CONCERNING TOPOLOGICAL TRANSFORMATIONS
IN $E_n$*

BY

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In my paper Concerning non-dense plane continua† I showed that if in the plane $S$ the set $M$ is the sum of a countable number of closed sets containing no domain then there exists a topological transformation $\Pi$ of the plane $S$ into itself such that if $L$ is any straight line whatsoever the point set $L \cdot \Pi(M)$ is totally disconnected. The principal object of the present paper is to prove this result with “plane $S$” replaced by “euclidean space of $n$ dimensions.” In the proof here given use is made of a general theorem concerning transformations in a locally compact, complete metric space.

Theorem I. Suppose that $S$ is a locally compact complete metric space and, for every positive integer $n$, $e_n$ is a positive number and $\Pi_n$ is a topological transformation of $S$ into itself‡ such that $\delta(P, \Pi_n(P)) \leq e_n$ for every point $P$ of $S$. For each point $P$ of $S$ let $P^1$ denote $\Pi_1(P)$ and in general let $P^{n+1}$ denote $\Pi_{n+1}(P^n)$. Suppose the series $e_1 + e_2 + e_3 + \cdots$ converges. For each point $P$ of $S$ let $\Pi(P)$ denote the sequential limit point of the sequence $P^1, P^2, P^3, \cdots$. Then $\Pi$ is a single-valued continuous transformation§ of $S$ into itself. Furthermore if $\Pi^{-1}$ is single-valued it is continuous. A necessary and sufficient condition that $\Pi^{-1}$ be single-valued is that for every positive integer $m$ if $P$ and $Q$ are points of $S$ then there is an integer $n$ ($n>m$) such that if $\delta(P, Q) > 1/n$ then $\delta(P^n, Q^n) > \sum_{i=n+1}^{\infty} 3e_i$.

Since the series $e_1 + e_2 + e_3 + \cdots$ converges the sequence of transformations $\Pi_1, (\Pi_2 \Pi_1), \cdots, (\Pi_n \Pi_{n-1} \cdots \Pi_1)$ is uniformly convergent, and thus $\Pi$, the limit of this sequence, is continuous. If $Q$ is any point of $S$ then there exists a sequence of points $P_1, P_2, P_3, \cdots$ of $S$ such that for each $n$ (with the notation as in the statement of the theorem) $(P_n)^n = Q$. Let $k$ be a positive number such that the domain $S(Q, k)$ is compact. There exists a positive in-

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‡ That is, a continuous single-valued transformation with continuous single-valued inverse.
Moreover $\Pi(S) = S$.
§ If $A$ and $B$ are points of $S$ then $\delta(A, B)$ denotes the distance from $A$ to $B$.
¶ This does not imply that $\Pi^{-1}$ (the inverse of $\Pi$) is either single-valued or continuous.
∥ If $Q$ is a point and $k$ is a number then $S(Q, k)$ denotes the set of all points whose distance from $Q$ is less than $k$.
integer \( m \) such that if \( T \) is any point and \( i \) any integer \( (i \geq 0) \) then \( \delta(T^m, T^{m+i}) < k \).
In particular \( \delta((P_{m+i})^m, (P_{m+i})^{m+i}) < k \) \((i \geq 0)\). But \( (P_{m+i})^{m+i} \) is \( Q \). Thus 
\( (P_{m+i})^m \) \((i \geq 0)\) belongs to the compact domain \( S(Q, k) \). Let \( K \) denote \( \sum_{i=1}^\infty (P_{m+i})^m \), and let \( K' \) denote \( \sum_{i=1}^\infty (P_{m+i+1})^m \). The infinite set \( K \) has a limit point. Thus there is a point \( P \) such that \( P^m \) is a limit point of \( K \). Then \( P \) is a limit point of \( K' \). As \( \Pi \) is continuous, \( \Pi(P) \) is a limit point of \( \Pi(K') \). Now 
\[ \delta(Q, \Pi(P)) \leq \delta(Q, (P_n)^n) + \delta((P_n)^n, \Pi(P_n)). \]
Now \( (P_n)^n = Q \), whence 
\[ \delta(Q, (P_n)^n) = 0. \] Moreover \( \delta((P_n)^n, \Pi(P_n)) < (\epsilon_{n+1} + \epsilon_{n+2} + \epsilon_{n+3} + \cdots) \). Thus 
\( Q \) is a sequential limit point of the sequence \( \Pi(P_1), \Pi(P_2), \Pi(P_3), \ldots \). Then no point except \( Q \) is a limit point of the point set \( \Pi(P_1) + \Pi(P_2) + \Pi(P_3) + \cdots \).
But \( \Pi(P) \) is a limit point of this set. Then \( \Pi(P) = Q \). Hence for each point \( Q \) of \( S \) there is a point \( P \) such that \( \Pi(P) = Q \), whence \( \Pi(S) = S \).

Now suppose that \( \Pi^{-1} \) is single-valued. Suppose \( R \) is a point set and \( Q \) is a limit point of \( R \). Let \( \Pi^{-1}(Q) = P \) and \( \Pi^{-1}(R) = M \), so that \( \Pi(P) = Q \), \( \Pi(M) = R \), and \( \Pi(P) \) is a limit point of \( \Pi(M) \). It is to be shown that \( P \) is a limit point of \( M \). By hypothesis there exists a positive number \( k \) such that 
\( S[\Pi(P), k] \) is compact. Let \( n \) be an integer such that \( \sum_{i=1}^n \epsilon_i < k/2 \). For each \( i \) let \( X_i \) denote a point of \( \Pi(M) \) such that 
\[ \delta(X_i, \Pi(P)) < k/(2i). \]
Since \( X_i \) belongs to \( \Pi(M) \) it follows that there exists a unique point \( Y_i \) in \( M \) such that 
\( \Pi(Y_i) = X_i \). Let \( K \) denote the point set \( (Y_1)^* + (Y_2)^* + (Y_3)^* + \cdots \), and let \( K' \) denote \( Y_1 + Y_2 + Y_3 + \cdots \). For each \( i \), \( \delta((Y_i)^*, X_i) < k/2 \), and \( \delta(X_i, \Pi(P)) < k/2 \), whence every point of \( K \) belongs to the compact domain \( S[\Pi(P), k] \). Let \( W \) denote a point such that \( W^n \) is a limit point of \( K \). Then \( W \) is a limit point of \( K' \) and \( \Pi(W) \) is a limit point of \( \Pi(K') \). But \( \Pi(P) \) is the only limit point of \( \Pi(K') \). Hence \( \Pi(P) = \Pi(W) \) and by hypothesis \( P = W \). But \( W \) is a limit point of the subset \( K' \) of \( M \). Hence \( P \) is a limit point of \( M \).

