CONTINUOUS TRANSFORMATIONS OF ABSTRACT SPACES*

BY
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Introduction

In the study of continuous transformations of abstract sets it is customary to restrict both the set itself and its transforms to a particular type of space. This has been done by Fréchet,† Hausdorff,‡ and Alexandroff.§

It is apparent that properties of continuous transformations are in reality properties of the range of the function and the functional values. For this reason we propose to study transformations on general ranges, namely, the topological space. The fundamental theory of topological spaces has been given by Chittenden¶ and Sierpinski.|| Chittenden also considered the relationship between the properties of the class of all continuous real-valued functions on a topological space and the properties of the space.

The first chapter of this paper is a discussion of the definition of a continuous transformation and the difficulties involved. A theorem of Hausdorff is extended to a more general type of space, and a necessary condition for a transformation to be continuous is obtained.

The second chapter is devoted to the invariants of topological spaces, that is, those properties of a space which are properties of every continuous transform. They are not invariants in the strict meaning of the word, but as Sierpinski** remarks, they are invariants in a sense. Invariants under biunivocal and under bicontinuous transformations are also considered. Invariants under biunivocal bicontinuous transformations are not discussed since invariants under such transformations have been extensively studied.

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† Fréchet, (I) *Esquisse d'une théorie des ensembles abstraits*, Sir Asutosh Mookerjee Silver Jubilee Volumes, vol. II, p. 363, Calcutta, Baptist Mission Press, 1922, (II) *Les Espaces Abstraits*, Paris, Gauthier-Villars, 1928.

‡ Hausdorff, *Grundzüge der Mengenlehre*, pp. 358–369, Leipzig, Veit, 1914.

§ Alexandroff, *Über stetige Abbildungen kompakter Räume*, Mathematische Annalen, vol. 96 (1926), pp. 555–571.

¶ Chittenden, *On general topology and the relation of the properties of the class of all continuous functions to the properties of space*, these Transactions, vol. 31, No. 2.

|| Sierpinski, *La notion de dérivée comme base d'une théorie des ensembles abstraits*, Mathematische Annalen, vol. 97 (1926), pp. 321–337.

** Sierpinski, loc. cit., p. 330.

The last chapter may be regarded as a discussion of the following problem: Characterize the most general space such that there exists a non-constant continuous transformation to a given type of space. Necessary and sufficient conditions are found for the existence of continuous transformations of a space to neighborhood, accessible, and L-spaces. The case for a non-constant continuous real function has been solved by Chittenden.*

The following notation will be used.† The term space or topological space denotes a system (P, K) composed of an abstract set P and a relation $E'K\,E$ between subsets E, E' of P. The set E' is unique and determined for each set of P. Thus the relation $E'=K(E)$ defines a single-valued, set-valued set function, whose range is the class of all subsets of P, and whose values are also subsets of P. The points of E' are called K-points of E. Different set functions $K(E)$, $J(E)$, relative to the same set P determine different spaces. By $L(E)$ we denote those points of P which are K-points of some subset of E. When no ambiguity arises we shall use E' for $K(E)$. The complement of a set E with respect to the space is denoted by $C(E)$. The symbol $\subset$ means "is included in."

I. Definition of a continuous transformation

1. **Univocal continuous transformations.** Fréchet‡ defines a continuous transformation in a neighborhood space as follows: A transformation of the space P to the space Q is continuous at the point a if, whatever be the subset G of P having a for a point of accumulation, the transform b of a is a point of accumulation of the transform H of G or belongs to H. A transformation is a continuous transformation of P to Q if it is continuous at each point of P.

Sierpinski§ defines a continuous transformation for a topological space as follows: Let P and Q be two sets for whose subsets the derived sets are defined. Suppose that the function f determines an application of the set $G_0 \subset P$ on the set $H_0 \subset Q$. The function f is continuous in G_0 for the element a of this set if for every subset $G \subset G_0$ such that $a \subset G'$ one has the formula

$$f(a) \subset \{f(G-a)+[f(G)]'\}.$$

This definition is seen to be a modification of the Fréchet definition. In neighborhood spaces the two definitions are equivalent.

There are several properties of continuous real functions which we think should hold for continuous transformations of abstract spaces. They are as follows: (1) A constant function is continuous if the functional value has a

* Chittenden, loc. cit., p. 310.

† Chittenden, loc. cit.

‡ Fréchet II, p. 177.

