:

\begin{equation}
\sum_{n=1}^{\infty} c_n^{-1/(2n)} \text{ diverges} \quad \left( c_n = \int_{-\infty}^{\infty} x^n dF(x) \right);
\end{equation}

\begin{equation}
dF(x) = p(x)dx \quad (p(x) \geq 0 \text{ on } (a, b))
\end{equation}

with \(p(x) < M |x|^{-1-\lambda}e^{-\kappa x} \) for \(|x| \geq x_0\), sufficiently large (\(M, \alpha, \kappa\) are positive constants); \(\lambda \geq \frac{1}{2}\) for \((a, b) = (0, \infty)\), \(\lambda \geq 1\) for \((a, b) = (-\infty, \infty)\).

On the other hand, the moments-problem is indeterminate if \(dp(x) = p(x)dx\), and for \(|x|\) sufficiently large (see (1))

\(p(x) > e^{-\kappa |x|}\) for \((0, \infty)\), \(\lambda < 1\) for \((-\infty, \infty)\).

2. A generalized statement of the second limit-theorem. Given a sequence of laws of probability \(F_n(x)\) \((n = 1, 2, \ldots)\), with the following properties: (i) the moments \(m_n^{(r)} = \int_{-\infty}^{\infty} x^rdF_n(x)\) of all orders \(r = 0, 1, \ldots\) exist for \(n = 1, 2, \ldots\), or at least from a certain rank \(n\) on (possibly depending on \(r\)); (ii) the quantities \(m_1^{(r)}, m_2^{(r)}, \ldots\), for any \(r = 0, 1, \ldots\), lie, when they exist, between two fixed limits independent of \(n\) (but possibly dependent on \(r\)). Then a subsequence \(\{C_i(x) = F_{n_i}(x)\}\) \((i = 1, 2, \ldots; n_1 < n_2 < \cdots; n_i \to \infty)\) can be extracted such that \((\alpha)\lim_{n \to \infty} \int_{-\infty}^{\infty} x^rdC_i(x)\) exists \((= m_r)\),\(\S\) for \(r = 0, 1, \ldots\); (\(\beta\) the subsequence \(\{C_i(x)\}\) converges for any \(x\) to one fixed law of probability \(\psi(x)\), save, perhaps, at its points of discontinuity; (\(\gamma\)) \(\int_{-\infty}^{\infty} x^rd\psi(x)\) exists and \(= m_r\).\(\|\)

The proof will be arranged in several steps.

3. Existence of \(m_r\) \((r = 1, 2, \ldots)\). We apply here the classical "diagonal method." The hypothesis of the uniform boundedness of \(\{m_n^{(r)}\}\) for all \(r = 1, 2, \ldots\), enables us to extract from the sequence \(\{m_n^{(1)}\}\) a subsequence \(\{m_n^{(1)}\}\) converging to a finite limit \(m_1\). The sequence \(\{m_n^{(2)}\}\) gives rise to a subsequence \(\{m_n^{(2)}\}\) converging to a finite limit \(m_2\), and so on. We thus get a sequence \(\{m_n^{(r)}\}\) converging to a finite limit \(m_r\), for any


\(\|\) It follows that there is necessarily at least one solution of the moments-problem corresponding to the data \(\{m_r\}\).
The corresponding laws of probability $\gamma_1(x) = F_{\gamma_1}(x)$, $\gamma_2(x) = F_{\gamma_2}(x)$, \ldots clearly satisfy (a). The reasoning still holds if none of the $F_n(x)$ has all of its moments finite.

4. Existence of a limiting law of probability $\psi(x)$. This follows by applying to $\{F_n(x)\}$ the Montel-Helly* theorem on monotonic functions. We state it in a slightly generalized form:

* If a family $\{f(x)\}$ of functions, non-decreasing on $(-\infty, \infty)$, is uniformly bounded in any finite interval (i.e. $|f(x_0)| < A(x_0)$ at any finite point $x_0$, $A(x_0)$ being the same for all $f(x)$), then from any infinite sequence of this family we can extract a subsequence which converges, for any $x$, to a non-decreasing function. Moreover, the convergence is uniform in any interval, where the limit-function is continuous.