We come now to the proof of the last sentence of Theorem I. Suppose that for every positive integer \( m \) and pair of points \( P \) and \( Q \) of \( S \) there is an integer \( n \) \((n > m)\) such that if \( \delta(P, Q) > 1/n \) then \( \delta(P^n, Q^n) > \sum_{i=n+1}^\infty \epsilon_i \). Let \( P \) and \( Q \) be distinct points, and let \( m \) be an integer such that \( \delta(P, Q) > 1/m \).
Then there exists an integer \( n \) \((n > m)\) such that if \( \delta(P, Q) > 1/n \) then 
\[ \delta(P^n, Q^n) > \sum_{i=n+1}^\infty \epsilon_i. \]
But as \( n > m \), \( \delta(P, Q) > 1/n \), whence 
\[ \delta(P^n, Q^n) > \sum_{i=n+1}^\infty \epsilon_i. \]
Obviously \( \delta(P^n, \Pi(P)) \leq \delta(P^n, P^{n+1}) + \delta(P^{n+1}, P^{n+2}) + \cdots < (\epsilon_{n+1} + \epsilon_{n+2} + \epsilon_{n+3} + \cdots) \). That is, both \( \delta(P^n, \Pi(P)) \) and \( \delta(Q^n, \Pi(Q)) \) are less than \( \sum_{i=n+1}^\infty \epsilon_i \). It follows that 
\[ \delta[\Pi(P), \Pi(Q)] > \epsilon_{n+1} \text{ and hence } \Pi(P) \neq \Pi(Q). \]

Now suppose \( \Pi^{-1} \) is single-valued. Then \( \Pi \) is a topological transformation of \( S \) into itself. Let \( M \) be any positive integer and let \( P \) and \( Q \) be distinct points of \( S \). Let \( \epsilon \) denote \( \delta[\Pi(P), \Pi(Q)] \).
Since \( \sum \epsilon_i \) converges it follows that there exists an \( n \) \((n > m)\) such that 
\[ \sum_{i=n+1}^\infty \epsilon_i < \epsilon/3, \delta[\Pi(P), P^n] < \epsilon/3, \text{ and} \]

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\[ \delta[II(Q), Q^n] < \epsilon/3. \] Then \[ \delta(P^n, Q^n) > \sum \zeta_{n+1} 3e. \] This completes the proof of Theorem I.

Let \( x^1, x^2, \ldots, x^n \) denote the coordinates of a point in \( E^n \). If \( \epsilon \) is any positive number and \( (a^1, a^2, \ldots, a^n) \) is any point of \( E^n \) then the set of points for which \( x^i = a^i, a^i - \epsilon \leq x^i \leq a^i + \epsilon \) \( (i \leq n, j = 1, 2, \ldots, i-1, i+1, \ldots, n) \) will be called an \((n-1)\)-cell. A point of such a cell for which \( a^i - \epsilon < x^i < a^i + \epsilon \) for every \( j \) \( (j \leq n) \) will be called an interior point of that cell.

**Theorem II.** If \( E^n \) denotes euclidean space of \( n \) dimensions, \( H \) and \( K \) are mutually exclusive closed and compact point sets in \( E^n \), and \( \epsilon \) is any positive number, then there exists in \( E^n - (H + K) \) a finite set \( G \) of mutually exclusive \((n-1)\)-cells each of diameter less than \( \epsilon \) and such that any straight line interval, with end points in \( H \) and \( K \) respectively, contains an interior point of at least one cell of the set \( G \).

Let \( t \) be a positive number such that the product \( t \cdot n \) is the lower distance from \( H \) to \( K \). For each \( i \) \( (i \leq n) \) let \( A_i \) denote the point set containing every point \( P \) whose lower distance from \( H \) lies between the numbers \( i \cdot \epsilon \) and \( (i+1) \cdot \epsilon \). Then the sets \( A_1, A_2, \ldots, A_n \) are mutually exclusive domains, and every straight line interval with end points in \( H \) and \( K \) respectively contains segments lying in \( A_1, A_2, \ldots, A_n \) respectively. As \( H \) and \( K \) are separable there exist sequences of points \( P'_1, P'_2, P'_3, \ldots \) and \( Q'_1, Q'_2, Q'_3, \ldots \) such that \( H \) is the set \((P'_1 + P'_2 + P'_3 + \cdots) \) plus its limit points, and \( K \) is \((Q'_1 + Q'_2 + Q'_3 + \cdots) \) plus its limit points. There exist points \( P_1, P_2, P_3, \ldots \) and \( Q_1, Q_2, Q_3, \ldots \), such that (1) for every \( i \) there exist numbers \( j \) and \( k \) such that \( P_i = P'_i \) and \( Q_i = Q'_i \), and (2) for every pair of integers \( j \) and \( k \) there is an integer \( i \) such that \( P_i = P'_j \) and \( Q_i = Q'_k \). Let \( i \) denote the smallest integer \( (i \leq n) \) such that \( x^i \) is not constant on the interval \( P_1Q_1 \). Let \( C_1 \) denote a point of \( P_1Q_1 \) in \( A_i \), and let \( D_1 \) denote an \((n-1)\)-cell with center \( C_1 \), lying in \( A_i \) and in the set with equation \( x^i = x^i_{C_1}. \) Then not only does \( D_1 \) contain a point of \( P_1Q_1 \), but there exist spherical neighborhoods \( E_{P_1} \) and \( E_{Q_1} \) of \( P_1 \) and \( Q_1 \) respectively such that any interval with end points in \( E_{P_1} \) and \( E_{Q_1} \) respectively contains an interior point of the cell \( D_1 \). Clearly there is a greatest number \( \delta_{D_1} \) such that for every positive number \( \nu \) the domains \( E_{P_1} \) and \( E_{Q_1} \) can be taken of diameter greater than \( \delta_{D_1} - \nu \). Let \( \delta^* \) be the upper limit of \( \delta_{D_1} \) for all such cells \( D_1 \), and let \( D_1^* \) denote a cell \( D_1 \) such that \( \delta_{D_1} > \delta^*/2 \).