§ Sierpinski, loc. cit., p. 325.

null derived set; (2) If G is a connected set, then $f(G') \subset [f(G)]'$ unless $[f(G)]'$ is null, in which case $f(G') \subset f(G)$; (3) If there is a continuous transformation of the set, it is also a continuous transformation of every subset; (4) If there is a one-to-one correspondence between two sets and a transformation which is continuous both ways, then the sets are abstractly identical.

The Sierpinski definition does not have the first property if the space P has a point a such that $a \subset a'$, for then $f(a) \subset f(a-a)+[f(a)]'=[f(a)]'$ and the constant transformation $f(P)=Q$, where $Q=q$, a single point whose derived set is null, is not continuous. We call a point a such that $a \subset a'$ a singular point.

Neither does the Sierpinski definition always have the second property, as is shown by the following example: $P=a_1+a_2+a_3$, $Q=b_1+b_2$. Non-null derived sets are given by $a_1'=a_2$, $b_1'=b_2$. The transformation is $f(a_1+a_2)=b_1$, $f(a_3)=b_2$.

The third and fourth conditions always hold under the Sierpinski definition.* The following theorem is easily proved.

THEOREM 1. *Under the Sierpinski definition, every continuous transformation of P to Q possesses the four properties if P has no singular points and if Q has the first and third Riesz† properties.*

Definitions other than the Sierpinski definition may be made, but neither do they agree with all of our intuitive notions. Since the Sierpinski definition seems to be the more fruitful and since objections to it disappear except in unusual spaces, we shall recognize it as the definition of a continuous transformation in a topological space.

2. **Other types of transformations.** A transformation is called biunivocal if it establishes a one-to-one correspondence between the elements of the two ranges. If a transformation is biunivocal the Sierpinski condition that it be continuous reduces to the condition that for every set G such that $a \subset G'$, $f(a) \subset [f(G)]'$. This condition may be stated $f(G') \subset [f(G)]'$.

If we have a continuous transformation $f(P)=Q$ we shall denote by $g(b)$ the set of all points of P to which b corresponds under f, and call g the inverse of f. The inverse transformation g will be called continuous at b if for every set H of which $b \subset H'$, $g(b) \subset g(H-b)+[g(H)]'$. This reduces to the condition $g(b) \subset [g(H)]'$. A univocal continuous transformation whose inverse g is continuous is called univocal bicontinuous.

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* Sierpinski, loc. cit., p. 325.

† Riesz, F., *Stetigkeitsbegriff und abstrakte Mengenlehre*, Atti del 4o Congresso Internazionale dei Matematici, Roma, vol. 2, 1910, p. 18.

A biunivocal bicontinuous transformation is one which is both biunivocal and bicontinuous. A necessary and sufficient condition that a biunivocal transformation be bicontinuous is that for every set E,

$$f(E') = [f(E)]'.$$

This is easily derived from the Sierpinski definition. If a bicontinuous biunivocal transformation exists between two spaces, they are said to be homeomorphic, or topologically equivalent.

3. **Immediate consequences of definitions.** These definitions take interesting forms in certain spaces, and lead to theorems which later prove useful. For these reasons we prove the following theorems.

THEOREM 2. *If f is a continuous transformation such that $f(P)=Q$, and b is a point interior to $B\subset Q$, then $g(b)$ is interior to $g(B)$.*

The proof is by contradiction. Assume $g(b)$ is not interior to $g(B)$. Then there exists a point a of $g(b)$ such that $a\subset G'$ where G is a subset of $C[g(B)]$. Since f is continuous,

$$f(a) \subset f(G) + [f(G)]',$$

$$b \subset [f(G)]',$$

but

$$f(G) \subset C(B)$$

and b is not interior to B as given by hypothesis.

Theorem 2 is restated more strikingly in Theorems 3 and 4.

