ON SUMS OF TWO OR FOUR VALUES OF A QUADRATIC FUNCTION OF \( x^* \)

BY

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1. We shall consider sums of values of the function

\[ q(x) = \mu x^2 + \nu x + c, \]

where \( \mu > 0, \nu \) and \( c \) are real. If \( q(x) \) is an integer for every integer \( x \), it is of the form

\[ \frac{1}{2} m x^2 + \frac{1}{2} n x + c, \quad m > 0, \]

\( m, n, \) and \( c \) being integers, \( m+n \) even; and conversely.

Denote the least number represented by

\[ q(x_1) + q(x_2) + \cdots + q(x_s) \quad (s \text{ given}) \]

for integers \( x_i \geq w \) by \( \lambda = \lambda(w, q(x), s) \). In the case (2) our problem may be stated as follows: to determine the magnitude of the largest stretch of consecutive integers \( \geq \lambda \) not represented by (3) for integers \( x_i \geq w \).

We shall obtain a reasonably comprehensive solution of this problem for \( s = 2 \) and \( s = 4 \) (see end of this Section). Many known facts concerning sums of four values of quadratic functions of one variable are corollaries. Thus the Fermat-Cauchy polygonal number theorem is implied by Theorem 2 for the range \(-\mu < \nu \leq -\frac{1}{3} \mu \); for this special case, however, a simpler proof, much like that given here for the range \( 0 < \mu < \nu \), exists.

§2 is the only one relating to \( s = 2 \). I know of nothing general for \( s = 3 \).

For \( s \geq 5 \) Professor L. E. Dickson\(^{\dagger} \) gave a complete solution, by ingenious methods depending upon conditions for solving the equations

\[ a = x_1^2 + x_2^2 + x_3^2 + x_4^2, \quad b = x_1 + x_2 + x_3 + x_4. \]

The basic lemma, due to Cauchy,\(^{\dagger\dagger} \) is as follows.

**Lemma 1.** Necessary and sufficient conditions that equations (4) be solvable in integers \( x_i \) are

\[ a \equiv b \pmod{2}, \quad 4a - b^2 = \text{a sum of 3 squares}. \]

* Presented to the Society, November 29, 1930; received by the editors May 1, 1931.


If $b \geq 0$ these $x_i$ are necessarily $\geq 0$ if (5) holds and

$$b^2 + 2b + 4 > 3a.$$  

New methods, involving values $a$, $b$ below the limit $b^2 + 2b + 4 > 3a$ of Lemma 1, must be invented for all except the simplest cases already done by Professor Dickson.

Let $F_k(x \geq -k)$, or simply $F_k$, denote the table of all sums of four values of

$$f(x) = \mu x^2 + \nu x \quad (\mu > 0)$$

for integers $x \geq -k$; or, what is the same thing, the class of all quantities $\mu a + \nu b$, where $a$ and $b$ are integers such that equations (4) are solvable in integers $x_i \geq -k$. For any $\mu$, $\nu$ the entries of $F_k$ may be arranged in order of magnitude. By a gap in $F_k$ we mean a difference of consecutive entries. If $\gamma_0$ is the largest gap for, say, the first thousand entries, we try to show that it is large enough to bridge the entire table. If a larger gap be later encountered it may be taken as a new standard. A difference of entries is called allowable if it is $\leq$ some gap already discovered in the table.

Since sums of four squares or triangular numbers become very numerous for large numbers it might be expected by analogy that the largest gap should always occur early in the tables $F_k$. In $F_\infty$, in fact, by Theorem 5, the largest gap occurs among the first six entries. In all cases of this sort (Theorems 2, 3, 5, 6, 7, 8) we shall solve our problem completely.

But there are two distinct stretches of values of $\mu$ and $\nu$ for which the largest gap in $F_k$ occurs arbitrarily far out.

The first case of this is completely solved in Theorem 4. The values $\mu$, $\nu$ in question satisfy $\nu < 0$, $\mu \geq 5|\nu|$.

Finally in the really difficult case with $\mu$, $\nu$ in the vicinity of $\mu = (3/2)|\nu|$ the distribution of large gaps is extremely complicated. For every $k > 0$ there are, roughly speaking, infinitely many gaps larger than any near the beginning. I content myself (§§10 and 13) in this case with devising a method for a finite exhaustive determination of the gaps, showing plainly by explicit formulas where the gaps are situated.

Many properties, new and old, are developed of the sum of the roots $\sum x_i$ with $x_i \geq -k$, when $a = \sum x_i^2$.

2. The case $s = 2$ is trivial, at least for (2).

**Theorem 1.** If, for $q(x)$ in (2), the values of $q(x) + q(y)$ for all integer pairs $x$, $y$ be arranged in order of magnitude, arbitrarily large gaps will occur.

By (2) the equations

$$N = q(x) + q(y),$$

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8mN + 2n^2 - 16mc = (2mx + n)^2 + (2my + n)^2

are equivalent. Hence (8) is not solvable in integers x, y for r consecutive positive integers N, provided that x^2 + y^2 fails to represent 8m(r+1) consecutive integers > 2n^2 - 16 mc. The last fact follows from

**Lemma 2. Any binary quadratic form**

\[ \phi = Ax^2 + Bxy + Cy^2, \]

with integer coefficients and discriminant \(d = B^2 - 4AC\) not a square, fails to represent any given number \(k\) of consecutive positive integers.

For \(\phi \neq N\) if \((d | p) = -1\) for any odd prime \(p\) dividing \(N\) to an odd exponent. Since \(d\) is not a square there exist infinitely many odd primes \(p_1, p_2, \ldots\), such that \((d | p_i) = -1\). The congruences

\[ N \equiv -h \pmod{p_h} \quad (h = 1, 2, \ldots, k) \]

whose moduli are relative prime in pairs have a solution

\[ N = N_0 + p_1 p_2 \cdots p_k, \]

where \(l\) is an arbitrary integer. Choosing \(l\) so that

\[ l \equiv 0 \pmod{p_k} \text{ if } N_0 \neq -h \pmod{p_k}, \]

\[ l \equiv 1 \pmod{p_k} \text{ if } N_0 \equiv -h \pmod{p_k}, \]

we get integers \(N\) such that \(N+1, \ldots, N+k\) are not represented by \(\phi\).

3. The problems for \(q(x)\) and \(f(x)\) are seen to be equivalent. Also, for any \(\omega\),

\[ \mu x^2 + \nu x = \mu(x - \omega)^2 + (\nu + 2\mu\omega)(x - \omega) + \mu\omega^2 + \nu\omega. \]

Hence, by altering the range of \(x\), we can obtain

\[ -\mu < \nu \leq \mu. \]

The following classification is therefore exhaustive:*  

\[ F_0 \equiv F_4(x \geq 0), \text{ when } 0 < \mu < \nu; \]

\[ F_0, \text{ when (12) holds; } \]

\[ (13a) \quad \mu x^2 + \nu x, \quad x \geq x_0, \text{ where } x_0 \leq 0 \text{ and } -\mu < \nu \leq \mu. \]

By (11) the table for \(\mu x^2 + \nu x, x \geq x_0\), may be replaced by the table for \(\mu x^2 + (\nu + 2\mu\omega)x, x \geq x_0 - \omega\). If \(\nu + 2\mu\omega > \mu\) we choose \(\omega = x_0\); otherwise we can choose a unique integer \(\omega \geq x_0\) such that \(-\mu < \nu + 2\mu\omega \leq \mu\). In the first case we obtain (13), in the second (13a).
(15) \[ F_k = F_4(x \geq -k), \quad k \geq 1, \text{ when } |\nu| \leq \mu < 3|\nu|; \]
(16) \[ F_\nu, \text{ when } \mu \geq 3|\nu|. \]

4. Some properties of \( \sum x_i \) in \( a = \sum x_i^2 \). We use \( a \) and \( e \) in the rest of this paper to denote positive integers only. For \( k \geq 0 \) and any \( a \) we define \( L_k(a) \) to be the class of all values \( b \) such that (4) are solvable in integers \( x_i \geq -k \); but we use it also in the sense of the class of all entries of \( F_k \) of which the coefficient of \( \mu \) is \( a \). We drop the subscript 0 from \( L_0(a) \), and define

\begin{align*}
B_a & = \text{largest } b \text{ on } L(a); \\
ba & = \text{least } b \equiv a \pmod{2} \text{ such that } b^2 + 2b + 4 > 3a; \\
(17) & b_\nu = \text{least } b \equiv a \pmod{2} \text{ such that } b^2 + 4b + 16 > 3a; \\
b_a(a) & = \text{least } b \text{ on } L_k(a).
\end{align*}

We outline a proof of Lemma 1. We perceive the equivalence of the systems (4) and

\begin{equation}
4a - b^2 = (x_1 + x_2 - x_3 - x_4)^2 + (x_1 - x_2 + x_3 - x_4)^2 + (x_1 - x_2 - x_3 + x_4)^2, \\
b = x_1 + x_2 + x_3 + x_4. \tag{18}
\end{equation}

Hence there is a \((1,1)\) correspondence between the sets of integers \( x_i \) of (4) and \( y_h \) of

\begin{equation}
4a - b^2 = y_1^2 + y_2^2 + y_4^2, \quad b + y_2 + y_3 + y_4 = 0 \pmod{4}. \tag{19}
\end{equation}

For odd \( a, b \), (19) holds by choice of sign of one of the (odd) \( y_i \); for given even \( a, b \), (19) is a consequence of (19). The statement about (5) follows.

As to (6): \( \sum x_i \geq 0, x_i < 0 \) imply

\[ (\sum x_i + 1)^2 \leq (x_2 + x_3 + x_4)^2 \leq 3(x_2^2 + x_3^2 + x_4^2), \quad (b + 1)^2 \leq 3(a - 1). \]

We do not use here the fact that, when \( a, b \) are even, the signs of the \( y_h \) are at our disposal, and the preceding can be modified to show that (4) are then solvable in integers \( x_i \geq 0 \) if (5) holds and merely

\begin{equation}
b \geq 0, \quad 3b^2 + 8b + 16 > 8a. \tag{20}
\end{equation}

If \( a, b \) are odd, \( 4a - b^2 \equiv 3 \pmod{8} \); if \( b \) is even and \( a \equiv 2 \pmod{4} \), \( a - \frac{1}{4}b^2 \equiv 2 \pmod{4} \). Hence (5) holds provided only that

\begin{equation}
4a \equiv b^2. \tag{21}
\end{equation}

**Lemma 3.** If \( e \) is odd or double an odd, \( L_\infty(e) \) consists of every \( b \equiv e \pmod{2} \) satisfying \( 4e \geq b^2 \); and \( L(e) \) contains every \( b \equiv e \pmod{2} \) satisfying

\begin{equation}
b_\ast \leq b \leq B_\ast. \tag{22}
\end{equation}
The last part is clear from Lemma 1 and (17). By considering (52) we readily verify

**Lemma 4.** Let \( a = 4A, \ A \text{ odd}, \ 4a \geq b^2 \). Then

- \( b = 16w \) belongs to \( L_\omega(a) \) unless \( A \equiv 7 \pmod{8} \),
- \( b = 4w + 2 \) belongs to \( L_\omega(a) \) for every \( A \),
- \( b = 16w + 8 \) belongs to \( L_\omega(a) \) unless \( A \equiv 3 \pmod{8} \),
- \( b = 4B, \ B \text{ odd}, \) belongs to \( L_\omega(a) \) unless \( A - B^2 = \Lambda \),

where \( \Lambda \) denotes the linear form \( 4^4(8w+7) \). *

Now a positive \( b \) of the same parity as \( a \) satisfies

\[
2a^{1/2} \geq b, \ b + 1 \geq (3a)^{1/2}
\]

for every \( a \geq 1 \). By Lemmas 1 and 3 this gives the first part of

**Lemma 5.** If \( a \not\equiv 0 \pmod{4}, \ B_a \) is the maximum \( b \equiv a \pmod{2} \) satisfying (21). Also, \( B_a \geq b_a \). If \( a \) is even,

\[
(B_a)^2 \geq 3a
\]

unless

\[
a = 2^{2h-1}A, \ A = 1, 3, 7, 11, 17, \ h \geq 1.
\]

In these cases

\[
B_a = 2^h z, \ z = 1, 2, 3, 4, 5,
\]

and \( B_a \) is the maximum \( b \) satisfying (52).