Now consider the second pair of points \( P_2Q_2 \). Let \( i \) be the smallest integer \( (i \leq n) \) such that \( x^i \) is not constant on the interval \( P_2Q_2 \). Let \( C_2 \) be a point of \( P_2Q_2 \) lying in \( A_i \), and such that \( x^i_{C_2} < x^i_{C_1^*} \), where \( C_1^* \) denotes the center of

\[ \text{If } i \text{ is an integer } (i \leq n) \text{ and } C \text{ is a point, then by } x^i_C \text{ is meant the } i \text{th coordinate of the point } C. \]
the \((n-1)\)-cell \(D_i^*\). Let \(D_2\) be an \((n-1)\)-cell with center \(C_2\), lying wholly in \(A_i\) and in the set with equation \(x^i = x^i_{C_i}\). Let \(\delta^*_i\) and \(D_i^*\) denote respectively a number and an \((n-1)\)-cell obtained from \(P_iQ_i\) and the cells \((D_i)\) in the same manner that \(\delta^*_i\) and \(D_i^*\) were obtained from \(P_1Q_1\) and the cells \((D_1)\). This process may be continued indefinitely. Thus there exists an infinite set of numbers \(\delta^*_1, \delta^*_2, \delta^*_3, \ldots\), and an infinite set of \((n-1)\)-cells \(D_1^*, D_2^*, D_3^*, \ldots\) such that, for every \(m\), (1) \(D_m^*\) lies in \(A_i\) for some \(i (i \leq n)\) and in the set with equation \(x^i = k\) (\(k\) being a constant), (2) if \(D_m^*\) and \(D_n^*\) both lie in the set with equation \(x^i = w\) (\(w\) being a constant) then \(h = k\), and (3) if \(E_{P_m}\) and \(E_{Q_m}\) are spherical neighborhoods of \(P_m\) and \(Q_m\) respectively, then (a) if the diameters of \(E_{P_m}\) and \(E_{Q_m}\) are less than \(\delta_{m^*}/2\) every straight line interval with end points in \(E_{P_m}\) and \(E_{Q_m}\) respectively contains a point in the interior of \(D_m^*\), but (b) if \(E_{P_m}\) and \(E_{Q_m}\) are both of diameter greater than \(\delta_{m^*}\) and \(D\) is any \((n-1)\)-cell lying in \(A_i\) (\(i \leq n\)) and in the set with equation \(x^i = k\), and no cell \(D_k^*\) with \(h\) less than \(m\) lies in the set \(x^i = k\), then there exists a straight line interval with end points in \(E_{P_m}\) and \(E_{Q_m}\) respectively, which does not contain any point of \(D\).

If now we suppose the theorem false there exists a sequence of pairs of points \(R_1, S_1; R_2, S_2; R_3, S_3; \ldots\) such that (1) for every \(m\), \(R_m\) is a point of \(H\) and \(S_m\) is a point of \(K\), (2) the interval \(R_mS_m\) contains no interior point of \(D_k^*\) \((k \leq m)\), and (3) the sequences \(R_1, R_2, R_3, \ldots\) and \(S_1, S_2, S_3, \ldots\) respectively have sequential limit points \(R\) and \(S\). Let \(i\) be an integer \((i \leq n)\) such that \(x^i\) is not constant on the interval \(RS\). In view of the fact that the point set \(RS.A_i\) is uncountable, and that for each \(m\) if \(D_m^*\) lies in \(A_i\), then it contains at most one point of \(RS\), it follows that there exists a point \(C\) lying in \(RS.A_i\) which does not belong to \(D_m^*\) for any \(m\). Let \(D\) denote any \((n-1)\)-cell with center \(C\) and lying in \(A_i\) and in the set with equation \(x^i = x^i_{C_i}\). Let \(n_1, n_2, n_3, \ldots\) denote a sequence of numbers such that the sequence \(P_{n_1}, P_{n_2}, P_{n_3}, \ldots\) converges to \(R\), and the sequence \(Q_{n_1}, Q_{n_2}, Q_{n_3}, \ldots\) converges to \(S\). Since for every \(i\) the interval \(RS\) contains no interior point of \(D_{n_i}\), it follows that the sequence of numbers \(\delta_{n_1^*}, \delta_{n_2^*}, \delta_{n_3^*}, \ldots\) converges to zero. But there is a positive number \(\delta^*\) such that, if \(E_R\) and \(E_S\) denote spherical neighborhoods of \(R\) and \(S\) respectively of diameter \(\delta^*\), then every interval with end points in \(E_R\) and \(E_S\) respectively contains a point in the interior of \(D\). There exists a positive number \(m'\) such that if \(m > m'\), then the distances \(P_{n_m}R\) and \(Q_{n_m}S\) are each less than \(\delta^*/4\). Then, for the moment writing \(k = n_m\), the spherical neighborhoods \(E_{P_k}\) and \(E_{Q_k}\) of \(P_k\) and \(Q_k\) respectively which are of diameter \(\delta^*/4\) are subsets of \(E_R\) and \(E_S\), respectively. Hence any interval with end points in \(E_{P_k}\) and \(E_{Q_k}\) respectively contains an interior point.
of the \((n-1)\)-cell \(D\), whence \(\delta_k^{*} \geq \delta^*/8\). But \(\lim_{m \to \infty} \delta_k^* = 0\). Thus the supposition that Theorem II is false has led to a contradiction.