THEOREM 3. *A necessary condition that a transformation be continuous is that for each point a and its transform b, the inverse image of every neighborhood of b is a neighborhood of a, i.e.*

$$g(V_b) = V_a.$$

THEOREM 4. *A necessary condition that a transformation be continuous is that for every neighborhood of b, the transform of a, there exists a neighborhood of a whose transform is contained in the neighborhood of b.*

In the transformations between two neighborhood spaces the condition of Theorem 3 is seen to be sufficient by the Fréchet definition, so we have

THEOREM 5. *In neighborhood spaces a necessary and sufficient condition that a transformation be continuous is that for each point a and its transform b, the inverse image of every neighborhood of b is a neighborhood of a.*

A necessary and sufficient condition* in a V-space that neighborhoods

* Fréchet II, p. 188.

may be considered as open sets is that $\overline{E}=E+E'$ be closed. If we add this condition the previous theorem becomes

THEOREM 6.* *A necessary and sufficient condition that a transformation between two V-spaces in which $\overline{E}$ is closed, be continuous is that the inverse image of every open (closed) set be an open (closed) set.*

That the words "open" may be replaced by "closed" is shown as follows. For any open set O its complement is closed, so we have

$$Q = O + F,$$
$$P = g(Q) = g(O + F) = g(O) + g(F).$$

Hence if the inverse of an open (closed) set is open (closed) then the inverse of a closed (open) set is closed (open).

THEOREM 7. *A necessary condition that a transformation be continuous is that the inverse of every open (closed) set be an open (closed) set.*

That the theorem is true in the case of open sets is seen immediately from Theorem 2. Since open and completely† closed sets are complementary it is obvious the theorem holds in the case of completely closed sets. For K closed we prove the theorem by contradiction.

Let $f(P)=Q$ be a continuous transformation. Consider B a closed set of Q. Assume $g(B)$ not closed. Then there exists a point $p \subset [g(B)]'-g(B)$. By the definition of continuity

$$\begin{aligned} f(p) &\subset f[g(B) - p] - \{f[g(B)]\}' \\ &\subset f[g(B)] + \{B\}' \\ &\subset B + B' \subset B. \end{aligned}$$

Hence p is a point of $g(B)$ contrary to assumption.

The fact that this condition is not sufficient is shown by the following example: $P=a_1+a_2+a_3$ with the single non-null derived set $a_1'=a_2$, $Q=b_1+b_2+b_3$ with non-null derived sets $b_1'=b_3$, $b_3'=b_1+b_2$. The transformation is $f(a_1)=b_1, f(a_2)=b_2, f(a_3)=b_3$.

The following theorem is a direct consequence of the definition of a continuous transformation.

THEOREM 8. *If $f(P)=Q$ is a continuous transformation, then the addition of points of Q to the derived sets of Q leaves the transformation continuous.*

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* This is a generalization of a theorem of Hausdorff, loc. cit., p. 361.

† A completely closed set is one which contains the K-points of all its subsets.

II. Conditions imposed on transforms by the original space

4. Invariants of univocal transformations. If we have a given space P and a property of P, we wish to determine whether or not the property is true for every possible continuous transform of P. By considering various properties the following theorems are obtained.

THEOREM 9. *If a space P has the property that for every monotonic sequence of closed (completely closed) sets $F_1 \supset F_2 \supset \cdots \supset F_n \supset \cdots$ there exists at least one point common to the sets of this sequence, then every continuous transform Q of P possesses the same property.*

Let $F_1 \supset F_2 \supset \cdots \supset F_n \supset \cdots$ be any monotonic decreasing sequence of closed (completely closed) sets contained in Q. Then $g(F_1) \supset g(F_2) \supset \cdots \supset g(F_n) \supset \cdots$ is a monotonic sequence of such sets in P. Then there is a point a common to all $g(F_n)$ and its transform $f(a)$ is common to all F_n.

THEOREM 10. *Every continuous transform of a self-nuclear* set is self-nuclear.*

Let $f(P) = Q$ be a continuous transformation and E be a self-nuclear subset of P. If $f(E)$ is finite it is self-nuclear. If it is not finite, choose an infinite subset $\sum b_\alpha$ of $f(E)$. Let a_α be a point of $[g(b_\alpha)] \cdot E$. Since $\sum a_\alpha$ is an infinite subset of E, there is a point a such that every neighborhood of a contains an infinite subset of $\sum a_\alpha$ of order $|\sum a_\alpha|$. By Theorem 4, every neighborhood of $f(a)$ contains a subset of $\sum b_\alpha$ of order $|\sum b_\alpha|$.

THEOREM 11. *The continuous transform of a separable space is separable.*†

Let $P = N + N'$ where N is enumerable:

$$f(P) = f(N + N') = f(N) + f(N'),$$
$$f(N') \subset f(N) + [f(N)]',$$
$$f(P) = f(N) + [f(N)]'.$$

Since $f(N)$ is enumerable, the theorem is true.