Suppose that \( a \) is even. Write

\[
a = 2^h A, \ A \text{ odd}, \ g = 2h \text{ or } 2h - 1,
\]

so that \( h \geq 1 \). If \( 2^g \) does not divide \( b \), \( 4a - b^2 = \Lambda \) and is not a sum of three squares. Hence set \( b = 2^g y \).

The conditions \( 3a \leq b^2 \leq 4a \) are

\[
3A \leq y^2 \leq 4A \quad \text{(if } g \text{ is even)},
\]

\[
(3/2)A \leq y^2 \leq 2A \quad \text{(if } g \text{ is odd)}.
\]

An odd \( y \) satisfies (28) if \( (4A)^{1/2} - (3A)^{1/2} \geq 2, \ A \geq 57 \). An integer \( y \) satisfies (29) if \( (2A)^{1/2} - (3A/2)^{1/2} \geq 1, \ A \geq 29 \). By the remarks leading to (21), \( 4a - b^2 \) is then a sum of three squares.

* It is of interest to note that there are precisely 52 odd numbers \( A \) such that every square \( z^2 \leq A \) occurs in the representations of \( A \) as a sum of four squares. These are

\[

Table I is a list* of all values $y$ for which the equations

$$\tau a = \sum x_i^2, \ 2y = \sum x_i \quad (\tau = 4 \text{ or } 2)$$

are solvable in four integers $x_i \geq 0$; i.e. $L(a)$ consists of all values $2^b y$. Write

$$z = \text{maximum } y \text{ for a given } a,$$
$$w = \text{second largest } y \text{ for a given } a \text{ (if any exists)}.$$ 

Thus, $B_a = 2^b z$. In the column $y(4A)$ we verify $z^2 \geq 3A$ if $1 \leq A \leq 55$; in the column $y(2A)$ we find $z^2 \geq (3/2)A$ if $1 \leq A \leq 27$, except for $A = 1, 3, 7, 11, 17$, when $z = 1, 2, 3, 4, 5$ respectively. In these five cases $z^2$ is the largest square $\leq 2A$.

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Lemma 6. For any $k \geq 1$ the largest $b$ on $L_k(a)$ is $B_a$.

When $a$ is odd this is evident from Lemma 5 and the necessity of (21) for $(5_2)$. When $a$ is even it follows from the last clause of Lemma 5 if $(B_a)^2 < 3a$, and, since (6) holds for $b \geq B_a$, from the last part of Lemma 1 if (24) holds.

Lemma 7. Equations (4) are solvable in integers $x_i \geq -k$ if the following hold:

\[(32) \quad b \geq -4k, \quad b^2 + 2(k+1)b + 4(k+1)^2 > 3a.\]

This appears out of Lemma 1 if we replace $x_i$ by $x_i - k$ in (4) and obtain the equations

\[a + 2kb + 4k^2 = \sum x_i^2, \quad b + 4k = \sum x_i,^*\]

which are to be solvable in $x_i \geq 0$. (Cf. (17a).)

Lemma 8. For any even $a$,

\[(33) \quad B_a \leq B_{a-1} + 1, \quad B_a \leq B_{a+1} + 1.\]

For, by the maximal property of $B_{a-1}$ (Lemma 5),

\[(34) \quad (B_{a-1} + 2)^2 > 4a \quad \text{ (a even).}\]

If $(33_1)$ were false we should have

\[B_a \geq B_{a-1} + 3, \quad (B_a)^2 > 4a,\]

contrary to $(5_2)$. Similarly for $(33_2)$ with $a+2$ for $a$ in (34).

Lemma 9. For any even $a$ except $(25)$,

\[(35) \quad b_{a-1} \leq B_a + 1, \quad b_{a+1} \leq B_a + 1,\]

\[(36) \quad b_{a-1} \leq B_a - 1, \quad b_{a+1} \leq B_a - 1.\]

In fact, by the definitions (17) of $b_{a+1}, b_{a+1},$

\[(37) \quad (b_{a+1} - 1)^2 \leq 3a, \quad (b_{a+1})^2 \leq 3a - 9 \quad \text{ (a even).}\]

If $(35_2)$ is false, $(37_1)$ gives

\[b_{a+1} \geq B_a + 3, \quad (B_a)^2 < 3a,\]

contrary to (24). Similarly for $(35_1)$ with $a-2$ for $a$ in $(37_1)$, and for (36) by use of $(37_2)$.

* It follows that there exists a $(1, 1)$ correspondence between the sets $(a, b)$ such that (4) is solvable in integers $x_i \geq 0$, and the sets $(a', b')$ such that (4) is solvable in integers $x_i \geq -k$. This is defined by

\[a = a' + 2kb' + 4k^2, \quad b' + 4k = b.\]
Lemma 10. For any even $a$,

$B_{a+1} - 2 \leq B_{a-1} \leq B_{a+1}$ \hspace{1cm} (a even).

Also,

$B_{a+1} = B_{a-1} + 2$

if and only if an odd square lies between $4(a-1)$ and $4(a+1)$. When (39) holds,

$B_a = B_{a-1} + 1$ or $B_{a+2} = B_{a+1} - 1$,

according as $a \equiv 2$ or $a \equiv 0 \pmod{4}$.

This is evident from Lemma 5, the parities involved, and the fact that only one odd square can lie between $4a-4$ and $4a+4$.

5. We examine the existence of values $b < b_a$ on $L(a)$. For odd integers $a < 720$ the largest ratio $b^2/a$, for $b < 2a^{1/2}$ and such that $b$ is missing from $L(a)$, occurs when $a = 347$ and $b = 31$. Then $31 = b_a - 2$. But usually the sequence of $b$’s extends without a break some distance below $b_a$.

We derive Lemma 11 as a corollary of Lemmas 12 and 13. Lemma 12 is easily proved by the calculus.

Lemma 11. If $e$ and $x$ are positive integers, $e \geq x^2$, write

$g_0(x) = x + (e - x^2)^{1/2}$,

$g_1(x) = x + (1.8)^{1/2}(e - x^2)^{1/2}$,

$g_2(x) = x + 3^{1/2}(e - x^2)^{1/2}$.

Then, if $e - x^2$ is a sum of three squares, $L(e)$ contains a value $b$ such that

$g_0(x) \leq b \leq g_2(x)$.*

If also $e - x^2 \equiv 1 \pmod{3}$, $L(e)$ contains such a value with

$g_1(x) \leq b \leq g_2(x)$.

Lemma 12. Let $\xi, \eta, \zeta$ run over all real numbers such that

$\xi^2 + \eta^2 + \zeta^2 = c$ \hspace{1cm} (c > 0),

and (I) $\xi \geq \eta \geq \zeta \geq 0$; (II) $2\eta + 2\xi \geq \xi \geq \eta \geq \zeta \geq 0$. In Case I the maximum value of $\xi + \eta + \zeta$ is $(3c)^{1/2}$ and the minimum value is $c^{1/2}$; in Case II the minimum value is $(9c/5)^{1/2}$, obtained when $\xi = 2\eta, \zeta = 0$.

Lemma 13. If $c \not\equiv 1 \pmod{3}$ and $c$ is a sum of three (integral) squares, then $c$ is of the form

$c = X^2 + Y^2 + Z^2, \hspace{1cm} 2Y + 2Z \geq X \geq Y \geq 0, \hspace{0.5cm} X \geq Z \geq 0$.

* If $x$ is the largest integer such that $e - x^2$ is a sum of three squares, then (42) holds for $b = b_0(e)$.
For suppose that

\[(46) \quad c = t^2 + u^2 + v^2, \quad t \geq u \geq v \geq 0, \quad t > 2u + 2v.\]

Then 3 divides at least one of \(t+u+v, t+u-v, t-u+v, t-u-v\). In the respective cases write

\[X = \frac{1}{3}(2t + 2u - v), \quad Y = \frac{1}{3}(2t - u + 2v), \quad Z = \frac{1}{3}(t - 2u - 2v);\]
\[X = \frac{1}{3}(2t + 2u + v), \quad Y = \frac{1}{3}(2t - u - 2v), \quad Z = \frac{1}{3}(t + 2u - 2v);\]
\[X = \frac{1}{3}(2t + u + 2v), \quad Y = \frac{1}{3}(2t - 2u - v), \quad Z = \frac{1}{3}(t + 2u - 2v);\]

\(X\) and \(Z\) are the larger and smaller respectively of \(\frac{1}{3}(2t + u - 2v)\) and \(\frac{1}{3}(t + 2u + 2v)\),

\[Y = \frac{1}{3}(2t - 2u + v).\]

We see (1) that \(X, Y, Z\) are positive integers; (2) that \(X\) is the largest of the three; (3) that \(X \leq 2Y + 2Z\).

The derivatives of the \(g_1(x)\) are negative in the interval \(0.6e^{1/2} \leq x \leq e^{1/2}\). Hence the \(g_1(x)\) reach their greatest values at the beginning and their least values at the end of any interval

\[(47) \quad \rho e^{1/2} \leq x \leq \sigma e^{1/2}, \quad 0.6 \leq \rho < \sigma \leq 1.\]

By Lemma 11 a value \(b\) in the interval

\[(48) \quad g_1(\sigma e^{1/2}) \leq b \leq g_2(\rho e^{1/2})\]

will exist on \(L(e)\) if \(e\) and \(x\) satisfy (47) and

\[(49) \quad e - x^2 \equiv 1 \pmod{3}, \quad e - x^2 \not\equiv \Lambda.\]

We use five pairs \((\rho_i, \sigma_i)\) \((i = 1, \ldots, 5)\):

\[(50) \quad (0.87, 0.89), (0.885, 0.905), (0.915, 0.935), (0.955, 0.975), (0.980, 1).\]

Write \(R_i = e^{-1/2}g_1(\sigma_i e^{1/2}), S_i = e^{-1/2}g_2(\rho_i e^{1/2})\). By (41) and (50),

\[(51) \quad R_i \geq 1.5017, 1.4757, 1.4107, 1.2730, 1; \quad S_i \leq 1.7240, 1.6915, 1.6139, 1.4688, 1.3247.\]

If \(e = \pm 1 \pmod{8}\), (49) hold for at least one of any \(q\) consecutive integers \(x\), where \(q = 2, 3, 4, 6\) in the respective cases

\[(52) \quad e = 9 \text{ or } 15, e = 7, e = 1, e = 17 \text{ or } 23 \pmod{24}.\]

Indeed (49) is true of

any even \(x\) if \(e = 9\), any odd \(x\) if \(e = 15 \pmod{24}\),

any \(x = 1, 2, 5, 7, 10, \text{ or } 11 \pmod{12}\) if \(e = 7 \pmod{24}\),
any \( x \equiv 2, 4, 8, \) or 10 (mod 12) if \( e \equiv 1 \) (mod 24),
any \( x \equiv 0 \) (mod 6) if \( e \equiv 17 \), any \( x \equiv 3, 6, \) or 9 (mod 12) if \( e \equiv 23 \) (mod 24).