The theorem holds, with proper modifications, for families of functions of bounded variation.

In order to prove $\beta$, it suffices to apply this theorem to the sequence $\{\gamma_i(x)\}$, since $0 \leq \gamma_i(x) \leq 1$ for $-\infty \leq x \leq \infty$. We extract then from it a sequence $\{C_p(x)\}$ converging everywhere to a non-decreasing function $\phi(x)$, and we take $\psi(x) = \phi(x - 0)$.

The following remarks are important:

(i) The limit-function $\psi(x)$ of the $\{C_p(x)\}$ varies effectively from 0 to 1, i.e.

$$\psi(-\infty) = 0, \quad \psi(\infty) = \int_{-\infty}^{\infty} d\psi(x) = 1 \dagger$$

(ii) For any $r = 0, 1, \ldots$, the convergence of

$$\lim_{a \to -\infty, b \to +\infty} \int_a^b x^r dF_n(x) \to \int_{-\infty}^{\infty} x^r dF(x)$$

is uniform with respect to $n$, at least from a certain rank $n$ on.

(iii) $\lim_{x \to -\infty} x^s(1 - F_n(x)) = 0$, $\lim_{x \to +\infty} |x|^s F_n(x) = 0$ ($n = 1, 2, \ldots$; $s > 0$ arbitrary).

(i) follows from the inequalities

$$\int_a^b x^r dF_n(x) \leq \frac{m_n}{b^{r+2}} (b > 1); \quad \int_{-\infty}^a x^r dF_n(x) \leq \frac{m_n}{a^{r+2}} (a < -1),$$

\dagger This is by no means obvious. Take, for example, $F_n(x) = 0 (x < -n)$, $= \frac{1}{2} (-n \leq x \leq n)$, $= 1 (x > n)$. Here $\lim_{n \to \infty} F_n(x) = \psi(x) = \frac{1}{2}$ for $-\infty \leq x \leq \infty$.

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which, applied to \( \{ C_p(x) \} \), yield, for \( r = 0 \) and \( p \to \infty \),

\[
(5) \quad (0 \leq b) 1 - \psi(b) \leq \frac{1 + m_2}{b^2}, \quad 0 \leq \psi(a) \leq \frac{1 + m_2}{a^2} \quad (b, -a > 1),
\]

and this proves (i) by letting \( b \to \infty \), \( a \to -\infty \).

(ii) follows directly from (4), taking into account the uniform boundedness of \( \{ m_n^{(r+s)} \} \) \( (n = 1, 2, \ldots) \).

In order to establish (iii), we write

\[
1 - F_n(b) = \int_b^\infty dF_n(x) \leq \int_b^\infty \left( \frac{x}{b} \right)^{2r} dF_n(x); \quad b^r[1 - F_n(b)] \leq \frac{m_n^{(2r)}}{b^{2r+s}}
\]

\( (b > 0, 0 < s < 2r) \),

and a similar expression for \( |a|^r F_n(a) (a < -1) \).

5. \( \int_{-\infty}^{\infty} x^r d\psi(x) \) exists and = \( m_n(r = 0, 1, \ldots) \). This statement being the fundamental part of the theorem, we give for it two proofs.

**Proof I.** We apply the two following theorems due respectively to Hamburger* and to Helly,† the proofs of both of which are very elementary.

**Hamburger's Theorem.** Suppose (i) \( \int_{-\infty}^{\infty} [1/(z+x)] d\psi(x) \) converges for \( z = iy \) with \( y > 0 \), \( \psi(x) \) denoting a function non-decreasing in \( (-\infty, \infty) \); (ii) \( F(z) = \int_{-\infty}^{\infty} [1/(z+x)] d\psi(x) \) has, for \( z = iy \to \infty \), an asymptotic expansion (in Poincaré's sense) \( F(z) \sim \sum_{r=1}^{\infty} (-1)^r c_r / z^{r+1} \) (\( c_r \) real). Then \( \int_{-\infty}^{\infty} x^r d\psi(x) \) exists and = \( c_r \) \( (r = 0, 1, \ldots) \).