THEOREM 12. *If a space P has the property that every covering of P by open sets is reducible, then every continuous transform Q of P has the same property.*

Let $O = \sum O_\alpha$ be a covering of Q by open sets. Since by Theorem 7, $g(O_\alpha)$ is an open set, then

$$g(O) = g(\textstyle\sum O_\alpha) = \sum g(O_\alpha)$$

is a covering of P by open sets and hence is reducible. This reduced set may

* Chittenden, loc. cit., p. 297.

† A related theorem is stated for homeomorphic transformations in Fréchet II, p. 241.

be denoted by $\sum g(O_k)$. Since this covers $P, f[\sum g(O_k)] = \sum O_k$ is a reduced covering of Q by open sets.

Furthermore if Q is restricted to be a space in which, for every set H, $H+L(H)$ is completely closed, then Q is bicompact.

THEOREM 13. *The continuous transform of a singular point is a singular point.*

The following theorem is a consequence of Theorem 8.

THEOREM 14. *No property of a space which can be destroyed by the addition of points of the space to derived sets of the space is an invariant.*

THEOREM 15. *The following are not invariants*:

(1) *the property of the space being compact*;*

(2) *the property of the space being perfect*;

(3) *open and closed sets*;

(4) *the four Riesz properties*,† *the second Hausdorff property*,‡ *closure of derived sets, non-compactness, and the properties of a space being accessible, Hausdorff, regular, normal, L, or S.*

The proof consists of an example in which the property is not invariant. The proof for part k is given by example k.

Example 1. $P = \sum_1^\infty a_\alpha + \sum_1^\infty b_\alpha$. Derived sets are given by the following: every infinite set which contains a_α or b_α contains b_α or a_α in its derived set respectively. The transformation is $f(a_\alpha + b_\alpha) = c_\alpha$ where $\sum c_\alpha = Q$. Every derived set in Q is null.

Example 2. $P = a_1 + a_2$ with the derived set relations $a_1' = a_2$, $a_2' = a_1$, $(a_1+a_2)' = 0$. $Q = b$ with $b' = 0$. The transformation is $f(a_1+a_2) = b$.

Example 3. $P = a_1 + a_2$ with $a_1' = 0$, $a_2' = 0$, $(a_1+a_2)' = 0$. $Q = b_1 + b_2$ with $b_1' = b_2$, $b_2' = 0$, $(b_1+b_2)' = 0$. The continuous transformation $f(a_1) = b_1$, $f(a_2) = b_2$ carries the open set a_2 into the non-open set b_2 and the closed set a_1 into the non-closed set b_1.

Example 4. P consists of all the rational points greater than or equal to zero. Derived sets are given by the ordinary metric relationships. Q is likewise composed of the rational numbers greater than or equal to zero. Derived sets are given by the following:

(1) If a is an element of a derived set of E under the metric derived set relationship, $a \subset K(E)$.

* A space is compact if every infinite set of points has a non-null derived set.

† Riesz, loc. cit.

‡ Hausdorff, loc. cit. This is the axiom that neighborhoods are enumerable.

(2) The point 1 is in the derived set of every set having an irrational number in its derived set.

(3) The point 2 is added to the derived set of any infinite subset Y of the set $Z=(1/2, 1/4, 1/8, \cdots, 0)$ but not to $(Y+E)$ where $E(Q-Z)\neq 0$.

(4) If E contains the set $[3, 5, 7]$ then $K(E)$ contains the point 3.

(5) Let 9 be a K-point of every set containing an infinite subset of the positive even integers.

(6) If E contains the point 11 then $K(E)$ contains the point 11.

(7) If $15\subset K(E)$, then 13 is also.

(8) Let 10 be a K-point of every set containing an infinite sequence whose derived set is null.

The transformation $f(P)=Q$, given by $f(x)=x$ as x ranges over the rational points greater than or equal to zero, is a continuous transformation carrying P into Q where P possesses all the properties listed in the fourth part of the above theorem.

The following theorem gives a sufficient condition for every continuous transform of a compact space to be compact.