Now the interval \( \rho \varepsilon^{1/2} \leq x \leq \sigma \varepsilon^{1/2} \) contains \( q \) integers \( x \) if \((.02)^2 \varepsilon \geq q^2\),
that is,
\[
(53) \quad \varepsilon \geq 10000, 22500, 40000, 90000
\]
in the various cases (52). Hence

**Lemma 14.** For \( e \) satisfying (53) in the respective cases (52), \( L(e) \) contains a
value \( b = b(i) \) satisfying
\[
(54) \quad R_i \varepsilon^{1/2} \leq b(i) \leq S_i \varepsilon^{1/2} \quad (i = 1, 2, \ldots, 5),
\]
where the \( R_i \) and \( S_i \) satisfy (51).

6. Table \( F_0(x \geq 0), 0 < \mu < \nu \). We prove the following theorem:

**Theorem 2.** If \( \mu > 0 \) and \(-\mu < \nu \), then
\[
(55) \quad \gamma = \mu + |\nu|
\]
is the largest gap in Table \( F_0 \).

If \( \nu \geq 0 \), \( \gamma \) is the very first gap. Let \( \nu < 0 \). Then \( 7\mu + 5\nu \) evidently exceeds
every entry of \( F_0 \) with \( a \leq 6 \). Since \( B_a \leq a - 4 \) for every \( a \geq 8, 8\mu + 4\nu \) is the least
entry with \( a \geq 8 \). But \( 8\mu + 4\nu - 7\mu - 5\nu = \mu - \nu \). Hence \( \gamma \) is actually a gap in \( F_0 \).

For any \( a > 0 \) set \( \xi_a = a\mu + B_a\nu \), which is an entry of \( F_0 \).

Let \( 0 < \mu < \nu \). Then \( \mu + \nu, 2\mu, \) and \( \nu - \mu \) are allowable differences. Let \( m \) be
odd and positive. Then (39) holds for \( a = e \) and \( f \), where
\[
4e = m^2 - 1, \quad 4f = (m + 2)^2 - 1.
\]
By Lemma 10, we can pass from \( \xi_{e+1} \) by successive increments \( 2\mu \) over
\( \xi_{e+3}, \ldots \) to \( \xi_{f-1} \). If \( f \equiv 2 \) (mod 4) we proceed to \( \xi_f \) and \( \xi_{f+1} \) by two increments
\( \mu + \nu \). If \( f \equiv 0 \) we pass by the increments \( 2\mu, \mu + \nu, \nu - \mu \) to \( \omega = \xi_{f+1} - 2\nu \),
\( \xi_{f+2}, \xi_{f+1} \), provided \( \omega \) is an entry of \( F_0 \). This is certainly the case if three integers
\( b \) lie within the limits (23) for \( a = f + 1 \); hence, if \( f = 20 \) and \( f \geq 56 \). In the
sole remaining case, \( 4f = 49 - 1, f + 1 = 13 = 3^2 + 2^2 \), whence \( \omega = 13\mu + 5\nu \) is an
entry of \( F_0 \).

7. Table \( F_0, \mu \geq |\nu| > 0 \). Now \( 2 |\nu| \) is an allowable difference.

Write \( (a, b) = a\mu + b\nu \). If \( a \neq 0 \) (mod 4) each of
\[
(56) \quad \lambda_a = (a, b_a), (a, b_a + 2), \ldots, (a, B_a) = \xi_a
\]
is an entry of \( F_0 \), and we can pass allowably between any two such entries for the same \( a \).

By Lemma 8 and the first half of Lemma 9 it is clear then that all gaps in \( F_0 \) are allowable except for the values \( a \) in (25) as follows:

\[
\begin{align*}
\text{between } (a, B_a) \text{ or } (a-1, B_{a-1}), \text{ and } (a+1, b_{a+1}), & \text{ if } v > 0; \\
\text{between } (a-1, b_{a-1}) \text{ and either } (a, B_a) \text{ or } (a+1, B_{a+1}), & \text{ if } v < 0.
\end{align*}
\]

If \( v > 0 \), let \( \xi_{a+1} \) denote an entry \((a+1, x_{a+1})\). Suppose that all gaps in \( F_0 \) between \( \xi_{a+1} \) and \( \lambda_{a+1} \) are known to be allowable. Then, in further progress from \( \xi_a \) to \( \xi_{a+1} \) we can suppose \( \xi_{a+1} - \xi_{a-1} > \gamma \), or

\[
\begin{align*}
(581) \quad & \mu + |v| > (B_{a-1} - x_{a+1} + 2) |v|,
\end{align*}
\]

since otherwise we can pass directly to \( \xi_{a+1} \). The corresponding condition for \( v < 0 \) is

\[
\begin{align*}
(582) \quad & \mu + |v| > (2 + x_{a-1} - x_{a+1} + 2) |v|,
\end{align*}
\]

where \( \xi_{a-1} \) is an entry allowably approachable from \( \lambda_{a-1} \).

In the cases (52) let \( \epsilon \) satisfy (53). Write \( S_0 = 3^{1/2} \), \( b^{(0)} = b_\epsilon \). We shall use the preceding with

\[
(59) \quad x_\epsilon = b^{(i-1)} \quad (i = 1, 2, \ldots, 5),
\]

where \( b^{(0)} \) is any value \( b \) on \( L(\epsilon) \) in (54). By (34), (37), (58), and (54), we have

\[
(60) \quad B_f - b^{(i-1)} + 2 \geq (2 - S_{i-1}) e^{1/2} - 1 \quad (e \geq 3),
\]

where

\[
(61) \quad e = a + j, \quad f = a - j,
\]

and

\[
(62) \quad j = 1 \text{ if } v > 0, \quad j = -1 \text{ if } v < 0.
\]

By (54) and \( b^{(0)} \leq S_0 e^{1/2} + 1 \),

\[
(63) \quad b^{(i-1)} - b^{(i)} \leq (2 - S_{i-1}) e^{1/2} - 1
\]

if \( 2 \leq (R_{i+2} - 2S_{i-1}) e^{1/2} \), and hence by (51) if \( e \geq 5250 \).

Since \((1.4142 - S_0) e^{1/2} \geq 1\) if \( e \geq 145 \),

\[
(64) \quad b^{(i)} \leq B_a - 1 \text{ if } (B_a)^2 \geq 2a,
\]

for any even \( a \geq 146 \). Since \((B_a)^2 \geq 2a \) in each case (25) we can then pass between \( \xi_a \) and \( e \mu + b^{(0)} \nu \) by an increment \( \leq \mu - |v| \).

(a) \( v < 0 \). There remain the early values \( A = 1 \) or \( 3, h \leq 7; A = 7, h \leq 6; A = 11 \) or \( 17, h \leq 5 \). By calculation, \( b_{a-1} = B_a - 1 \) if \( h = 1, A = 1, 3, 11, \) or \( 17; \)
If $A = 1$, $h = 2, 3$; $A = 3, h = 2, 3, 4, 5$. Finally we give a table for the remaining cases, entries being established by giving $x_i$ satisfying

$$a - 1 = x_1^2 + \cdots + x_4^2, \quad b = x_1 + \cdots + x_4.$$ 

We give $W = B_{a+1} - b_{a-1} + 2$ where necessary.

<table>
<thead>
<tr>
<th>$a$</th>
<th>$b$</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$x_4$</th>
<th>$b_{a-1}$</th>
<th>$W$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2^7</td>
<td>17</td>
<td>1</td>
<td>1</td>
<td>5</td>
<td>10</td>
<td>19</td>
<td>4</td>
</tr>
<tr>
<td>2^9</td>
<td>31</td>
<td>1</td>
<td>3</td>
<td>3</td>
<td>22</td>
<td>39</td>
<td>8</td>
</tr>
<tr>
<td>2^{11}</td>
<td>65</td>
<td>1</td>
<td>1</td>
<td>26</td>
<td>37</td>
<td>79</td>
<td>12</td>
</tr>
<tr>
<td>2^{11}</td>
<td>71</td>
<td>1</td>
<td>7</td>
<td>29</td>
<td>34</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2^{13}</td>
<td>129</td>
<td>1</td>
<td>17</td>
<td>26</td>
<td>85</td>
<td>157</td>
<td>26</td>
</tr>
<tr>
<td>2^{13}</td>
<td>145</td>
<td>5</td>
<td>14</td>
<td>59</td>
<td>67</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3 \cdot 2^6</td>
<td>63</td>
<td>6</td>
<td>7</td>
<td>15</td>
<td>35</td>
<td>67</td>
<td>12</td>
</tr>
<tr>
<td>3 \cdot 2^{11}</td>
<td>129</td>
<td>1</td>
<td>27</td>
<td>38</td>
<td>63</td>
<td>135</td>
<td>22</td>
</tr>
<tr>
<td>3 \cdot 2^{13}</td>
<td>257</td>
<td>1</td>
<td>43</td>
<td>102</td>
<td>111</td>
<td>271</td>
<td>44</td>
</tr>
<tr>
<td>7 \cdot 2^7</td>
<td>49</td>
<td>1</td>
<td>7</td>
<td>19</td>
<td>22</td>
<td>51</td>
<td>10</td>
</tr>
<tr>
<td>7 \cdot 2^9</td>
<td>95</td>
<td>10</td>
<td>13</td>
<td>17</td>
<td>55</td>
<td>103</td>
<td>18</td>
</tr>
<tr>
<td>7 \cdot 2^{11}</td>
<td>191</td>
<td>7</td>
<td>37</td>
<td>41</td>
<td>106</td>
<td>207</td>
<td>34</td>
</tr>
</tbody>
</table>

(b) For $\nu > 0$ we give an alternative proof for all cases (25), rather than a table for the small cases, which would occupy almost as much space. Let $t, u, v$ have the values 0, 1 in

$$2^{a-1} - t)^2 + (2^{a-1} + t)^2 + u^2 + v^2.$$

For $a = 2^{a-1}$ we thus perceive the entries

$$\xi_a + r\mu + R\nu, \quad (r, R) = (1, 1), (2, 0), (2, 2), (3, 1), (4, 2).$$

From $14 = 3^2 + 2^2 + 1^2$ we establish at once the entry $\xi_a + \mu + \nu$ for $A = 7$; from $6 = 2^2 + 1^2 + 1^2$, $22 = 3^2 + 3^2 + 2^2$, $34 = 4^2 + 3^2 + 3^2$, we see the entries $\xi_a + \mu + \nu$, $\xi_a + 2\mu$ for $A = 3, 11, 17$.