**Theorem III.** If \(T_1\) and \(T_2\) are countable point sets, dense in \(E_n\), and \(M\) is the sum of a countable number of closed point sets lying in \(E_n\) and containing no domain, then there exists a topological transformation \(\Pi\) of \(E_n\) into itself such that \(\Pi(T_1) = T_2\), and if \(L\) is any straight line the set \(L \cdot \Pi(M)\) is totally disconnected.

To facilitate the proof of Theorem III, I will establish two lemmas.

**Lemma 1.** If, in \(E_n\), \(L\) is any finite point set, \(e\) is any positive number, \(P_1, P_2, P_3, \ldots\) and \(Q_1, Q_2, Q_3, \ldots\) are countable sets dense in \(E_n\), and \(i\) is an integer such that \(P_i\) and \(Q_i\) are not in \(L\), then there exist integers \(n_i\) and \(m_i\), and a topological transformation \(C\) of \(E_n\) into itself, such that (1) for every point \(U\) the distance \(d[U, C(U)] < e\), (2) \(C(P_i) = Q_{n_i}\), and \(C(P_m) = Q_i\), and (3) if \(U\) is any point of \(L\) then \(C(U) = U\).

Let \(n_i\) be any integer such that the length \(P_iQ_{n_i} < e/6\), and also less than \(1/6\) of the lower distance from \(P_i\) to \(Q_i + L\). Let \(t\) denote three times the distance \(P_iQ_{n_i}\), and let \(R\) denote the point such that the interval \(RQ_{n_i}\) is bisected by the point \(P_i\). Let \(X\) denote any point and \(x\) denote its distance from the point \(R\). If \(x > t\) let \(Y_X\) denote \(X\). If \(x < t\) let \(Y_X\) denote the point on the ray \(RX\) whose distance \(y\) from \(R\) is given by the equation \(2y = t(3x^2/t^2 + 5x/t)\). Let \(C_1\) be the transformation throwing \(X\) into \(Y_X\) for every \(X\). For the point \(P_i\) we have \(x = t/3\), and for \(Q_{n_i}\), \(x = 2t/3\). It is then easily verified that \(C_1(P_i) = Q_{n_i}\). Thus \(C_1\) is a topological transformation of \(E_n\) into itself which reduces to the identity outside the sphere with \(R\) as center and radius \(t\), and which throws \(P_i\) into \(Q_{n_i}\). In a similar manner there exists a topological transformation \(C_2\) of \(E_n\) into itself, and an integer \(m_i\), such that \(C_2(P_m) = Q_i\), and \(C_2\) reduces to the identity outside a sphere \(S\) so chosen that (1) it does not contain any point of \(L\) or any point of the sphere with center \(R\) and radius \(t\) and (2) its radius is less than \(e/2\). Then the product transformation \(C_2C_1\) satisfies the requirements of the lemma.

**Lemma 2.** If \(H\) and \(K\) are mutually exclusive closed and compact point sets, \(e\) is any positive number, \(R\) is a closed point set of dimension less than \(n\), and \(L\) is any finite point set, then there exists a topological transformation \(\beta\) of \(E_n\) into

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\[\dagger\] With the help of this lemma and Theorem I, a very short proof can be given of the following well known theorem: If \(T_1\) and \(T_2\) are countable point sets dense in \(E_n\), then there is a topological transformation of \(E_n\) into itself throwing \(T_1\) into \(T_2\). See Fréchet, Mathematische Annalen, vol. 68 (1910), p. 83. Also see Urysohn, *Sur les multiplicités Cantoriennes*, Fundamenta Mathematicae, vol. 7 (1925), pp. 30–137, and Menger, *Dimensionstheorie*, p. 264.
itself, and a positive number $\epsilon'$, such that (1) if $P$ is a point of $L$ then $\beta(P) = P$, (2) if $P$ is any point of $E_n$ then $\delta[P, \beta(P)] < \epsilon$, (3) $\beta$ reduces to the identity transformation outside some sphere, and (4) if $\rho$ is any topological transformation of $E_n$ into itself such that $\delta[P, \rho(P)] < \epsilon'$ for every point $P$ of $E_n$ then any straight line interval containing a point both of $H$ and of $K$ contains a point of $E_n - \rho[\beta(R)]$.

Since the point set $L$ is finite it can readily be shown, with the help of Theorem II, that there exist $k$ mutually exclusive $(n - 1)$-cells $s_1, s_2, \ldots, s_k$, lying in $E_n - (H + K)$, such that no point of $L$ belongs to any $s_i$ ($i \leq k$) and every straight line interval containing a point of $H$ and a point of $K$ contains an interior point of $s_i$ for some $i$ ($i \leq k$). Let $L'$ denote $L - L \cdot (H + K)$. Let $\epsilon$ denote a positive number less than $\epsilon$, and less than every number $\delta(X, Y)$, where $X$ and $Y$ are points of distinct sets of the sequence $H$, $K$, $L'$, $s_1, s_2, \ldots, s_k$. For each $i$ ($i \leq k$) let $Q_i$ be a spherical domain in the complement of $R + s_i$, every point of which is at a distance less than $\epsilon'/4$ from some point of $s_i$. There exists a topological transformation $T_i$ of $E_n$ into itself such that (1) $T_i$ reduces to the identity on $s_i$ and for every point of $E_n$ at a distance greater than $\epsilon/2$ from every point of $s_i$, (2) if $l$ is any straight line which contains a point of $s_i$ then $l$ contains a point of $T_i(Q_i)$. Let $\beta$ be the product transformation $T_1T_2T_3\cdots T_k$. Then $\beta$ is a topological transformation of $E_n$ into itself such that if $l$ is any straight line interval containing a point of $H$ and a point of $K$ then $l$ contains a point of $\beta(Q_i)$ for some $i$ ($i \leq k$). Now since $H + K$ is a closed point set while $\beta(Q_i)$ is open ($i \leq k$), it follows that there exists a number $\epsilon'$ such that if $\rho$ is any topological transformation of $E_n$ into itself such that for each point $P$ of $E_n$ the distance $\delta[P, \rho(P)]$ is less than $\epsilon'$ then if $l$ is any straight line interval with end points in $H$ and $K$ respectively, $l$ contains a point of $\rho[\beta(Q_i)]$ for some $i$ ($i \leq k$). Then the transformation $\beta$ and the number $\epsilon'$ thus obtained satisfy the conclusions of the lemma.