THEOREM 16. *The continuous transform of a compact space possessing the first three Riesz properties is compact.*

Let P be compact and possess the first three properties of Riesz. Let $f(P)=Q$ under a continuous transformation. In Q let $B=\sum b_\alpha$ be any infinite set of points. Let a_α be a point chosen from each $g(b_\alpha)$. Call $\sum a_\alpha = A$. Since A' is not null it contains at least one point, say c. Now

$$f(c) \subset f(A-c) + [f(a)]' = f(A-c) + B'.$$

If $f(c)\subset f(A-c)$ then there is one point d of A such that $f(d)=f(c)$. Now $c\subset(A-d+c)'$. Hence $f(c)\subset f(A-d)+f(A-d+c)'$. Since $f(c)$ is not included in $f(A-d)$, $f(c)\subset[f(A-d-c)]'=B'$ and the theorem is proved.

5. **Invariants of biunivocal transformations.** We here wish to consider what properties of a topological space remain properties of every transform of the space under biunivocal continuous transformations.

Consider two topological spaces (P, K) and (Q, J) and let $f(P, K)=(Q, J)$ be a biunivocal continuous transformation. There is then a one-to-one correspondence between the elements of P and Q. Let G be any set of P and let H be the corresponding set in Q. Then

$$f(G) = H,$$
$$f[K(G)] \subset J(H).$$

To each set H of Q there corresponds a unique set $g(H)=G$ in P. But to G

there is a unique set $K(G)$, and for $K(G)$ there corresponds a unique set $f[K(G)] \subset Q$. Hence for each set H we can make correspond another set $f[K(G)]$ of Q. Denote $f[K(G)]$ by $K_1(H)$. Then the derived set function J may be expressed as

$$J(H) = K_1(H) + (J - K_1)H = K_1(H) + J_1(H)$$

if we denote the function $(J - K_1)$ by J_1.

THEOREM 17. *If a biunivocal continuous transformation exists between two topological spaces (P, K) and (Q, J) such that $f(P, K) = (Q, J)$, then the derived set function J may be expressed as the sum $K_1 + J_1$ where the space (Q, K_1) is homeomorphic to (P, K).*

Since invariants under univocal transformations remain invariants under biunivocal transformations, Theorems 9, 10, 11, 12, 13 and 14 hold for biunivocal transformations, and the corresponding properties are invariant. Also since the examples given to prove parts 3 and 4 of Theorem 15 are biunivocal, these theorems hold for biunivocal transformations.

THEOREM 18. *The biunivocal continuous transform of a set dense in itself is dense in itself.**

If G is a subset of P which is dense in itself, i.e., $G \subset G'$, then $f(G) \subset f(G') \subset [f(G)]'$ and the theorem is proved.

As a corollary we have

THEOREM 19. *The biunivocal continuous transform of a perfect space is perfect.*

THEOREM 20. *The biunivocal continuous transform of a compact space is compact.*

Let $\sum b_\alpha$ be any infinite set of points in Q. Then $g(\sum b_\alpha) = \sum g(b_\alpha)$ is an infinite set of points in P. Since P is compact there is at least one point, say a, in $[\sum g(b_\alpha)]'$. Then $f(a) \subset \{f[g(\sum b_\alpha)]\}' \subset (\sum b_\alpha)'$ and every infinite set of points in Q has a non-null derived set.

6. **Invariants of bicontinuous transformations.** Invariants under univocal continuous transformations are obviously invariants under bicontinuous transformations.

For any set $G \subset P$ we have from the continuity of f that

$$f(\overline{G}) \subset \overline{f(G)},$$

and for any set $H \subset Q$, we have from the continuity of g that

* This theorem is stated for homeomorphic transformations in Fréchet II, p. 241.

$$g(\overline{H}) \subset \overline{g(H)}.$$

Applying the first formula to the set $\overline{g(H)}$ gives $f[\overline{g(H)}] \subset \overline{f[g(H)]} = \overline{H}$. Taking the inverse of these sets gives $g(f[\overline{g(H)}]) \subset g(\overline{H})$. But $\overline{g(H)} \subset g(f[\overline{g(H)}])$ and $g(\overline{H}) \subset \overline{g(H)}$. Hence $g(\overline{H}) \subset \overline{g(H)} \subset g(f[\overline{g(H)}]) \subset g(\overline{H})$, and $g(\overline{H}) = \overline{g(H)} = g(f[\overline{g(H)}])$. Furthermore $\overline{H} = f[\overline{g(H)}]$.