**Lemma 15.** Let $s = 1$ or $-1$, $0 \leq (2-s)\nu \leq \mu$, $a$ even, $r, R, B$ integers, $r > 0$.

*Write*

$$\theta = (a + r)\mu + (B + R)\nu.$$

*Then*

$$\lambda_{a+1} - \max (\theta, \xi_{a-1}) \leq \mu + s\nu$$

*if, for some value $p = 0, 1, \cdots, r-1,$

$$\left((r - p)B_{a-1} + B + R + (2 - s)\nu + 1 > (r - p + 1)(b_{a+1} - s)\right)$$
For, \( \eta = \theta - \rho \{ \mu - (2-s)\nu \} \leq \theta \). Hence (69) holds if \( \lambda_{n+1} - \max (\eta, \xi_{n-1}) \leq \mu + sv \). If \( \lambda_{n+1} - \xi_{n-1} \geq \mu + sv \); i.e. \( \mu > (B_{n-1} - b_{n+1} + s)\nu \), then \( \lambda_{n+1} - \eta \leq \mu + sv \), or \( (r - \rho)\mu \geq (b_{n+1} - B - R - (2-s)p - s)\nu \), if (70) holds.

We apply the lemma with \( s = 1, B = B_a \). Then, by (34), (37), (25) and (26), (70) follows from

(71) \( a^{1/2} \{ 2(r - \rho) + s(2/A)^{1/2} - 3^{1/2}(r - \rho + 1) \} > 2r - R - 3\rho - 1 \).

By (25) and (26) the coefficient of \( a^{1/2} \) is positive if \( r - \rho > 1.2, A, .5, .1, .1 \) respectively. The choices \( r = 4, 2, 1, 2, 2, \rho = 2, 1, 0, 1, 1 \) make the right member of (71) zero or negative.

8. Table \( F_a, k \geq 1, \mu \geq 3|v| > 0 \). Since the gap

(72) \( \Gamma \equiv \mu - |v| \)

occurs at the beginning of the table, \( 2|v| \) is allowable and part of the treatment is like that of §7. We use \( b_a \) instead of \( b_a \), \( \Gamma \) instead of \( \gamma \), (36) instead of (35), and see that, in addition to the cases (25), we have to consider the possibilities \( a \) even and

(73) \( B_a = B_{a-1} + 1 \) if \( v > 0 \), \( B_a = B_{a+1} + 1 \) if \( v < 0 \).

When \( w \), of (31), exists,

\( w = \omega_a = \lambda \mu + 2^hwv \equiv (a, 2^hw) \)

is the entry of \( L(a) \) just below \( \zeta_a \). Clearly \( w \leq z - 1 \). Hence, in cases (73), \( \omega_a - (a-1, B_{a-1}) \leq \Gamma \); and, in cases (732), \( (a+1, b_{a+1}) - \omega_a \leq \Gamma \). If, in these cases,

(74) \( (2^hw)^2 \geq 3a \),

then \( 2^h w > b_e, \ e = a \mp 1 \), by (37), and, respectively, \( (a+1, b_{a+1}) - \omega_a \leq \Gamma \), \( \omega_a - (a-1, B_{a-1}) \leq \Gamma \).

When (73) holds, \( B_a \) is, by (34), the largest \( b \leq (2a)^{1/2} \), whence \( z \) is the greatest \( y \leq (rA)^{1/2} \), where \( r = 4 \) if \( g = 2h \), \( r = 2 \) if \( g = 2h - 1 \). Now

(75) \( (z - 2)^2 \geq 3A \) (if \( g = 2h \))

if \( (4A)^{1/2} - (3A)^{1/2} \geq 3, A \geq 127 \), and

(76) \( (z - 1)^2 \geq (3/2)A \) (if \( g = 2h - 1 \))

if \( (2A)^{1/2} - (3A/2)^{1/2} \geq 2, A \geq 113 \). Hence \( w \) exists for these values \( A \), and

(761) \( 2^{2h}(\tau A - z^2) \geq 2^{h+1}z + 5 \).
and (73a) fails to hold if \(4(a+1) \geq (2^h+1)^2\), or
\[
(76a) \quad 2^h(a^2 - z^2) \geq 2^{h+1}z - 3.
\]
Consider column \(y(4A)\) of Table I. Observing the entry \(f(3)+2f(1)+f(-1) = 12\mu + 4\nu\) of \(F_1\), we define
\[
w = 2 \text{ if } A = 3, h = 1, g = 2h.
\]
We see that \(w^2 \geq 3A\) except for
\[
(77) \quad 1 \leq A \leq 19, A = 23, 29, 35, 43, 71, 79.
\]
When \(\tau = 4\), (76i) holds in all these cases except
\[
(78) \quad h = 1, A = 3, 5, 7; h = 1, 2, A = 13, 17; h = 1, 2, 3, A = 43; A = 1, 9.
\]
When \(\tau = 4\), (76a) holds in all cases (77) except
\[
(79) \quad h = 1, A = 7, 17; h = 1, 2, A = 13; h = 1, 2, 3, A = 43; A = 1, 9.
\]
Consider column \(y(2A)\) of Table I. In case \(h = 1, A = 1, 3, 15, 21, 27, 35, 43, 45, 55\);
\[
\begin{align*}
&h = 1, 2, A = 5, 9, 19; h = 1, 2, 3, A = 13, 25, 33; \\
&\text{in all of which cases } w \text{ is defined. Also } (76a) \text{ holds unless}
\end{align*}
\]
(82) \(h = 1, A = 9, 27, 35, 43; h = 1, 2, A = 5, 19, 33; h = 1, 2, 3, A = 13, 25.
\]
I. Remaining cases (73i), \(g\) even: namely, (78). Since \(w\) exists, \(\omega_a - \alpha \leq \Gamma\), \(\alpha = (a - 1, B_{a-1})\). Let \(\beta = (a + 1, B_{a+1})\).
\[
(a) \quad A = 3, 5, 7, 13, 17, h = 1. \text{ Then } z = w + 1.
\]
\[
(b) \quad A = 13, 17, h = 2. \text{ Then } \omega_a = 164\mu + (A + 11)\nu = \beta - \Gamma.
\]
\[
(c) \quad A = 43. \text{ We observe the entries } \omega_a + \mu \mp \nu, \omega_a + 2\mu \text{ since}
\]
\[
43 - 2^h + 1 = (5 \cdot 2^h)^2 + (3 \cdot 2^h)^2 + (3 \cdot 2^h)^2 + (\mp 1)^2,
\]
\[
43 - 2^h + 2 = (5 \cdot 2^h)^2 + (3 \cdot 2^h + 1)^2 + (3 \cdot 2^h - 1)^2 + 0^2.
\]
If \(h = 1\) or \(2, \xi - (\omega_a + 2\mu) \leq 2\nu, \text{ since } \mu \geq 3\nu. \text{ If } h = 3, \omega + \mu + \nu = 2753\mu + 89\nu = \beta.
\]
\[
(d) \quad A = 1. \text{ Then } \omega = 2^{2h} + 2\nu, \alpha = (2^{2h} - 1)\mu + (2^{h+1} - 1)\nu. \text{ To establish}
\]
\[
\omega + \mu \mp \nu, \omega + 2\mu, \omega + 3\mu \mp \nu, \omega + 3\mu + 3\nu \text{ as entries of } F_1, \text{ give the } x_i \text{ in } 2^{2h} + x_i^2
\]
\[
\text{values } -1, 0, \text{ or } 1. \text{ Finally set } x_2 = 2, x_3 = x_4 = 0, \text{ for the entry}
\]
\[
\theta = \omega + 4\mu + 2\nu. \text{ We use Lemma 15 with } s = -1, a = 2^{2h}, B = 2^h, r = 4, R = 2,
\]

$b$ in place of $b$. When $p = 1$, (70) becomes $7 \cdot 2^h - 1 > 4b_{a+1}$, and follows from (37a) for every $h \geq 1$.

(e) $A = 9$. Then $\omega = 9 \cdot 2^h + 5 \cdot 2^u$, $\alpha = (9 \cdot 2^h - 1)\mu + (6 \cdot 2^h - 1)\nu$. The quantities $\omega + \mu + \nu$, $\omega + 2\mu$, $\omega + 3\mu - \nu$, $\theta = \omega + 3\mu + \nu$ are entries of $F_1$. E.g. if $v = 2^{h-1}$,

$$36\alpha^2 + 3 = (4v + 1)^2 + (4v - 1)^2 + (2v)^2 + (-1)^2.$$  
We use Lemma 15 with $s = -1$, $a = 9 \cdot 2^h$, $B = 5 \cdot 2^h$, $r = 3$, $R = 1$, $b$ for $b$.

When $p = 0$, (70) is $23 \cdot 2^h - 5 > 4b_{a+1}$, which follows from (37a).

II. Remaining cases (73i), $g$ odd; i.e. (81). Again $w$ exists.

(a) $h = 1$, all $A$'s in (81). Then $z = w + 1$, $\xi = \omega = 2v$.

(b) $A = 2, A = 5, 9, 19, 13, 25, 33$. Then $4w = 8, 12, 20, 16, 24, 28$. In view of $8A = 6^2 + 2^4$, $6^2 + 2^4$, $10^2 + 2^4 + 2^4$, $8^2 + 6^2 + 2^4$, $10^2 + 8^2 + 2^4$, $10^2 + 10^2 + 2^4$, the sum of the square roots being $4w$, we have the entries of $F_1$, $\omega + \mu - \nu$, $\theta = \omega + \mu + \nu$, by adding $(\mp 1)^2$. Since $\xi = \omega = 4v$, $\xi - \theta \leq 2 |v|$.  
(c) $h = 3$, $A = 13, 25, 33$. Precisely as in (b), $\theta = \omega + \mu + \nu$ is an entry. Also, $b_{a+1} = 35, 47, 55$. Hence $\beta - \theta = 2v$, $-2v$, $-2v$.

III. Cases (25), $v > 0$. We use Lemma 15 with $s = -1, B = B_a$, $b$ for $b$.

Then (70) becomes

$$(r - p)B_{a-1} + B_a + R + 4p - r > (r - p + 1)b_{a+1},$$

which, by (34), (25), (26), and (37a), follows from

$$(83) \quad a^{1/2} \left\{ 2(r - p) + z(2/A)^{1/2} - 3^{1/2}(r - p + 1) \right\} > 3r - R - 6p.$$

To reobtain the $r, R$ of (67) we need merely interpolate some entries among those exhibited in (b) of §7, by changing some of the $x_i = 1$ to $x_i = -1$. We can then evidently reach $\theta = \xi + \gamma \mu + R \nu$ from $\xi$ by increments $\leq \Gamma$ or $2\nu$.

Now (83) is again trivially true, except when $A = 7$. Then it becomes

$$a^{1/2}(2 + (2/7)^{1/2} - 2 \cdot 3^{1/2}) > 2, \quad a \geq 202.$$  
If $a = 14$, $\xi = 14\mu + 6\nu = \beta - \gamma$; if $a = 56$, $\theta = 57\mu + 13\nu = \beta - 2\nu$.

Hence we have proved

**Theorem 3.** If $0 < \nu \leq \frac{1}{2}\mu$, $\Gamma = \mu - |\nu|$ is the largest gap in $F_1$, and hence in every table $F_k$, $k \geq 1$.