Proof of Theorem III. Let $R_1, R_2, R_3, \ldots$ denote the set of all spherical domains with centers and radii rational. There exists a sequence of pairs of integers $n_1, m_1; n_2, m_2; \ldots$ such that (1) for every $i$ the sets $R_{n_i}$ and $R_{m_i}$ are mutually exclusive, and (2) if $h$ and $k$ are integers such that $R_h$ and $R_k$ are mutually exclusive then there exists an integer $i$ such that $n_i = h$ and $m_i = k$. Define new symbols $S_1, S_2, S_3, \ldots$ as follows: $S_1 = R_{n_1}$, $S_2 = R_{m_1}$, $S_3 = R_{n_2}$, $S_4 = R_{m_2}$, $\ldots$, $S_{2k-1} = R_{n_k}$, $S_{2k} = R_{m_k}$. Then the sequence $S_1, S_2, S_3, S_4, \ldots$ contains every pair of domains of the set $R_1, R_2, R_3, \ldots$ which with their boundaries are mutually exclusive.

Let $P_1, P_2, P_3, \ldots$ and $Q_1, Q_2, Q_3, \ldots$ denote the points of $T_1$ and $T_2$ respectively. Suppose $M$ is the set $M_1 + M_2 + M_3 + \cdots$, where for every $k$
the set $M_k$ is closed and furthermore $M_k$ is a subset of $M_{k+1}$. With the help of Lemma 1 it can be seen that there exists a topological transformation $C_1$ of $E_n$ into itself such that (1) there exist integers $n_1$ and $m_1$ such that $C_1(P_i) = Q_i$, and $C_1(P_{m_1}) = Q_{m_1}$, (2) if $U$ is any point then $\delta [U, C_1(U)] < 1/2$, and (3) $C_1$ reduces to the identity transformation outside some sphere. Let $C_2$ denote a transformation and $\epsilon_1'$ a number satisfying the conclusion of Lemma 2, where $H$ and $K$ denote $\overline{S}_1$ and $\overline{S}_2$, $\epsilon = 1/2$, $R$ is the set $C_1(M_1)$ and $L$ is $P_1 + Q_1 + P_{m_1} + Q_{m_1}$. Let $\Pi_1$ be the product transformation $C_2C_1$. There exists a number $\epsilon_1'$ such that if $U$ and $V$ are points and $\delta(U, V) > 1$ then $\delta(\Pi_1(U), \Pi_1(V)) > \epsilon_1'$. Then, letting $\epsilon_1$ equal 1, the following properties hold true: (1) $\Pi_1(P_i) = Q_i$, and $\Pi_1(P_{m_1}) = Q_{m_1}$, (2) $\delta(U, \Pi_1(U)) < \epsilon_1$, (3) if $U$ and $V$ are points and $\delta(U, V) > 1$ then $\delta(U, V) > \epsilon_1$, and (4) if $\rho$ is any topological transformation such that for each $U$, $\delta(U, \rho(U)) < \epsilon_1''$, then any straight line interval containing a point of $\overline{S}_1$ and of $\overline{S}_2$ contains a point of $E_n - \rho[\Pi_1(M_1)]$. Moreover $\Pi_1$ reduces to the identity outside some sphere.

Let $\epsilon_2$ be any positive number less than each of the numbers $\epsilon_1/12, \epsilon_1'/12$, and $\epsilon_1''/12$. Again by the use of Lemma 1 it can be seen that there exist integers $n_2$ and $m_2$, and a continuous transformation $C_3$ of $E_n$ into itself such that (1) $C_3\Pi_1(P_i) = Q_i$, and $C_3\Pi_1(P_{m_1}) = Q_{m_1}$, (2) the distance $\delta(U, C_3(U)) < \epsilon_2/2$ for every point $U$, and (3) $C_3$ reduces to the identity outside some sphere. Let $C_4$ denote a transformation, and $\epsilon_2'$ a number, satisfying the conclusion of Lemma 2, where $H$ and $K$ denote $\overline{S}_3$ and $\overline{S}_4$, $\epsilon = \epsilon_2/2$, $R$ is the set $C_3\Pi_1(M_2)$, and $L$ is $P_1 + Q_1 + P_{m_1} + Q_{m_1}$. Let $\Pi_2$ denote the product transformation $C_4C_3$. There exists a number $\epsilon_2'$ such that if $U$ and $V$ are points and $\delta(U, V) > 1/2$, then $\delta(\Pi_2(U), \Pi_2(V)) > \epsilon_2'$. Let $\epsilon_2''$ be less than $\epsilon_1/12$ and $\epsilon_2'$. Then the following properties obtain: (1) $\Pi_2\Pi_1(P_i) = Q_i$, and $\Pi_2\Pi_1(P_{m_1}) = Q_{m_1}$, (2) $\delta(U, \Pi_2(U)) < \epsilon_2$ for every point $U$, (3) if $U$ and $V$ are points and $\delta(U, V) > 1/2$ then $\delta(\Pi_2(U), \Pi_2(V)) > \epsilon_2'$, and (4) if $\rho$ is any topological transformation of $E_n$ into itself such that $\delta(U, \rho(U)) < \epsilon_2''$ for every point $U$, then any straight line interval containing a point of $\overline{S}_3$ and of $\overline{S}_4$ contains a point of $E_n - \rho[\Pi_2\Pi_1(M_2)]$, and (5) each of the numbers $\epsilon_2, \epsilon_2', \epsilon_2''$ is less than each of the numbers $\epsilon_1/12, \epsilon_1'/12, \epsilon_1''/12$.