**Helly's Theorem.** Given a sequence \( \{ V_n(x) \} \) of functions of bounded variation on a finite interval \( (a, b) \) such that (i) the total variations of all \( V_n(x) \) on \( (a, b) \) are uniformly bounded; (ii) \( \lim_{n \to \infty} V_n(x) = v(x) \) exists for \( a \leq x \leq b \), with the possible exception of a countable set of points not including \( a, b \). Then \( \lim_{n \to \infty} \int_a^b f(x) dV_n(x) = \int_a^b f(x) dv(x) \), for any continuous function \( f(x) \).

Going back to the given sequence \( \{ F_n(x) \} \) and to the above function \( \psi(x) = \lim_{p \to \infty} \{ C_p(x) \} \), we notice, first, that \( \int_{-\infty}^{\infty} f(x) d\psi(x), \int_{-\infty}^{\infty} f(x) dF_n(x) \) certainly exist if \( f(x) \) is bounded on \( (-\infty, \infty) \) and continuous on any finite interval. Furthermore, since, as we have seen, \( \int_{-\infty}^{\infty} dC_p(x) \) converges uniformly (with respect to \( p \)), an easy application of Helly's theorem yields

† Loc. cit.
‡ Necessarily of bounded variation, by virtue of the Montel-Helly theorem extended to this class of functions.
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\[ \lim_{p \to \infty} \int_{-\infty}^{\infty} f(x) dC_p(x) = \int_{-\infty}^{\infty} f(x) d\psi(x), \]

\[ \lim_{p \to \infty} \int_{-\infty}^{\infty} \frac{dC_p(x)}{x + z} = \int_{-\infty}^{\infty} \frac{d\psi(x)}{x + z} \quad (z = iy; y > 0). \]

Consider now the expression

\[ f_p(z) = \int_{-\infty}^{\infty} \frac{dC_p(x)}{z + x} = \sum_{r=0}^{\infty} (-1)^r \frac{\alpha_p(\gamma)}{\gamma^{r+1}} + \frac{(-1)^r}{\gamma^{r+1}} I_{r,p} \quad (\gamma \text{ arbitrary}), \]

\[ I_{r,p} = \int_{-\infty}^{\infty} x^r \left( \frac{z}{z + x} \right) dC_p(x), \quad \alpha_p(\gamma) = \int_{-\infty}^{\infty} x^r dC_p(x). \]

Letting \( p \to \infty \), using (7) and the property

\[ \lim_{p \to \infty} \alpha_p(\gamma) = m_r \quad (r = 0, 1, \ldots), \]

we get

\[ F(z) = \int_{-\infty}^{\infty} \frac{d\psi(x)}{x + z} = \sum_{r=0}^{\infty} (-1)^r \frac{m_r}{\gamma^{r+1}} + \frac{(-1)^r}{\gamma^{r+1}} \lim_{p \to \infty} I_{r,p}. \]

Observing that \( |z/(z+x)| \leq 1 \) \((-\infty \leq x \leq \infty, z = iy, y \geq y_0 > 0\), we get

\[ |I_{2s,p}| \leq \alpha_p(2s); \]

\[ |I_{2s-1,p}| \leq \left| \int_{-1}^{1} x^{2s-1} dC_p(x) \right| + \int_{1}^{\infty} x^{2s} dC_p(x) + \int_{-\infty}^{-1} x^{2s} dC_p(x) \]

\[ \leq 1 + \alpha_p(2s), \]

\[ \lim_{p \to \infty} I_{2s-1,p} \leq \lim_{p \to \infty} (1 + \alpha_p(2)) = 1 + m_{2s} \quad (s = 0, 1; s = 1, 2, \ldots), \]

\[ \lim_{s \to \infty} \left| z^s \left[ F(z) - \sum_{r=0}^{\infty} (-1)^r \frac{m_r}{\gamma^{r+1}} \right] \right| = \lim_{s \to \infty} \frac{1}{z} \lim_{p \to \infty} I_{r,p} = 0. \]