THEOREM 21. *The bicontinuous transform of the interior of a set is contained in the interior of the transform of the set.*

Denote by $I(A)$ the interior of a set $A \subset P$. Let a be a point of $I(A)$. Assume $f(a)$ is not an element of $I[f(A)]$. Then $f(a) \subset H'$ where $H' \subset C[f(A)]$ and $a \subset g[f(a)] \subset [g(H)]'$. But $g(H) \subset C(A)$ and a is not interior to A contrary to hypothesis.

As a corollary we have

THEOREM 22. *The bicontinuous transform of an open set is an open set.*

THEOREM 23. *Every bicontinuous transform of a bicompact space is bicompact.*

Let P be a bicompact space and $f(P) = Q$ be a bicontinuous transformation. If $R = \sum R_\alpha$ is any proper covering of Q, $g(R) = g(\sum R_\alpha) = \sum g(R_\alpha)$ is a proper covering of P by Theorem 2. Since P is bicompact, $g(R)$ is reducible. Denote the sets of the reduced covering by $g(\sum R_k)$. Each element of P is interior to a set of $g(\sum R_k)$ and by Theorem 20 each element of Q is interior to some set of $\sum R_k$. Hence $\sum R_k$ is a proper covering of Q, R is reducible, and Q is bicompact.

THEOREM 24. *None of the four Riesz properties are invariant under bicontinuous transformations.*

The proof consists of examples in which the properties are not invariant. The non-invariance of property k is shown by example k.

Example 1.* $P = a_1 + a_2 + a_3$ with non-null derived sets $a_2' = a_1$, $(a_1 + a_2)' = a_1 + a_2$, $(a_2 + a_3)' = a_1$, $(a_1 + a_2 + a_3)' = a_1 + a_2$. $Q = b_1 + b_2$, with the non-null derived set $b_1' = b_1$. The transformation is $f(a_1 + a_2) = b_1$, $f(a_3) = b_2$.

Example 2. $P = a_1 + a_2 + a_3$ with non-null derived sets $a_1' = a_2$, $a_2' = a_1$, $(a_2 + a_3)' = a_1$, $(a_1 + a_3)' = a_2$, $(a_1 + a_2)' = a_1 + a_2$, $(a_1 + a_2 + a_3)' = a_1 + a_2$. $Q = b_1 + b_2$ with the non-null derived set $(b_1 + b_2)' = b_1$. The transformation is $f(a_1 + a_2) = b_1$, $f(a_3) = b_2$.

Example 3. $P = a_1 + a_2 + a_3$ with non-null derived sets $(a_1 + a_2)' = a_3$.

* Examples may be constructed without the use of singular points.

$Q=b_1+b_2$ with non-null derived sets $b_1'=b_2$. The transformation is given by $f(a_1-a_2)=b_1, f(a_3)=b_2$.

Example 4. $P=a_1+a_2+a_3$. Non-null derived sets are $a_1'=a_2$, $a_2'=a_1$, $(a_1+a_2)'=a_1+a_2+a_3$. $Q=b_1+b_2$ with the non-null derived set $b_1'=b_1+b_2$. The transformation is $f(a_1+a_2)=b_1, f(a_3)=b_2$.

THEOREM 25. *The closure of derived sets is not an invariant of bicontinuous transformations.*

Example. $P=a_1+a_2+a_3+a_4$ with non-null derived sets $(a_1+a_4)'=a_2+a_4$, $a_2'=a_3$. $Q=b_1+b_2+b_3$ with non-null derived sets $b_1'=b_2$, $b_2'=b_3$. The transformation is $f(a_1+a_4)=b_1, f(a_2)=b_2, f(a_3)=b_3$.

However, properties corresponding to the first two Riesz properties and the closure of derived sets are invariant. These are the corresponding statements in terms of the closure* of a set.

THEOREM 26. *If P has the property that for every set A and B such that $A\subset B$, $\overline{A}\subset\overline{B}$, then every bicontinuous transform Q of P has the same property.*

Let $R\subset S$ be sets of Q. Then

$$g(R)\subset g(S),\qquad \overline{g(R)}\subset\overline{g(S)},$$
$$\overline{R}=f[\overline{g(R)}]\subset f[\overline{g(S)}]=\overline{S}.$$

THEOREM 27. *If a space has the property that for every set E such that $E=A+B$, $\overline{E}\subset\overline{A}+\overline{B}$, then every bicontinuous transform of the space has the same property.*