This process can be continued indefinitely. Thus, there exist transformations $\Pi_1, \Pi_2, \Pi_3, \ldots$, three sequences of positive numbers $\epsilon_1, \epsilon_2, \epsilon_3, \ldots; \epsilon_1', \epsilon_2', \epsilon_3', \ldots$ and two sequences of positive integers $n_1, n_2, n_3, \ldots$ and $m_1, m_2, m_3, \ldots$ such that, for every integer $k$ (with the notation of Theorem 1) (1) $P_i^k = Q_i$, and $P_{m_i}^k = Q_{m_i}$, (2) if $U$ is any point then $\delta(U, \Pi_k(U)) < \epsilon_k$, (3) if $U$ and $V$ are points and $\delta(U, V) > 1/k$ then $\delta(U, V') > \epsilon_k'$, (4) if $\rho$ is any topological transformation of $E_n$ into itself such that, for each point $U$, $\delta(U, \rho(U)) < \epsilon_k''$, then any interval containing a
point both of $\mathbb{S}_{2k-1}$ and $\mathbb{S}_{2k}$ contains a point of $E_n - \rho [\Pi_{k} \Pi_{k-1} \cdots \Pi_{3} \Pi_{2} (M_k)]$, and (5) each of the numbers $\epsilon_{k+1}, \epsilon_{k+1}', \epsilon_{k+1}''$ is less than each of the numbers $\epsilon_k/12, \epsilon_k'/12$ and $\epsilon_k''/12$ and $\epsilon_{k+1} > \epsilon_{k+1}'$.

Let $\Pi$ be the transformation defined as in Theorem 1. For each $n$ let $\epsilon_n$ denote $\epsilon_n$. Then since $\epsilon_n' > 3 \sum_{i=n+1}^{\infty} \epsilon_i$, and $\epsilon_n > \epsilon_n'$, it follows from (3) above that the hypotheses of Theorem 1 are satisfied. Hence $\Pi$ is a topological transformation of $E_n$ into itself. From (1) it follows that $\Pi(T_1) = T_2$. Suppose $L$ is some straight line such that the point set $L \cdot \Pi(M)$ contains an arc $t$. Since the sum of a countable number of closed and totally disconnected sets is not connected it follows that there exists an integer $a$ and a subarc $t'$ of $t$ such that $t'$ is a subset of $\Pi(M_a)$. There exists an integer $k$ ($k > a$) such that the end points of $t'$ lie in the mutually separated sets $\mathbb{S}_{2k-1}$ and $\mathbb{S}_{2k}$. Let $\rho$ denote the transformation such that $\rho [\Pi_{k} \Pi_{k-1} \cdots \Pi_{3} \Pi_{2} (P)] = \Pi (P)$ for every point $P$. Then $\delta [P, \rho (P)] < \epsilon_{k+1} + \epsilon_{k+1}' + \epsilon_{k+1}'' + \cdots < \epsilon_t'$. Hence by (4) above, the interval $t'$ contains a point of $E_n - \rho [\Pi_{k} \Pi_{k-1} \cdots \Pi_{3} \Pi_{2} (M_k)]$. That is, $t'$ contains a point of $E_n - \Pi(M_k)$. But $t'$ is a subset of $\Pi(M_a)$ and therefore of $\Pi(M_{k+1})$, since $k > a$. Then the supposition that $L \cdot \Pi(M)$ contains a connected set has led to a contradiction and the theorem is proved.

It has been shown† that if $M$ is any continuous curve lying in a plane $S$, then there exists a topological transformation $\Pi$ of $S$ into itself such that if $K$ is the interior of the rectangle whose edges lie in the lines $x = r_1, x = r_2, y = s_1, y = s_2$, where $r_i$ and $s_i$ are rational ($i = 1, 2$), then the point set $K \cdot \Pi(M)$ is the sum of a finite number of connected sets. The following proposition does not hold true: If $M$ is a continuous curve in $E_3$ then there exists a topological transformation $\Pi$ of $E_3$ into itself such that if $K$ is the interior of a cube with sides in the planes $x = r_1, x = r_2, y = s_1, y = s_2, z = t_1, z = t_2$, where $r_i, s_i, t_i$ are rational ($i = 1, 2$) then the point set $K \cdot \Pi(M)$ is the sum of a finite number of connected sets.

**Example.** Let $(x, y, z)$ denote a general point of 3-dimensional space. For each $n$ ($n = 0, 1, 2, \cdots$) let $A_n, B_n, C_n,$ and $D_n$ be the points with coordinates $(0, 0, 0), (0, 1/2^n, 0), (1/2^n, 1/2^n, 0)$ and $(1/2^n, 1/2^{n+1}, 0)$. Let $E_n$ denote the midpoint of the interval $C_{n+1}D_{n+1}$. In the plane perpendicular to the $xy$ plane and passing through the points $D_n$ and $E_n$ let $G_n$ denote the circle with center $E_n$ and with diameter $1/2^{n+4}$. Let $F_n$ denote the first point of $G_n$ on the interval $D_nE_n$ in the order from $D_n$ to $E_n$. Let $K$ denote the continuum $\sum_{n=0}^{\infty} (A_nB_n + B_nC_n + C_nD_n + D_nF_n + G_n)$, where $A_nB_n$, etc., denote straight line intervals with end points as indicated. Then $K$ is a bounded regular curve of order 3. It will be shown below that if $H$ is any domain such that

† Cf. Roberts, loc. cit., Theorem 3.
no simple closed curve \( J \) in \( H \) is interlaced with any closed point set not in \( H \) and \( H \) contains the point \( A_0 \) but does not contain the point \( B_0 \), then the point set \( H \cdot K \) is not connected. The continuous curve \( M \) desired will be defined as the sum of a countable number of continua homeomorphic with \( K \).