Formula (10), where \( \gamma \) is arbitrary, gives the asymptotic expansion of \( F(z) \):

\[ F(z) \sim \sum_{r=0}^{\infty} (-1)^r \frac{m_r}{\gamma^{r+1}} \quad (z = iy \to \infty). \]

Hence, Hamburger's theorem is applicable and proves (γ).

Proof II. We restrict ourselves to the domain of real numbers, making use of the following extension of Helly's theorem to the infinite interval \((-\infty, \infty)\).
Given a sequence \( \{v_n(x)\} \) defined on \((-\infty, \infty)\) such that (i) \( v_n(x) \) is of bounded variation on any finite interval, (ii) all \( v_n(x) \) and their total variations are uniformly bounded on any finite interval, (iii) \( \lim_{n \to \infty} v_n(x) = v(x) \) exists for all \( x \), with the possible exception of a countable set of points, (iv) \( \int_{a}^{b} f(x)dv_n(x) \) converges uniformly (with respect to \( n \)) to \( \int_{a}^{b} f(x)dv(x) \) \( (a \to -\infty, b \to \infty) \), if \( f(x) \) is continuous everywhere (not necessarily uniformly). Then \( \int_{a}^{b} f(x)dv(x) \) exists and equals \( \lim_{n \to \infty} \int_{a}^{b} f(x)dv_n(x) \).

We notice, first, that \( v(x) \) is of bounded variation (see footnote on page 538) on any finite interval, secondly (by virtue of Helly’s theorem), that
\[
\int_{b}^{b'} f(x)dv(x) = \lim_{n \to \infty} \int_{b}^{b'} f(x)dv_n(x),
\]
where \( b, b' \), \( |b|, |b'| \) sufficiently large; \( \epsilon > 0 \) arbitrarily small; hence, \( \int_{a}^{b} f(x)dv(x) \) exists. Furthermore,
\[
\Delta_n = \left| \int_{-\infty}^{\infty} f(x)dv(x) - \int_{-\infty}^{\infty} f(x)dv_n(x) \right| \leq \left| \int_{-\infty}^{a} f(x)dv(x) \right| + \left| \int_{a}^{\infty} f(x)dv(x) \right| + \left| \int_{b}^{\infty} f(x)dv(x) \right|
\]
\[
+ \left| \int_{b}^{\infty} f(x)dv_n(x) \right| + \left| \int_{a}^{\infty} f(x)dv_n(x) \right| + \left| \int_{a}^{b} f(x)dv(x) - \int_{a}^{b} f(x)dv_n(x) \right| \quad (a < 0, \ b > 0)
\]
can be made as small as we please by taking \( -a, b, \) and then \( n \) sufficiently large. Hence \( \lim_{n \to \infty} \Delta_n = 0. \)

Remark. The new condition (iv) is essential. The following example shows that if (iv) is not satisfied, the theorem may not hold: \( f(x) = x, \ v_n(x) = 0 \ (x \leq 0), \ 1 - 1/2^n (0 < x \leq 4^n), \ 1(x > 4^n) \). Here
\[
\lim_{n \to \infty} v_n(x) = v(x) = 0 \ (x \leq 0), \ 1(x > 0);
\]
\[
\lim_{n \to \infty} \int_{-\infty}^{\infty} xdv_n(x) = \lim_{n \to \infty} 4^n \cdot \frac{1}{2^n} = \infty \neq \int_{-\infty}^{\infty} xdv(x) = 0. *
\]

It suffices to apply the above theorem, with \( f(x) = x^r (r = 0, 1, \ldots) \), to the above sequence \( \{C_p(x)\} \) which satisfies all four conditions stated, and \( (\gamma) \) is established.