We rework the part of §7 relating to $\nu < 0$. The condition corresponding to (58a) is here

$$(84) \quad \mu - |\nu| > (B_{a+1} - x_{a-1} - 2)|\nu|.$$  
Taking now $b^4 = b_{a-1}$, whence $b^3 \leq S_{d^{1/2}}$ by (37a), we require...
which holds if \( 4 \leq (R_i + 2 - 2S_{-1})e^{1/2} \), or \( e \geq 21000 \).

IV. Cases (25), \( \nu < 0 \). Since \( 11 \cdot 2^{11} > 21000 \) the same early values remain as in (a) of \( \S 7 \). If \( a = 2^8, 3 \cdot 2^6, 7 \cdot 2^2, 11 \cdot 2^4, 17 \cdot 2^2, 17 \cdot 2^7 \), and \( 17 \cdot 2^7 \), we find \( b_{a-1} = b_{a-1} - 2 = B_{a-1} \). For \( a = 2^7, 2^{11}, 2^{18}, 3 \cdot 2^{11}, 3 \cdot 2^{18}, 7 \cdot 2^7 \), we get the entry of \( L_1(a - 1) \) with \( b = B_{a-1} - 1 \) by changing \( x_1 = 1 \) to \( x_1 = -1 \) below (65). In addition we have the following entries.

\[
\begin{array}{cccccccc}
a & b & x_1 & x_2 & x_3 & x_4 & a-1 & B_{a+1} - b_{a-1} - 2 \\
2^8 & 7 & -1 & 1 & 2 & 5 & 9 \\
2^9 & 35 & -1 & 5 & 14 & 17 & 39 & 4 \\
3 \cdot 2^7 & 31 & 2 & 3 & 9 & 17 & 33 & 4 \\
7 \cdot 2^8 & 23 & 3 & 3 & 14 & 25 & 2 \\
\end{array}
\]

V. Remaining cases (732), \( g \) odd: (82).

(a) all \( h = 1 \): \( w = z - 1, \omega - \xi = 2\nu \).
(b) \( h = 2, A = 33, 25 \): \( 2^hw = 8, 24; b_{a-1} = 27, 23 \).
(c) \( h = 2, A = 5, 19, 13 \): \( 2^hw = 8, 20, 16; b_{a-1} = 9, 21, 17; 39 = 6^2 + 1^2 + 1^2 + (-1)^2 \)
\[ + 151 = 11^2 + 5^2 + 2^2 + 1^2, 103 = 7^2 + 7^2 + 2^2 + (-1)^2; b = 7, 19, 15 \).
(d) \( h = 3, A = 25 \): \( 2^hw = 48, b_{a-1} = 47 \).
(e) \( h = 3, A = 13 \): \( 2^hw = 32, b_{a-1} = 35, 13 \cdot 2^2 - 1 = (\pm 1)^2 + 5^2 + 10^2 + 17^2 \);
\( b = 31, 33 \).

VI. All remaining cases except \( a = 2^h, h \geq 3 \): (79).

(a) \( h = 1, A = 7, 17, 13, 43, 1, 9 \): \( w = z - 1, \omega - \xi = 2\nu \).
(b) \( h = 1, A = 43 \): \( 2^hw = 22, b_{a-1} = 21 \).
(c) \( h = 2, A = 13 \): \( 2^hw = 24, b_{a-1} = 23 \).
(d) \( h = 2, A = 43 \): \( 2^hw = 44, b_{a-1} = 45, 2^4 43 - 1 = 1^2 + 6^2 + 17^2 + 19^2, b = 43 \).
(e) \( h = 3, A = 43 \): \( 2^hw = 88, b_{a-1} = 89, 43 \cdot 2^2 - 1 = 1^2 + 15^2 + 37^2 + 34^2, b = 87 \).
(f) \( A = 9, g = 2h \). By (64) and \( \S 7 \) there remain only \( h \leq 6 \). For these we have the following table.

\[
\begin{array}{cccccccc}
a & 2^hw & b & x_1 & x_2 & x_3 & x_4 & b_{a-1} & B_{a+1} \\
9 \cdot 2^4 & 20 & 19 & -1 & 5 & 6 & 9 & 19 & 23 \\
9 \cdot 2^6 & 40 & 39 & 2 & 7 & 9 & 21 & 41 & 47 \\
9 \cdot 2^8 & 80 & 79 & 5 & 15 & 17 & 42 & 83 & 95 \\
9 \cdot 2^{10} & 160 & 159 & 15 & 26 & 33 & 85 & 165 & 191 \\
9 \cdot 2^{12} & 320 & 319 & 35 & 54 & 59 & 171 & 331 & 383 \\
\end{array}
\]

(g) \( A = 1, h = 2 \). \( L_1(15) \): 7, 5, 3;
\( L_1(16) \): 8, 4; and \( 3 < 4 \).
Thus, if $\mu \geq 3 |\nu|$ and $\nu < 0$, all gaps in $F_1$ are $\leq \Gamma$ except possibly those necessary to pass the points $a = 2^h, h \geq 3$.

9. Gaps associated with $a = 2^h, h \geq 1, h \geq 3, \nu < 0$. Let $b_{h,k}$ denote the least $b$ on $L_k(2^h - 1)$. For any $e$ write $b_k(e)$ for the least $b$ on $L_k(e)$. Set

$$\alpha_{h,k} = (2^h - 1)\mu + b_{h,k}\nu, \quad \beta_h = 2^h\mu + 2^h+1\nu,$$

(85)

$$\gamma_h = (2^h + 1)\mu + (2^{h+1} - 1)\nu, \quad \omega_h = 2^h\mu + 2^h\nu.$$

If $\mu \geq |\nu|$, $\beta_h$ is evidently the least entry of all $L_k(a')$ with $a' \geq 2^h+1$. We have $\beta_h - \alpha_{h,k} \leq \Gamma$ if

(86)

$$\mu \leq C_{h,k} |\nu|, \quad C_{h,k} = 2^{h+1} - 2 - b_{h,k}.$$

Suppose that, when $\mu > C_{h,k} |\nu|$, $\alpha_{h,k}$ is the largest entry of all $L_k(a')$ with $a' \leq 2^h-1$, that is to say,

(87)

$$b_k(2^h - r) \geq b_{h,k} - (r - 1)C_{h,k} \quad (r = 2, 3, 4, \ldots).$$

If $\omega_h - \alpha_{h,k} \leq \Gamma$ we can pass first to $\omega_h$, then to $\beta_h$. The contrary case is equivalent to

(88)

$$b_{h,k} \geq 2^h + 1, \text{ or } C_{h,k} \leq 2^h - 3.$$

If both (87) and (88) hold, $F_k$ contains the gap

$$\Gamma_{h,k} = \min (\beta_h, \omega_h) - \max (\alpha_{h,k}, \gamma_h)$$

(89)

$$= \min (2^h |\nu|, \mu + |\nu|, 2\mu - (C_{h,k} + 1) |\nu|).$$

The greater of $\Gamma$ and $\Gamma_{h,k}$ is

$$\Gamma \text{ if } \mu \leq C_{h,k} |\nu| \text{ or } \mu \geq (2^h + 1) |\nu|,$$

(90)

$$2^h |\nu| \text{ if } (2^h - 1) |\nu| \leq \mu \leq (2^h + 1) |\nu|,$$

$$\mu + |\nu| \text{ if } (C_{h,k} + 2) |\nu| \leq \mu \leq (2^h - 1) |\nu|,$$

$$2\mu - (C_{h,k} + 1) |\nu| \text{ if } C_{h,k} |\nu| \leq \mu \leq (C_{h,k} + 2) |\nu|.$$

Since $2^h - 1 - (2^h - 1)^2 \equiv 1 \pmod{3}$, Lemma 13 shows that

(91)

$$b_{h,k} \leq 2^h - 1 + (9/5)^{1/2}(2^{h+1} - 2)^{1/2}.$$

Hence $b_{h,k} < 2^{h+1} - 2$, and indeed

(92)

$$C_{h,k} \geq ((2^h - 1)^{1/2} - 1)^2.$$

We readily find a lower limit for $b_k(e)$. Let $A$ denote the greatest integer such that $e - A^2$ is a sum of three squares. If $e > 4k^2$ and $k \geq 1$,

(93)

$$b_k(e) \geq -3k + (e - 3k^2)^{1/2}.$$
Indeed, if \( e - A^2 > 3k^2 \), we have
\[
(94) \quad b_k(e) \leq A - 2k + (e - A^2 - 2k^2)^{1/2}.
\]
Both (93) and (94) can be improved in special cases by various considerations.

If \( e = 2^h - 1, A = 2^h - 1 \). Hence by (94), \( b_k, h > 2^h \) as soon as \( 2^h - 1 > 3k^2 \), and possibly for smaller values of \( h \). We find also that (87) holds in virtue of (86a), (92), (93), and (94), as soon as \( 2^{k+1} - 2 > 3k^2 \). Hence the gap \( \Gamma_k, h \) occurs in \( F_k \) for every \( h \) such that \( 2^h - 1 > 3k^2 \), and possibly for smaller values of \( h \). It exceeds \( \Gamma \) only within the range
\[
(95) \quad C_{k,i} |v| < \mu < (2^h + 1) |v|.
\]
If \( b_k, h > 2^h \) but \( 2^{k+1} - 2 < 3k^2 \) it is necessary, under the present analysis, to verify whether (87) holds. If (87) did not hold, \( \Gamma_k, h \) would have to be changed by the introduction of new entries in the max term subtracted in (89).

It is easy to see by Lemma 14 and an argument like that employing (84) that we can pass by increments \( \leq \Gamma \) in \( F_h \) from \( e\mu + b_k\nu \) to \( e\mu + b_k, h\nu \), where \( e = 2^{k+1} - 1 \), at least if \( h \geq 8 \); and as we shall see, for all \( e \).

(i) \( A = 3 \).

\[
L_2(63) : 15, 13, 11_t, 9_t, 7_s; \quad L_2(64) : 16, 8.
\]
Here the terms without subscripts belong to \( L_0(a) \), and those with subscript \( j \) belong to \( L_j(a) \) but not to \( L_{j-1}(a) \).

Thus \( b_1, 1 = 9 > 8 \), and \( F_1 \) contains the gap \( \Gamma_{1,i} = 8 |v| \) if \( 7 |v| \leq \mu \leq 9 |v|, 2\mu - 6 |v| \) if \( 5 |v| \leq \mu \leq 7 |v|, \mu + |v| \) if \( \mu = 7 |v| \).

(ii) \( A = 4 \).

\[
L_4(256) : 32, 16; \quad L_4(255) : 31, 29, 27, 25, 23, 21_t, 19_s, 17_s, 15_t, 13_s, 11_s, 9_s, 7_s, 1_s.
\]
Hence \( b_4, 1 = 21, b_4, k = 17 \) \((k = 2, 3, 4)\). There is no difficulty in passing from \( 255\mu + 17\nu \) to \( 255\mu + 13\nu \) when \( k \geq 5 \), since we can assume \((257\mu + 31\nu) - (255\mu + 17\nu) > \mu + \nu \). To assure (87) for \( k = 4 \), we verify that
\[
b_4(254) \geq 4, \quad b_4(253) \geq -9, \cdots.
\]
Since \( C_{4,i} = 9, C_{4,k} = 13 \) \((k = 2, 3, 4)\), the gap (90) is easily written down.