For each pair of integers \( n > 0 \) and \( k (0 < k < 2^n) \) let \( T_{kn} \) denote the transformation such that if \( T_{kn}(x, y, z) = (x', y', z') \) then \( x' = x/2^n \), \( y' = (y + k)/2^n \), and \( z' = z/2^n \). This transformation may be thought of as dividing every distance to the origin by \( 2^n \), and then moving space upward (along the \( y \)-axis) a distance \( k/2^n \). Let \( M' \) denote \( K + \sum_{n=1}^{\infty} \sum_{k=1}^{2^n-1} T_{kn}(K) \). Let \( T \) denote the transformation such that if \( T(x, y, z) = (x', y', z') \) then \( x' = -x \), \( y' = 2^{1/2}y \), and \( z' = z \). Set \( M'' \) equal to \( T(M') \), and \( M \) equal to \( M' + M'' \). Then \( M \) is the continuum desired.

Let \( H \) denote any domain containing \( A_0 \) but not every point of \( A_0 B_0 \) and such that no simple closed curve in \( H \) is interlaced with any closed point set containing no point in \( H \). It will be shown that the point set \( H \cdot K \) is not connected. Suppose on the contrary that \( H \cdot K \) is connected. For each \( n \) let \( Q_n \) denote the set consisting of the circle \( G_n \) plus its interior in the plane which contains \( G_n \). Let \( k \) denote the smallest integer for which there exists an arc \( A_0 E_k \) such that (1) \( A_0 E_k \) lies in \( H \) and has only the point \( E_k \) in common with the set \( \sum_{n=0}^{n=k} (B_n C_n + C_n D_n + D_n E_n + Q_n) \), (2) \( A_0 E_k + (A_0 B_{k+1} + B_{k+1} C_{k+1} + C_{k+1} + E_k) \), where \( A_0 B_{k+1} \), etc., denote straight line intervals, is a simple closed curve \( J_k \) and is interlaced with \( G_k \). Now there exists in \( H \cdot Q_k \) an arc from \( E_k \) to some point of the circle \( G_k \). For if we suppose the contrary then the common part of \( Q_k \) and the boundary of \( H \) must contain a continuum \( L_k \) which separates \( E_k \) from \( G_k \); then \( J_k \) is not connected with \( L_k \), contrary to the definition of \( H \). Let \( E_k N_k \) denote a simple continuous arc lying in \( H \cdot Q_k \), where \( N_k \) is on \( G_k \). Then since, by supposition, the set \( H \cdot K \) is connected, the point set \( N_k F_k + F_k D_k + D_k C_k + C_k B_k + B_k A_k \) lies in \( H \), where \( N_k F_k \) denotes one of the arcs into which \( N_k \) and \( F_k \) divide \( G_k \) (or \( N_k F_k \) denotes the point \( F_k \) in case \( N_k \) and \( F_k \) are identical). But then \( A_0 E_k + E_k N_k + N_k F_k + F_k D_k + D_k E_k \) is an arc \( A_0 E_{k-1} \) satisfying the conditions given above. Thus the supposition that \( H \cdot K \) is connected has led to a contradiction.

Let \( S \) denote the first point of the interval \( A_0 B_0 \) which lies on the boundary of \( H \).

Case 1. Suppose the \( y \)-coördinate of \( S \) (call it \( y_S \)) is irrational. If \( k/2^n < y_S \)

\[ \text{See Mazurkiewicz and Straszewicz, Sur les coupures de l'espace, Fundamenta Mathematicae, vol. 9 (1927), p. 205. If } J \text{ is a simple closed curve and } L \text{ is a closed point set having no point in common with } J, \text{ then } J \text{ is said to be interlaced with } L \text{ provided there does not exist a continuous point function } x(t, w), \text{ defined for } 0 \leq t \leq 1, 0 \leq w \leq 1, \text{ such that (1) the point } x(t, w) \text{ does not belong to } L, \text{ (2) } x(0, w) = x(1, w) \text{ for every } w, \text{ (3) } x(t, 1) (0 \leq t \leq 1) \text{ generates the curve } J, \text{ and (4) } x(t, 0) = x_0, \text{ where } x_0 \text{ is a fixed point.} \]
<(k + 1)/2^n then \(T_{kn}(K)\) is such that \(T_{kn}(A_0)\) is within \(H\) but some point of \(T_{kn}(A_0B_0)\) is not in \(H\). Then by the preceding argument \(T_{kn}(K) \cdot H\) is not connected. There exists a sequence of distinct continua \(V_1, V_2, V_3, \cdots\) such that, for each \(i\), there exist integers \(k\) and \(n\) such that \(V_i = T_{kn}(K)\), \(T_{kn}(A_0)\) is in \(H\) but \(T_{kn}(A_0B_0)\) is not entirely in \(H\). Now any arc lying in \(M\) and connecting two points of a set \(V\) homeomorphic with \(K\) must lie in the set \(V\). Hence it follows that, since for each \(i\) there are at least two components of \(V_i \cdot H\), the number of components of \(H \cdot M\) is infinite.

Case 2. Suppose \(y_5\) is rational. Let \(W\) be the inverse of the transformation \(T\). Then \(W(M''') = M'\), and \(y_W(S)\) is irrational. The domain \(W(H)\) is such that no simple closed curve in it is interlaced with a closed point set not containing a point in \(W(H)\). Moreover \(W(H)\) contains the point \(A_0\) and does not contain every point of \(A_0B_0\). The point \(W(S)\) is the first point, in the order from \(A_0\) to \(B_0\) on the boundary of the domain \(W(H)\), and \(y_W(S)\) is irrational. Hence by Case 1 the set \(M'' - W(H)\) is not the sum of a finite number of connected sets. Then \(M' \cdot H\) is not the sum of a finite number of connected sets. Thus in any case \(M \cdot H\) is not the sum of a finite number of connected sets.

Theorem IV. If \(M\) is a continuous curve lying in \(E_n\) and \(G\) is any uncountable set of mutually exclusive hyperspheres, then there is at least one element \(g\) of \(G\) such that for each positive number \(e\) the set \(g \cdot M\) contains a subset \(T_{ge}\) such that \(M - T_{ge} = S_1 + S_2 + \cdots + S_k\), where \(s_i\) and \(s_j\) \((i \neq j)\) are connected, mutually separated sets, and \(s_i\) lies either within the hypersphere concentric with \(g\) and of radius equal to that of \(g\) increased by \(e\), or outside the hypersphere concentric with \(g\) and of radius equal to that of \(g\) decreased by \(e\).