Let H be any set in Q and let $H=A+B$, $A\neq 0$, $B\neq 0$. Then

$$g(H)=g(A)+g(B),$$
$$\overline{g(H)}\subset\overline{g(A)}+\overline{g(B)},$$
$$f[\overline{g(H)}]\subset f[\overline{g(A)}]+f[\overline{g(B)}],$$
$$\overline{H}\subset\overline{A}+\overline{B}.$$

THEOREM 28. *If a space P has the property that for every set E, $\overline{\overline{E}}\subset\overline{E}$, then every bicontinuous transform Q of P has the same property.*

Let H be any set of Q. Then

$$g(\overline{H})=\overline{g(H)},$$
$$g(\overline{\overline{H}})=\overline{g(\overline{H})}=\overline{\overline{g(H)}}\subset\overline{g(H)},$$
$$f[g(\overline{\overline{H}})]\subset f[\overline{g(H)}],$$
$$\overline{\overline{H}}\subset\overline{H}.$$

* Closure of sets as a basis for abstract spaces has been studied by Kuratowski, Fundamenta Mathematicae, vol. 3, p. 182.

III. Transformations to given spaces

7. **Transformations to V-spaces.** We wish to find a necessary and sufficient condition that for a topological space P there exist a neighborhood* space Q such that Q is the continuous transform of P.

From Theorem 13 it is necessary that no point of P be a singular point. This condition is also sufficient, for if P has no singular point, we can define a V-space Q which is a biunivocal continuous transform of P in the following manner. Let (P, K) be the given space. We define Q on the same class P as a space (P, J). The derived set function J is defined as follows: If $a \subset K(A)$, then $a \subset J(E)$ where E is any set containing $(A-a)$, (P, J) is a neighborhood space and the transformation $f(a)=a$ between (P, K) and (P, J) is continuous. We have then

Theorem 29. *A necessary and sufficient condition that for a topological space P there exist a neighborhood space Q, such that Q is the univocal (biunivocal) continuous transform of P, is that P does not contain a singular point.*

8. **Transformations to other spaces.** We assume in the following discussion that the transform space Q is connected and consists of more than one element, for otherwise the problem is trivial.

Let $f(P)=Q$ be a continuous transformation of a topological space P into a space Q possessing the first three Riesz properties. Since such a space Q is a V-space, it is necessary that P does not contain a singular point. Q consists of an infinite number of disjoined completely closed sets such that the sum of any finite number of them is completely closed, i.e., the points q_α of Q. By Theorem 7, P is the sum of such a family, namely, the $g(q_\alpha)$. It is apparent that if the following conditions hold in (P, K):

$$a \subset g(q_\alpha), \quad a \subset K(E), \quad \text{and} \quad [g(q_\alpha)-a]\cdot E=0,$$

then E has points in common with an infinite number of $g(q_\alpha)$. We can then derive from (P, K) a space (W, J) satisfying the following conditions:

(1) The elements W_α of W are the disjoined completely closed sets $g(q_\alpha)$ of P.

(2) If in P, $a \subset K(E)$ and $(W_\alpha - a)\cdot E=0$, then in W, $w_\alpha \subset J(\sum w_E)$ where w_α is the set w_α containing a, and $\sum w_E$ is the sum of the sets w_α containing points of E.

(3) In (W, J) every finite set has a null derived set.

Furthermore if from a space P we can derive a space (W, J) satisfying these three conditions, then there exists a continuous transform of P which

* Fréchet II, p. 192.

possesses the three Riesz properties, for instance, the space on the class W such that the derived set of every finite set is null, and the derived set of every infinite set is the entire space. Since this space is also accessible we have

THEOREM 30. *A necessary and sufficient condition that for a space P there exist a space Q possessing the first three Riesz properties* which is the continuous transform of P is*

(A) *no point of P is a singular point;*

(B) *a space (W, J) satisfying the three conditions stated above may be derived from P.*

Next we consider transformations to L-spaces. Let $f(P)=Q$ where Q is an L-space. P then satisfies conditions (A) and (B) above. In the derived space (W, J) consider an element $w_a \subset J(\sum w_E)$. Then

$$f(w_a) \subset [f(\sum w_E)]'.$$

Since Q is an L-space, there exists an infinite sequence $B \subset [f(\sum w_E)]'$ such that $f(w_a)$ is its unique limit point. Then $g(B)$ is an infinite set of $w_\alpha \subset \sum w_E$. Call $g(B)=A$. $J(A)=w_a$ or 0. It follows that if $w_\alpha \subset J(G)$, there is an enumerably infinite subset $H_\alpha \subset G$ such that $H_\alpha' = w_\alpha$ or 0. Furthermore for any point w_β, any corresponding H_β is such that $H_\alpha \cdot H_\beta$ is finite.