(iii) \( A = 5 \).

\[
L_5(1023) : 63, \cdots, 55, \cdots, 49, \cdots, 39_t, 37_s, 33_s, 31_s.
\]
Now \( C_{5,i} = C_{5,2} = 23, C_{5,3} = C_{5,4} = 29 \).
(iv) \( h = 6 \).

\[ L_6(4095) : 127, \ldots, 111, \ldots, 97, \ldots, 75_1, 73_2, 71_2, 69_6, 67_6, 63_6. \]

Now \( C_6_1 = 51, C_6_2 = C_6_3 = C_6_4 = 55, C_6_5 = 59 \).

(v) \( h = 7 \).

\[ L_7(16383) : 255, \ldots, 221, \ldots, 199, \ldots, 149, 147_2, 145_2, 143_7, 141_6, 139_6, 137_7, 135_6, 131_7, 127_7. \]

Hence \( C_7_1 = 105, C_7_2 = C_7_3 = C_7_4 = 109, C_7_5 = C_7_6 = 119 \).

Finally we note that \( C_8_1 = 229 \). Hence we have

**Theorem 4.** Let \( \mu \geq -3v > 0, k \geq 1 \). Let \( b_{h,k} \) denote the least \( b \) on \( L_k(2^{2k} - 1) \).

For every \( h \) such that \( b_{h,k} > 2^h \), which is true at least if \( 2^h - 1 > 3k^2 \), \( F_k \) contains a gap just preceding \( \min (\beta_h, \omega_k) \) of (85) which exceeds \( \Gamma \) for certain values of \( \mu, v \). This gap is given in (90), with \( C_{h,k} \) in (86), if (87) holds, which is true if \( 2^{h+1} - 2 > 3k^2 \). No other gaps in \( F_k \) exceed \( \Gamma \). In particular the largest gap in \( F_1 \) is \( \Gamma \) if \( 3 |v| \leq \mu \leq 5 |v| \),

\[ \Gamma \text{ if } (2^{h-1} + 1) |v| \leq \mu \leq C_{h,1} |v|, \]

\[ 2\mu - (C_{h,1} + 1) |v| \text{ if } C_{h,1} |v| \leq \mu \leq (C_{h,1} + 2) |v|, \]

\[ \mu + |v| \text{ if } (C_{h,1} + 2) |v| \leq \mu \leq (2^h - 1) |v|, \]

\[ 2^h |v| \text{ if } (2^h - 1) |v| \leq \mu \leq (2^h + 1) |v|, \]

where \( h = 3, 4, 5, \ldots \), and

(97) \( C_{3,1} = 5, C_{4,1} = 9, C_{5,1} = 23, C_{6,1} = 51, C_{7,1} = 105, C_{8,1} = 229, \ldots \).

The largest gap in \( F_2 \) is \( \Gamma \) if \( 3 |v| \leq \mu \leq 13 |v| \), and for the rest is given by (96) with \( C_{h,1} \) replaced by \( C_{h,2} \), \( h = 4, 5, 6, \ldots \), and

(98) \( C_{4,2} = 13, C_{5,2} = 23, C_{6,2} = 55, C_{7,2} = 109, C_{8,2} = 229, \ldots \).

The values \( C_{h,k} \) to be used in writing down the largest gaps in \( F_3, \ldots, F_6 \), in the above fashion, are

\[ C_{4,3} = 13, C_{5,3} = 29, C_{6,3} = 55, C_{7,3} = 109, \ldots; \]

\[ C_{h,4} = C_{h,3} \text{ (} h = 4, 5, 6, \ldots \text{)}; \]

\[ C_{8,5} = 59, C_{7,5} = 119, \ldots; C_{7,6} = 119, \ldots. \]

If \( k \geq 7 \) all gaps in \( F_k \) are \( \leq \Gamma \) if \( 3 |v| \leq \mu \leq 229 |v| \).

10. **Table** \( F_k \), \( |v| < \mu < 3 |v| \), \( k \geq 1 \). The writer has previously considered the functions \( 3x^2 \pm 2x \) (in papers to appear shortly in the Bulletin of the
The following theorem and three lemmas were proved.

**Theorem 5.** If $\mu \geq |\nu| > 0$ the largest gap in $F_\mu$ is

$$r = \mu - |\nu|, \quad s = 5|\nu| - 3\mu \quad \text{if } \mu \leq (3/2)|\nu|.$$  

Evidently $r$ and $s$ are gaps in every $F_k$. For let $j = \pm (\text{sign } \nu)$ or $1 \cdot (\text{sign } \nu)$. Then they occur from $4f(0)$ to $f(-j) + 3f(0)$, and from $4f(-j)$ to $f(j) + 3f(0)$.

**Lemma 16.** Although [by Theorem 5] every integer $p \geq 0$ is a sum of four values of $3x^2 + 2jx$ for integers $x$, there exist infinitely many integers $p > 0$, for any $k \geq 1$, such that

$$p \neq (3x_1^2 + 2jx_1) + \cdots + (3x_4^2 + 2jx_4), \quad x_i \geq -k.$$  

**Lemma 17.** If $k = j = 1$ the only odd $p > 0$ satisfying (100) are

$$9, 13, 25, 29, 45, 47, 69, 75, 79, 97, 109, 149, 165, 189, 235, 305, 509.$$  

If $k = -j = 1$ the only odd $p > 0$ such that (100) holds are

$$33, 59, 129.$$  

Every odd $p > 0$ is a sum of four values $3x^2 + 2jx$, $x \leq -2$.

**Lemma 18.** Let $k \geq 0$, $j = \pm 1$. The only even $p \geq 0$ not sums of four values $3x^2 + 2jx$ for integers $x \geq -k$ are $\frac{1}{2}(4t-4)$ where

1. $t = 4, 34, 52, 130, 148, 172, 202, 286, 298, 316, 340, 358, 394, 436, 490, 526, 580, 598, 676, 694, 766, 772, 844, 862, 898, 1102, 1252, 1306; 2^k \neq j \mod 3$,
2. $t = 58, 154, 178, 292, 310, 346, 382, 604, 622, 778, 814, 1006, 1198, 1276, 3676; 2^k = j, 2 \cdot 2^h > 3k-j$;
3. $t = 10, 28, 70, 124, 190, 226, 262, 430, 466; 2^h > 3k-j, or 2^h = j$ and $2 \cdot 2^h > 3k-j$;
4. $t = 94, 244; 2^h > 3k-j, or 2^h = j and 5 \cdot 2^h > 3k-j \geq 2^h$;
5. $t = 22, 106, 238; 2 \cdot 2^h > 3k-j, or 2^h = j and 4 \cdot 2^h > 3k-j \geq 2 \cdot 2^h$;
6. $t = 46, 142; 4 \cdot 2^h > 3k-j, or 2^h = j and 5 \cdot 2^h > 3k-j \geq 4 \cdot 2^h$;
7. $t = 82, 166, 220, 334; 2^h \neq j$ and $4 \cdot 2^h > 3k-j$;
8. $t = 76, 484, 652, 1564; 2^h = j$ and $5 \cdot 2^h > 3k-j$;
9. $t = 508, 1324; 2^h \neq j$ and $7 \cdot 2^h > 3k-j$.

It is seen, by continuity, that every $F_k$ contains a gap greater than

$$\epsilon = \max \left( r, s \right),$$
in a neighborhood of $\mu = (3/2)|\nu|$, and that the first such gap will occur as far out as we please for a sufficiently large $k$.  

To determine the least even values \( p \) for which (100) holds, write \( H \) for the least integer such that \( 2^H > 3k - j \). If \( 2^H \equiv j \pmod{3} \) the only \( 4^H \leq 6 \cdot 4^H \) in Lemma 18 are

\[
4^{H-2j'} \quad (j' = 46, 76, 88, 94).
\]

If \( 2^H \not\equiv j \pmod{3} \) the only \( 4^H \leq 3 \cdot 4^H \) are

\[
4^{H-2j''} \quad (j'' = 46^*, 76^*, 88^*, 94^*, 142^*, 160, 184),
\]

where the four starred numbers are to be omitted unless \( 5 \cdot 2^{H-3} > 3k - j \).

Write \( M_p = M(p, k, \mu, \nu) \) for the set of all numbers \( \mu a + \nu b \) such that

\[
4^H - v \leq \mu - j |v| \quad \text{and} \quad 6 |v| - 4^H \leq 5 |v| - 3 \mu,
\]

i.e. if \( 3 \mu \leq 5 |v| \) and \( |v| \leq \mu \).

The largest entry of \( M_p \) is

\[
\xi_+(p) = \mu a_+(p) + \nu b_{-j}(p) \quad \text{or} \quad \xi_-(p) = \mu a_-(p) + \nu b_+(p).
\]

according as

\[
\theta = \mu - (3/2) |v| \quad \text{is} \geq 0 \quad \text{or} \leq 0;
\]

and the least entry is the other.

In the next two sections we consider completely the cases \( k = 1 \) and 2, and \( k \geq 3, \mu \geq 5 |v|/3. \) We apply the preceding discussion in \( \S 13. \)

11. Cases \( k = 1 \) and 2, \( |v| < \mu < 3 |v| \). We prove the following theorems:

**Theorem 6.** The largest gap in \( F_1 \) is

\[
2\nu \quad \text{if} \quad \nu < \mu < 3 \nu,
\]

and, if \( \nu < 0 \), is
Theorem 7. The largest gap in $F_2$ is

$$5
4 \text{ if } v \leq \mu \leq (4/3) v, \quad 3 \mu - 3 \text{ if } (4/3) v \leq \mu \leq (7/5) v,$$

(113) $$4 \text{ if } (7/5) v \leq \mu \leq (3/2) v, \quad 2 \mu - 2 \text{ if } (3/2) v \leq \mu \leq (5/3) v, \quad 3 \mu - \mu \text{ if } (5/3) v \leq \mu \leq 2 v,$$

If $v < 0$ the largest gap in this table is

$$5 \text{ if } v \leq \mu \leq (4/3) v, \quad 3 \mu - 3 \text{ if } (4/3) v \leq \mu \leq (7/5) v,$$

(114) $$4 \text{ if } (7/5) v \leq \mu \leq (3/2) v, \quad 2 \mu - 2 \text{ if } (3/2) v \leq \mu \leq (5/3) v, \quad 3 \mu - \mu \text{ if } (5/3) v \leq \mu \leq 2 v.$$

No entries of $F_1$ come between $7\mu + 3v$ and $7\mu + 5v$ if $\mu > v > 0$; hence (111) is a gap in $F_1$. Similarly the gaps (112) for $v < 0$ occur in the following places: from max $(22\mu + 4v, 26\mu + 10v)$ to $25\mu + 7v$ if $|v| \leq 3|\mu|$, from $15\mu + 5v$ to min $(17\mu + 7v, 15\mu + 3v)$ if $(5/3)|v| \leq \mu \leq (7/4)|v|$, from $127\mu + 21v$ to $125\mu + 17v$ if $|v| \leq 3 \mu$.