6. Special case. A direct corollary is the following.

Theorem. If \( \lim_{n \to \infty} m_n^r \) exists \( (=m_r) \) for \( r = 0, 1, \ldots \), then at least one fixed law of probability, say \( F(x) \), exists such that \( m_r \) is its \( r \)th moment \( (r = 0, 1, \ldots) \), and a subsequence \( \{C_r(x) = F_{n_r}(x)\} \) can be extracted from the given

* \( v_n(x), v(x) \) have each a single saltus = \( 1/2^n, 1 \) at \( x = 4^n \), \( 0 \) respectively.

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sequence \( \{F_n(x)\} \) of laws of probability such that \( \lim_{n \to \infty} C_i(x) = F(x) \) for any \( x \).

If, in addition, the \( \{m_r\} \) are such that the corresponding moments-problem is determined, then the sequence \( \{F_n(x)\} \) itself converges, for \( n \to \infty \), to \( F(x) \) at any point of continuity of \( F(x) \).

We need a proof for the last part only. Assume that a point \( x_0 \) of continuity of \( F(x) \) exists such that \( \{F_n(x_0)\} \) does not converge to \( F(x_0) \). Hence, a subsequence \( \{C_k(x_0) = F_{n_k}(x_0)\} \) can be extracted such that \( C_k(x_0) \) converges, for \( k \to \infty \), to a certain number \( A \neq F(x_0) \). On the other hand, we have seen that the sequence \( \{C_k(x)\} \) gives rise to a subsequence \( \{d_i(x)\} \) which, for any \( x \), converges, as \( i \to \infty \), to a function \( d(x) \), having the same moments \( m_r (r = 0, 1, \cdots) \) as \( F(x) \), and therefore, since, by hypothesis, the moments-problem corresponding to \( \{m_r\} \) is determined,

\[
\lim_{i \to \infty} d_i(x_0) = d(x_0) = F(x_0)
\]

\( (F(x) \) being continuous at \( x = x_0) \), which is impossible, \( \{d_i(x_0)\} \) being a subsequence of \( \{C_k(x_0)\} \) which converges, but not to \( d(x_0) \). We have seen also ((i), p. 537) that \( F(-\infty) = 0, F(\infty) = 1 \).

The condition that the moments-problem corresponding to \( \{m_r\} \) be determined is not only sufficient for the validity of the limiting relation

\[
\lim_{n \to \infty} F_n(x) = F(x)
\]

(at any point of continuity of \( F(x) \)), but it is also necessary. For if \( F(x) \) and \( \phi(x) \) be two distinct solutions of the moments-problem in question, then \( F_n(x) \) should converge simultaneously to \( F(x) \) and \( \phi(x) \) at all points of continuity, while at least one such point \( x_0 \) exists where \( F(x_0) \neq \phi(x_0) \).

7. The classical case: \( m_r = \pi^{-1/2} \int_{-\infty}^{\infty} x^r e^{-x^2} dx (r = 0, 1, \cdots) \). Here

\[
\lim_{n \to \infty} \int_{-\infty}^{\infty} x^r dF_n(x) = \pi^{-1/2} \int_{-\infty}^{\infty} x^r e^{-x^2} dx \quad (r = 0, 1, 2, \cdots)
\]

implies

\[
\lim_{n \to \infty} \int_{-\infty}^{x} dF_n(x) = \pi^{-1/2} \int_{-\infty}^{x} e^{-x^2} dx \quad (x \text{ arbitrary}).
\]

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* Even everywhere in \((-\infty, \infty)\), \( F(x) \) being a law of probability, hence continuous to the left (see Introduction).