Let \(g\) be any element of \(G\) and let \(e\) be any positive number. Let \(h_1, h_2, \cdots, h_k\) denote a finite set of components of \(M - M \cdot g\) containing every component of \(M - M \cdot g\) which contains a point whose distance from \(g\) is as much as \(e\). Suppose that if \(Q\) is any point of \(M - \sum_{i=1}^{k} h_i\), then there exists in \(M\) an arc \(QR\), where \(R\) belongs to \(h_i\), for some \(i\) \((i \leq k)\), but no point of \(QR\) belongs to \(h_j\) \((j \neq i)\). Let \(h_k^*\) denote the component containing \(h_1\) of \(M - \sum_{i=1}^{k} h_i\). Let \(h_k^*\) denote the component containing \(h_2\) of \(M - (h_1^* + \sum_{i=3}^{k} h_i)\). In general let \(h_i^*\) denote the component containing \(h_i\) of \(M - (\sum_{i=1}^{i-1} h_i^* + \sum_{i=j+1}^{k} h_i)\). It is clear that the sets \(h_1^*, h_2^*, \cdots, h_k^*\) are mutually separated and connected. Let \(T_{ge}\) denote the set of all points common to \(h_i^*\) and \(h_j^*\) \((i \neq j; i, j \leq k)\). Now \(M = \sum_{i=1}^{k} h_i^*\), so in this case the theorem is proved.

Thus if we suppose the theorem false it follows that for every element \(g\) of \(G\) there is a positive number \(e_\varphi\) such that if \(h_1, h_2, h_3, \cdots, h_k\) is any set of components of \(M - M \cdot g\) containing every component of \(M - M \cdot g\) which contains a point whose distance from \(g\) is as much as \(e_\varphi\), then there exists a
point $Q$ in $M - \sum_{i=1}^{k} h_i$ such that if $QR$ denotes any arc in $M$, and $R$ is the only point of this arc in $\sum_{i=1}^{k} h_i$, then $R$ must belong to two sets $h_i$ and $h_j$ ($i \neq j$).

Let $P_1, P_2, P_3, \cdots$ denote the points of a countable point set dense in $M$.

Let $G^1$ denote an uncountable subset of $G$ and $e'$ a number such that, for every element $g$ of $G^1$, $2e' < e_0$. Let $g'$ be a condensation element of $G^1$ and let $p_1, p_2, p_3$ and $p_4$ be spheres concentric with $g'$ but with radii $r - e'/2, r - e'/4, r + e'/4$, and $r + e'/2$, respectively ($r$ being the radius of $g'$). Let $a_1, a_2, a_3, \cdots, a_m$ denote the components of $M - (p_2 + p_3)$ which contain points on $p_1 + p_4$. Let $G^2$ denote the uncountable subset of $G$ containing all elements of $G^1$ which lie entirely within $p_3$ and entirely without $p_2$. Let $g$ be any element of $G^2$ and let $h_1, h_2, \cdots, h_k$ denote the components of $M - g$ containing points on $p_1 + p_4$. Then there exists a point $Q_\phi$ (and this may be taken as a point of the countable set $P_1, P_2, P_3, \cdots$) such that if $Q_\phi R$ is any arc in $M$ such that $R$, but no other point of $Q_\phi R$, lies in $\sum_{i=1}^{k} h_i$, then $R$ must belong to two sets $h_i$ and $h_j$ ($i \neq j$). Hence there exists a point $Q$ and an uncountable subset $G^3$ of $G^2$ such that, for every $g$ in $G^3$, $Q_\phi = Q$.

For each element $g$ of $G^3$ let $a_{\phi 1}, a_{\phi 2}, \cdots, a_{\phi k_\phi}$ denote the components of $M - M \cdot g$ which contain points on $p_1 + p_4$. Then $k_\phi \leq m$, since, for each $i$ ($i \leq k_\phi$), $a_{\phi i}$ contains a point of $a_j$ ($j \leq m$). If $a_{\phi i}$ and $a_{\phi j}$ have a point in common, and both $a_{\phi i}$ and $a_{\phi j}$ lie inside (outside) $g$, then if $h$ is any element of $G^3$ outside (inside) $g$ the set $a_{\phi i} + a_{\phi j}$ is a subset of a single component of $M - M \cdot h$. It can thus be seen that there exists an uncountable subset $G^4$ of $G^3$ such that if $g$ is any element of $G^4$ and $h$ and $k$ are components of $M - M \cdot g$ having points on $p_1 + p_4$, then one and only one of the sets $h$ and $k$ lies inside $g$.

Let $g_1$ and $g_2$ denote two elements of $G^4$. Let the components of $M - M \cdot g_i$ with points on $p_1 + p_4$ be called $h_{1i}, h_{1i2}, \cdots, h_{1ik_i}$ ($i = 1, 2$). Let $QR$ be any arc in $M$ from $Q$ to a point $R$ in $a_1$. Let $W$ be the first point of $QR$ belonging to $\sum_{i=1}^{2} \sum_{j=1}^{k_i} h_{ij}$. The point $W$ obviously cannot belong both to $g_1$ and $g_2$. Moreover it must belong to one of these sets. Suppose $W$ belongs to $g_1$. Let $h_{1i}$ and $h_{1ij}$ be two sets ($i \neq j$) such that $W$ belongs to $h_{1i}, h_{1ij}$. One of the sets $h_{1i}$ and $h_{1ij}$ lies on the non-$g_2$ side of $g_1$. Hence $QW$ is an arc having no point in common with the set $h_{2i2}, h_{2i3} (i \neq j, i, j \leq k_2)$, and connecting $Q$ to a component of $M - M \cdot g_2$ having a point on $p_1 + p_4$. Thus we have reached a contradiction and the theorem is proved.

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