In Dickson's table II* multiply the terms free of $m$ by $t$, and write $n = 2\mu$, $t = \mu + v$, thereby obtaining table $F_1$ for $v < 0$. We easily verify that all gaps in $F_1$ are $\leq 4|v| - 2\mu = m - 4t$ or $3\mu - 3 |v| = 3t$, at least up to $130\mu + 22v = 54m + 22t$. Further, all gaps are $\leq 3 |v| = m - 3t$ or $2\mu - 2 |v| = 2t$, at least to this point. Finally, all gaps to this point are $\leq 2 |v| = m - 2t$ if $m \geq 3t$, i.e. $\mu \leq 3|v|$.

Proof of Theorem 6, by the divisions $L_1(a)$. Suppose first that $v > 0$. Then $(0, 2), (1, -1)$, and hence $(2, -4)$ are allowable increments. If $e \equiv 1 \pmod{4}$ we can pass from $(e, B_e)$ to $(e+1, B_{e+1})$, $(e+1, B_{e+2})$, $(e+2, B_{e+1})$, $(e+2, B_{e+2})$ by increments $\mu - v$ and $2v$, provided that $B_{e-2}$ belongs to $L_1(e)$ if $e \not\equiv 0 \pmod{4}$. If $e \equiv 3 \pmod{4}$ we pass from $(e, B_e)$ to $(e+2, B_{e-4})$, $(e+2, B_{e-2})$, $(e+2, B_{e+2})$, provided these quantities belong to $L_1(e+2)$. Finally we may verify

Lemma 19. If $a \not\equiv 0 \pmod{4}$, $B_a - 2$ belongs to $L_1(a)$ for every $a \geq 1$, and $B_a - 4$ does so except for

\[(115) \quad a = 1, 5, 9, 13, 14, 23, 29, 49, 71.\]

If $a \equiv 3$ and $B_{a+2} = B_a + 2$, $B_a - 4$ belongs to $L_1(a + 2)$ unless

\[(116) \quad a + 2 = 13, 21, 57, 157.\]

Crossing these points is found to introduce no new gaps.

Suppose second that $v < 0$. We prove

Lemma 20. If $\mu > |v|$, $v < 0$, and $a \equiv 2 \pmod{4}$, all gaps in $F_1$ are $\leq 4|v| - 2\mu$ or $2\mu - 2|v|$ between $(a, B_a)$ and $(a, B_a - 2)$.

For, if $B_{a+1} = B_a + 1$ we pass from $(a, B_a)$ to $(a+1, B_a + 1)$, $(a-1, B_a - 3)$, $(a, B_a - 2)$. If $B_{a+1} = B_a - 1$, then $B_{a-3} \leq B_{a-1} = B_a - 1$, and we pass to $(a-3, B_a - 5)$, $(a-1, B_a - 3)$, $(a, B_a - 2)$.

Lemma 21. If $\mu > |v|$, $v < 0$, and $a \equiv 2 \pmod{4}$, we can pass in $F_1$ from $(a, B_a - 2)$ to $(a+4, B_{a+4})$ by any of the following sets of increments:

I. $(-2, -4), (3, 3)$; II. $(-1, -3), (2, 2)$; III. $(0, -2), (1, 1), (2, 4)$.

I and II. We pass to $(a+1, B_a - 1)$. If $B_{a+3} \geq B_a + 1$ we proceed to $(a+3, B_a + 1)$, $(a+1, B_a - 3)$, $(a+3, B_a - 1)$, $(a+4, B_a)$. Otherwise, we use $(a-1, B_a - 5)$, $(a+1, B_a - 3)$, etc.

III. From $(a+1, B_a - 1)$ we pass to $(a+1, B_a - 3)$, either $(a+3, B_a + 1)$ or $(a+1, B_a - 5)$, $(a+3, B_a - 1)$, $(a+4, B_a)$.

This completes the proof of Theorem 6.

In Dickson's table IV* multiply the terms free of $m$ by $t$, and write $m = 2\mu$, $t = -\mu - v$, thus getting table $F_2$ for $v > 0$. The gap $3t$ is seen to occur from $9m - 3t$ to $9m$, if $7t \leq m$. Now $\Delta = m - 5t$ and $\Gamma = t$. We may verify that all gaps in $F_2$ are $\leq m - 5t$ or $2t$ from $0$ to $9m - 3t$, and from $9m$ to $198m - 21t = 375\mu + 21v$.

If $m \leq 7t$, i.e. $5\mu \leq 7v$, the gap $4\nu - 2\mu = m - 4t$ occurs in $F_2$ from $15\mu + 3\nu$ to $13\mu + 7v$. If $m \geq 5t$, i.e. $3\mu \leq 5v$, the gap $2t = 2\mu - 2\nu$ occurs from $15m - 7t$ to $15m - 5t$. If $4t \leq m \leq 5t$ the gap $m - 3t = 3v - \mu$ occurs from $14m - 2t$ to $15m - 5t$. If $3t \leq m \leq 4t$ the gaps $m - 3t$ and $4t - m$ are allowable; we verify that all differences in $F_2$ at least as far as $198m - 23t = 373\mu + 23v$ are $\leq m - 3t$ or $t$ or $4t - m$.

By this examination, one or the other of the following three sets of increments occurs in $F_2$ if $\nu \leq \mu \leq 3\nu$, and all differences to $373\mu + 23v$ are allowable

by any one of these sets: I \((-3, 5), (3, -3)\); II \((-2, 4), (2, -2)\); III \((-1, 3), (1, -1), (2, -4)\).

Let \(a = 1 \pmod{4}\). We can pass by I or II from \((a, B_a)\) to \((a+2, B_{a-2}), (a+4, B_{a-4})\), either \((a+1, B_{a+1})\) or \((a+5, B_{a-5}), (a+2, B_a), (a+4, B_{a-2})\); from \((a, B_{a-4})\) to either \((a-3, B_{a+1})\) or \((a+1, B_{a-5}), (a-2, B_a), (a, B_{a-2}), (a+2, B_{a-4}), (a+3, B_{a-5}), (a, B_a)\). By III we can pass from \((a, B_a)\) to \((a+1, B_{a+1}), (a+2, B_{a+2}), (a+1, B_{a+1}), (a+2, B_{a+1} - 1), (a+4, B_{a+1} - 5), (a+5, B_{a+1} - 6), (a+4, B_{a+1} - 3), (a+5, B_{a+1} - 4), (a+4, B_{a+1} - 1)\), etc. Since we can suppose \(a \geq 350\) we do not need quite all of

**Lemma 22.** If \(e \neq 0 \pmod{4}\), \(B_a - 6\) belongs to \(L_2(e)\) unless

\[
e = 1, 2, 11, 17, 35, 53, 71, 123, 239.
\]

If \(e = 2 \pmod{4}\), \(B_a - 8\) belongs to \(L_2(e)\) unless

\[
e = 2, 14, 46, 62, 74, 98.
\]

In Dickson’s table \(T_2^*\) write \(m = 2\mu, t = \mu + v\), obtaining \(F_2\) for \(v < 0\). Again \(\Delta = m - 5t\) and \(\Gamma = t\). The gap \(m - 4t = 4|v| - 2\mu\) is seen to occur from \(53m + 21t\) to \(54m + 17t\) if \(6t \leq m \leq 7t\), and from \(9m + 5t\) to \(10m + t\) if \(5t \leq m \leq 6t\). The gap \(6t - m\) occurs from \(28m + 2t\) to \(27m + 8t\) if \(4\frac{3}{4}t \leq m \leq 5t\), and the gap \(2m - 8t\) from \(25m + 16t\) to \(27m + 8t\) if \(m \leq 4\frac{3}{4}t\). Finally, \(3t\) occurs from \(54m + 14t\) to \(54m + 17t\) if \(m \geq 7t\).

The largest of these gaps occurring for the various intervals is shown in

(125). Hence the following sets of increments are allowable if \(|v| \leq \mu \leq 3|v|:\)

(i) \((-3, -5), (2, 2)\) if \(\mu \leq (3/2)|v|\);

(ii) \((-2, -4), (1, 1), (4, 6)\) if \((3/2)|v| \leq \mu \leq (7/4)|v|\);

(iii) \((-2, -4), (1, 1), (3, 5)\) if \((5/3)|v| \leq \mu \leq 2|v|\);

(iv) \((-1, -3), (1, 1), (2, 4)\) if \(\mu \geq 2|v|\).

An examination of \(T_2\) shows that the gaps (114) are the largest to \(54m + 17t\), and that we can pass from \(54m + 17t\) to \(144m + 36t = 324\mu + 36v\) with differences \(m - 5t\) and \(2t\), or \(m - 4t\) and \(t\).

Thus we may suppose \(a > 320\). Then each of \(B_a, B_{a-2}, \cdots, B_{a-8}\) belongs to \(L_2(a)\) if \(a \neq 0 \pmod{4}\). The passage from \((a, B_{a-4})\) to \((a+4, B_{a+4} - 4)\) by any of the sets of increments (i), (ii), (iii), (iv) is simple, and left to the reader. It is readily considered on graph paper, with \(a = 1 \pmod{4}\).

12. **Table** \(F_k, k \geq 3, 5|v| \leq 3\mu \leq 9|v|\). We prove the following theorem:

Theorem 8. The largest gap in $F_3$ is $\Gamma$ if $\nu<0$ and $(5/3)|\nu| \leq \mu \leq 3|\nu|$, $4\mu-6\nu$ if $(5/3)\nu \leq \mu \leq (7/4)\nu$, $8\nu-4\mu$ if $(7/4)\nu \leq \mu \leq (9/5)\nu$, $\Gamma$ if $(9/5)\nu \leq \mu \leq 3\nu$. The largest in $F_k$, $k \geq 4$, is $\Gamma$ if $(5/3)|\nu| \leq \mu \leq 3|\nu|$.

If $\mu \geq 2|\nu|$ this follows from Theorem 7. Let $(5/3)|\nu| \leq \mu \leq 2|\nu|$. Then both $(3, -5\nu)$ and $(-2, 4\nu)$ are $\leq \Gamma$, $j=\text{sign} \nu$. If $\nu<0$ the set of gaps (iii) of §11 is allowable, and, by results there obtained, it remains only to examine table $F_3$ to $54m+17t$. All gaps in $T_2$ to this point are $\leq t$ or $m-4t(=4|\nu|-2\mu)$ if we insert the entry $27m+7t$ of $F_3$.

Hence let $\nu>0$. Denote by $\pi_1$ the process of adding $(1, -1), (1, -1), (-2, 4)$ to an entry, and by $\pi_2$ that of adding $(1, -1), (1, -1), (3, -5)$, $(1, -1), (-2, 4)$. Let $a=1 \pmod{4}$. The process $\pi_2$ and two or three ($\pi_1$)'s brings us from $(a, B_a)$ to $(a+4, B_{a+4})$. For this procedure it is necessary that $B_a-8$ belong to $L_3(a+6)$, $B_a-7$ to $L_3(a+5)$, $\ldots$.