† The conditions imposed by different writers on the quantities \( \{m_r\} \) are such as to ensure the determined character of the corresponding moments-problem. In fact, one sees readily that the conditions of R. von Mises, Pólya, and Cantelli (loc. cit.)

\[
m_n = \left( \frac{n}{c_n} \right)^n (C, c = \text{const.}), \lim_{n \to \infty} m_n^{1/(2n)} / \psi(n) \text{ is finite, } \lim_{n \to \infty} \psi(n) n^{1/2n} < 1 \quad (n > n_0; \psi(n) \to 0)
\]

are but special cases of Carleman’s condition (2).
In fact, the moments-problem corresponding to \( \{ m_r \} \) is determined (by virtue of (2) or (3)), and

\[
F(x) = \pi^{-1/2} \int_{-\infty}^{x} e^{-t^2} dt
\]

is continuous for all \( x \).

We see that the classical case is but a very special case of the general second limit-theorem.

8. The determined character of the moments-problem in the classical case. The conditions (2, 3) ensuring the determined character of the moments-problem have been established by means of very profound, but also complicated, considerations (continued fractions, singular integral equations). It seems of interest to give an elementary proof involving a simple theorem of Pólya.*

We wish to prove the following

**Theorem.** Given a law of probability \( \psi_1(x) \) such that

\[
m_{2r} = \int_{-\infty}^{\infty} x^{2r} \psi_1(x) dx = \frac{\Gamma\left(\frac{2r + 1}{\lambda}\right)}{\Gamma\left(\frac{1}{\lambda}\right)} \int_{-\infty}^{\infty} x^{2r} e^{-x^2/\lambda} dx,
\]

(12)

\[
m_{2r+1} = \int_{-\infty}^{\infty} x^{2r+1} \psi_1(x) dx = 0 = \frac{\lambda}{2 \Gamma(1/\lambda)} \int_{-\infty}^{\infty} x^{2r+1} e^{-x^2/\lambda} dx
\]

(\( \lambda \geq 1; \ r = 0, 1, \cdots \)).

Then necessarily

\[
\psi_1(x) = \frac{\lambda}{2 \Gamma(1/\lambda)} \int_{-\infty}^{x} e^{-t^2/\lambda} dt
\]

for any \( x \). In other words, \( \psi_1(x) \) is uniquely determined by (12).

Assume the existence of two such functions \( \psi_1(x) \) and \( \psi_2(x) \). Employing the reasoning of §1 (footnote on page 534) and using property (iii), page 537.

\[
\lim_{x \to \pm \infty} x^s (1 - \psi_i(x)) = \lim_{x \to \pm \infty} |x|^s \psi_i(x) = 0 \quad (i = 1, 2; s > 0 \text{ arbitrary}),
\]

we conclude that

\[
(13) \int_{-\infty}^{\infty} x^l F(x) dx = 0 \quad (l = 0, 1, \cdots; F = \psi_1 - \psi_2 = (1 - \psi_2) - (1 - \psi_1)).
\]

On the other hand, a reasoning similar to that of §4 leads, making use of (12) and the asymptotic expression for the $\Gamma$ function, to

\[ |F(x_0)| < \frac{2m_{2n}}{x_0^{2n}} \quad (n \text{ very large, } x_0 \text{ arbitrary}), \]

\[ |F(x_0)| < C e^{-1/2} x_0^{-1/2} \quad \left( \frac{2n}{\lambda} \right)^{2/\lambda + 1/2n} < x_0^2 < \left( \frac{4n}{\lambda} \right)^{2/\lambda}, \]

$C = \text{const. independent of } n \text{ and } x_0$. Therefore,

\[ \int_{-\infty}^{\infty} |F(x)| e^{-c|x|} dx \quad (0 < c < \frac{1}{2}) \]

exists. But this is precisely the condition imposed by Pólya,* which leads to the required conclusion: $F(x) = 0$ at all points of continuity.†

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† After the present paper had been prepared for publication, we came across an interesting article by A. Wintner: Über den Konvergenzsatz der mathematischen Statistik, Mathematische Zeitschrift, vol. 28 (1928), pp. 470-480, some of the results of which are similar to those obtained above.