To show the above conditions are sufficient assume P satisfies them and construct Q as follows: Q consists of a set of points in one-to-one correspondence T with the elements w_α of W. Denote the point corresponding to w_α by q_α. Derived sets are given by the following rule: If $T(w_\alpha)=q_\alpha$, where $w_\alpha \subset J(G)$, then $q_\alpha \subset B$, any set containing an infinite subset of $T(H_\alpha)$. The set $T(H_\alpha)$ is enumerably infinite and $q_\alpha = [T(H_\alpha)]'$. The space Q is an L-space and the transformation of P given by $f(w_\alpha)=q_\alpha$ is continuous.

THEOREM 31. *Conditions* A *and* B *of Theorem* 30 *and the following condition* C, *form a necessary and sufficient condition that for a space P there exist an L-space Q which is the continuous transform of P:*

(C) *If, in (W, J), $w_\alpha \subset J(G)$ then there is an enumerably infinite subset $H_\alpha \subset G$ such that $H_\alpha' = w_\alpha$ or* 0, *and such that for the corresponding set H_β of any point w_β, $H_\alpha \cdot H_\beta$ is finite.*

Next consider a compact topological space P, and a biunivocal continuous transform Q which is an L-space. If in P, $a \subset G'$, then $f(a) \subset [f(G)]'$ and $f(a)$ is the limit element of a converging sequence in $f(G)$, say $[b_n]$. Now $g(b_n)$ is an

* It may be noted that the theorem also holds if the accessible property is added to the three Riesz properties.

infinite sequence in G and has (since P is compact) at least one point of accumulation d. Since $d \subset [g(b_n)]'$, $f(d) \subset (b_n)'$. But (b_n) has only one point of accumulation, and a and b must coincide. For every point a and every subset G of P such that $a \subset G'$, there exists a subset of G which is compact and has no other point of accumulation than a. P is then an L-space.

Consider any point of accumulation q of Q. There is an infinite sequence converging to q which is the only limit element of the sequence. Call the sequence $[q_n]$. Then $g(q_n)$ has a limit point since P is a compact L-space. Call the limit point p. Then $f(p)=q$. We have then that for every point q and set H such that $q \subset H'$,

$$g(q) \subset [g(H)]'.$$

Hence g is a continuous transformation and f is a biunivocal bicontinuous transformation.

THEOREM 32. *If there exists a biunivocal continuous transformation of a compact space P to an L-space, then the transformation is biunivocal and bicontinuous, and the spaces are homeomorphic.*

The theorem is not true in case Q is a space with only the first three Riesz properties. This is shown by an example. Let Q consist of an enumerable infinite set of points $[b_n]$. Let b_1+b_2 be the derived set of every infinite subset of Q. All other derived sets are null. If P is a compact enumerable infinite set of points with a single limit point p, then a transformation which carries p into b_1, and establishes a one-to-one correspondence between the remaining points of P and Q, is a biunivocal continuous transformation, but is not bicontinuous.

Let $f(P)=Q$ where Q is a Hausdorff space. If a and b are two points of Q, there are two disjoined open sets O_a and O_b to which a and b belong respectively. Then $g(O_a)$ and $g(O_b)$ are disjoined open sets containing $g(a)$ and $g(b)$ respectively. From the above argument and the fact that Q is accessible, we see that it is necessary that P consist of the sum of an infinite number of disjoined closed sets such that the sum of a finite number is closed, and any two of them are separated by open sets.

If Q is regular,* then for any point a and any open set O_a containing a, there is a closed set F_a to which a is interior. But the interior of F_a is an open set, say O. We thus obtain an infinite number of distinct decreasing open sets and closed sets to which a is interior. In P we have a corresponding family for each of the closed sets.

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* Fréchet II, p. 206.

If Q is normal,* there exists a normal family of open sets; such a family also exists in P. Hence a necessary condition is that P contain a normal family of open sets. It is also sufficient, for if P contains such a family we can define a non-constant continuous function on P, which is a continuous transformation to a normal space.†

* Fréchet II, p. 206.

† Chittenden, loc. cit., p. 310.

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