We find that five consecutive odd values $b$ satisfy

$$b^2 + 8b + 64 > 3a, 4a > b^2, b > -12,$$

for any odd $a$ such that

$$273 \leq a \leq 295, \quad 307 \leq a \leq 335, \quad 343 \leq a \leq 377, \quad a \geq 381;$$

and, permitting ourselves to use the extension of (20a) analogous to (32a) for $k=3$, that five consecutive even values $b$ satisfy

$$3b^2 + 32b + 256 > 8a, 4a > b^2,$$

for any $a=2 \pmod{4}$ such that $a \geq 6$. Since $379=17^2+9^2+3^2$, $29=B_a-8$ belongs to $L_3(a)$, $a=379$. Finally, six consecutive odd values $b$ satisfy (119) if $a=423, 467, 511, 555, 603, 655, 707, a \geq 757$, and $B_a-10$ belongs to $L_3(a=383)$. No further values $a=3 \pmod{4}$ and $>350$ satisfy $B_a=B_{a-8}+2$.

It remains only to examine table $F_3$ to $349\mu+18\nu$. We insert into table IV (Dickson, loc. cit.) the following entries of $F_3$:

$$4m + 3t, 14m - t, 18m - t, 20m - 5t, 24m - 5t, 25m - 4t, 25m - 3t, 32m - 5t,$$

$$48m - 4t, 48m - 4t, 49m - 8t (b = 7, 6, 4, 3), 50m - 8t, 50m - 5t.$$  

Then all gaps to $50m-10t$ are $\leq t$, $m-4t$, or $5t-m$, which are our allowable gaps. As in Dickson (American Journal of Mathematics, loc. cit., p. 44), we see that, if $(5/3)\nu \leq \mu \leq 2\nu$, $F_3$ contains a gap from $90\nu+10t=50m-10t$ to min $(86\nu+18\nu, 94\mu+4\nu)$. This fact gives the gaps other than $\Gamma$ in the theorem. From this point to $198m-23t=373\mu+23\nu$ all differences are $\leq t$, $m-4t$, or $5t-m$. Table $F_4$ contains the entry $91\mu+9\nu$ which bridges the above gap.
13. Table $F_\ell$, $k \geq 3$, $|\nu| < \mu < 5 |\nu|/3$. By definition of $a_{-j}(q)$ none of 
\[ (a_\nu, b_\nu) = (a_{-j}(q) - 4j\nu, b_+(q) + 6\nu) \quad (\nu = 1, 2, 3, \cdots), \]
belongs to $M_\nu$. It follows that 
\[ (b_+(q) + 6)^2 > 4(a_{-j}(q) - 4j) \quad (q \not\equiv 0, \text{ mod } 4). \]
For, in the contrary case, we must have 
\[ b_\nu^2 + 8b_\nu + 64 \leq 3a_\nu \]
for every $\nu \geq 1$ such that $4a_\nu \geq b_\nu^2$. Let $V$ denote the greatest $\nu$ for which (121) would hold. Then, simultaneously, 
\[ b_\nu^2 + 8b_\nu + 64 \leq 3a_\nu, \quad 4(a_\nu - 4j) < (b_\nu + 6)^2, \]
a contradiction for arbitrary $a_\nu > 0, b_\nu \geq 3$. It is to be noted that $b_\nu \geq b_+(q) + 6$, and to be verified that 
\[ b_+(q) \geq -2 - j \quad (q \not\equiv 0, \text{ mod } 4). \]
Since, then, $a_{-j}(q)$ is as small (if $j = 1$) or large (if $j = -1$) as the condition $4a \geq b^2$ permits, we must have 
\[ a_-(q - 1) \geq a_-(q) - 1, \quad a_-(q + 1) \geq a_-(q) - 3 \quad (j = 1), \]
\[ a_+(q + 1) \leq a_+(q) + 1, \quad a_+(q - 1) \leq a_+(q) + 3 \quad (j = -1), \]
provided $q \not= 0$. Some of these are of course vacuous if $M_\ell \pm 1$ is null.

Now, if $3a + 2jb + 1 = 3a' + 2jb'$, the inequality 
\[ (\mu a' + vb') - (\mu a + vb) \leq \epsilon \]
is readily seen to be equivalent to 
\[ a' \leq a + 1 \quad (\text{if } \theta \equiv 0), \quad a' \geq a - 3 \quad (\text{if } \theta \equiv 0). \]

Hence, if $p$ is even, all differences from the greatest entry $\beta_{p-1}$ of $M_{p-1}$ to the least entry $\sigma_{p+1}$ of $M_{p+1}$ are $\leq \epsilon$ if $M_p$ contains an entry $(a^*, b^*)$ such that 
\[ a^* \leq a_+(p-1) + 1, \quad a^* \geq a_-(p+1) - 1 \quad (\theta \equiv 0), \]
\[ a^* \geq a_-(p - 1) - 3, \quad a^* \leq a_+(p + 1) + 3 \quad (\theta \equiv 0). \]
While all of these hold with $a^* = a_j(p)$ in virtue of (123) and (124) if $p \equiv 2 \pmod{4}$, generally if $p \equiv 0$ only (127) and (128) hold if $j = 1$, and (127) and (128) if $j = -1$.

If $M_p$ is null there is always a gap $\gamma_p = \gamma(p, k, \mu, \nu)$ in $F_k$, from the greatest of the quantities $\beta_{p-r}$ to the least of the quantities $\sigma_{p+r} (r = 1, 2, 3, \cdots)$. By
continuity, as we have seen, this must exceed \( \epsilon \) for a neighborhood of \( \mu = (3/2) \cdot |v| \).

If \( M_p \) is not null, and \( p \equiv 0 \), the relations

\[
 a_- (p) \geq a_+ (p - 1) + 5, \quad a_+ (p) \leq a_- (p + 1) - 5,
\]

(129)

\[
 a_- (p) \geq a_+ (p + 1) + 7, \quad a_+ (p) \leq a_- (p - 1) - 7,
\]

in the respective cases

(130) \quad j = 1, \theta \geq 0; \quad j = -1, \theta \geq 0; \quad j = 1, \theta \leq 0; \quad j = -1, \theta \leq 0;

are necessary and sufficient conditions for the existence of a gap \( \gamma_p \) in \( F_k \) exceeding \( \epsilon \) for a neighborhood of \( \mu = (3/2) |v| \). This gap is given by

\[
 \min \{ \xi_{-i}(p), \xi_{-i}(p + r) \} - \max \xi_j(p - r), \text{ if } j\theta \geq 0;
\]

(131)

\[
 \min \xi_{-i}(p + r) - \max \{ \xi_{-i}(p), \xi_{-i}(p - r) \}, \text{ if } j\theta \leq 0.
\]

As a further consequence of (120) and of \( |v| < \mu < 5 |v| / 3 \), we have that

\[
 \min \xi_{-i}(p + r) = \xi_{-i}(p + 1) \text{ if } j\theta \geq 0,
\]

\[
 \max \xi_{-i}(p - r) = \xi_{-i}(p - 1) \text{ if } j\theta \leq 0,
\]

which yields a simplification of (131). It is conjectural that

\[
 \xi_i(p - 1) = \max \xi_i(p - r) \text{ if } j\theta \geq 0, \quad \xi_i(p + 1) = \min \xi_i(p + r) \text{ if } j\theta \leq 0,
\]

(132) will always hold when

\[
 \delta_p > \epsilon,
\]

where

\[
 \delta_p = \delta(p, k, \mu, v) = \xi_{-i}(p + 1) - \xi_{-i}(p - 1) \text{ if } j\theta \geq 0,
\]

(133)

\[
 = \xi_i(p + 1) - \xi_{-i}(p - 1) \text{ if } j\theta \leq 0.
\]

Then, if \( M_p \) is null, \( \gamma_p = \delta_p \); and if \( M_p \) is not null, \( \gamma_p \) is the smaller of \( \delta_p \) and

\[
 \xi_{-i}(p) - \xi_i(p - 1) \quad (j\theta \geq 0), \quad \xi_i(p + 1) - \xi_{-i}(p) \quad (j\theta \leq 0).
\]

If \( 3a' + 2b' + 2 = 3a'' + 2b'' \), then

\[
 (\mu a'' + vb'') - (\mu a' + vb') > \epsilon
\]

(136) is equivalent to

\[
 \mu(a'' + 1 - a') < \frac{1}{2} (3a' + 4 - 3a'') |v| \text{ if } \theta \geq 0,
\]

(137)

\[
 \mu(a'' + 3 - a) > \frac{1}{2} (3a'' + 8 - 3a) |v| \text{ if } \theta \leq 0.
\]

If \( 3a + 2b + r - 1 = 3a' + 2b', r \geq 2 \), then
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\[ \mu a + \nu b \leq \mu a' + \nu b' \]

holds at once if \((a' - a)\theta \geq 0\), and is a consequence of (137) if

\[ a'' < a' < a, \quad a + (r - 1)a'' \leq ra' + r - 1 \quad (\theta \geq 0); \]
\[ a' < a'' + 2, \quad a + (r - 1)a'' \geq ra' - 3r + 3 \quad (\theta \leq 0). \]

If \(3a'' + 2b'' + r - 1 = 3a + 2b, \ r \geq 2\), then

\[ \mu a + \nu b \geq \mu a'' + \nu b'' \]

holds at once if \((a - a'')\theta \geq 0\), and in consequence of (137) if

\[ a < a'' < a', \quad a + (r - 1)a'' \leq ra'' - r + 1 \quad (\theta \geq 0); \]
\[ a > a'' \geq a' - 2, \quad a + (r - 1)a'' \leq ra'' + 3r - 3 \quad (\theta \leq 0). \]

These formulas yield sufficient conditions for (132) to hold at least for the values \(\mu, \nu\) satisfying (133), which is of the form of (137).

Now (136) implies \(|4n\mu - 6n\nu| \leq \varepsilon\) if

\[ 4n \geq a' + 2 - a'' \quad (\theta \geq 0), \quad 4n \leq a'' + 6 - a' \quad (\theta \leq 0). \]

This makes it very probable that no gaps larger than the greatest of \(\varepsilon\) and the first three or four of the \(\gamma_{p}\) occur in \(F_{k}\), for any \(k \geq 3\), in the range \(|\nu| < \mu < 5 |\nu|/3\) remaining.

To see this let \(j = 1, \theta \geq 0\) for simplicity. If \(p \equiv 2 \pmod{4}\) we can always pass from \(\xi-.(p - 1)\) to \(\xi-.(p)\), thence to \(\xi-.(p + 1)\). This is clear from (123). For the italicized statement above we must be able to pass allowably from \(\xi-.(p + 1)\) to \(\xi+.(p + 1)\). Let

\[ \alpha_{v} = \xi-.(p + 1) + v(4\mu - 6\nu) \quad (v = h_{1}, h_{2}, \ldots, h_{t}), \]

where \(0 \leq h_{1} < h_{2} < \cdots < h_{t}\), denote the entries of \(F_{k}\) on \(M_{p+1}\). We can pass from \(\alpha_{h_{t}}\) to \(\alpha_{h_{t+1}}\) allowably if (by (142)) \(h_{t+1} - h_{t} \leq h_{t}\). By Lemma 2 and Theorem 1 of the writer’s paper Improvements of the Cauchy lemma on simultaneous representation, this is the case at least of \(r = 3p + 4 > 10\), since then the \(h_{t}\) are distributed in such a way as to satisfy the relation.

Thus we have still to examine the values \(r = 3p + 4\) less than \(10\), a finite though long problem. Enough has been said to indicate the nature of the gaps throughout \(F_{k}\).

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