

# A CHARACTERIZATION OF THE CLOSED 2-CELL\*

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1. Introduction. A number of characterizations have been given of the simple closed surface.‡ The proofs involve considerable point set difficulties. We give here a characterization of the closed 2-cell, that is, a point set homeomorphic with a circle and its interior. The fundamental theorem is partly of a combinatorial and partly of a continuity nature. It reads

**THEOREM I.** *Let  $R$  be a continuous curve § containing the simple closed curve  $J$ , such that*

(1)  *$J$  is irreducibly homologous to zero in  $R$ , and*

(2) *If  $\gamma$  is an arc with just its two end points  $a$  and  $b$  on  $J$ , then  $R - \gamma$  is not connected.*

*Let  $R'$  and  $J'$  be defined similarly. Then  $R$  and  $R'$  are homeomorphic, with  $J$  corresponding with  $J'$ .*

That  $R$  is a closed 2-cell then follows immediately from the following theorem. We note that  $J$  corresponds with the circle, that is,  $J$  is the boundary of  $R$ .

**THEOREM II.** *If  $I$  is a circle in the plane and  $S$  is  $I$  with its interior, then  $S$  and  $I$  satisfy the conditions prescribed for  $R$  and  $J$  in the above theorem.*

The exact meaning of Condition (1) of Theorem I is given in §4; a stronger condition is the following: For every  $\epsilon > 0$  and any two points  $a$  and  $b$  on  $J$ , there is a set of points  $a_{ij}$  in  $R$ ,  $1 \leq i \leq m$ ,  $1 \leq j \leq n$ , such that all points  $a_{1j}$  coincide with  $a$ , all points  $a_{mj}$  coincide with  $b$ , all points  $a_{i1}$  lie on one arc  $ab$  of  $J$ , all points  $a_{in}$  lie on the other arc  $ab$  of  $J$ , and||

$$\rho(a_{ij}, a_{i+1,j}) < \epsilon, \quad \rho(a_{ij}, a_{i,j+1}) < \epsilon;$$

moreover, this does not hold in any proper subset of  $R$  containing  $J$ .

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‡ That is, a point set homeomorphic with the surface of a sphere. See L. Zippin, *American Journal of Mathematics*, vol. 52 (1931), pp. 331-350; these *Transactions*, vol. 31 (1929), pp. 744-770; C. Kuratowski, *Fundamenta Mathematicae*, vol. 13 (1929), pp. 307-318; also references in these papers.

§ See Lemma A.

||  $\rho(p, q)$  = distance from  $p$  to  $q$ , or in general, distance between two point sets;  $\delta(S)$  = diameter of  $S$ ;  $V_\epsilon(S)$  = those points  $p$  for which  $\rho(p, S) < \epsilon$ ;  $W_\epsilon(S)$  = those points  $p$  for which  $\rho(p, S) \leq \epsilon$ .

Notations and preliminary theorems are given in §§2, 3 and 4; an outline of the proof of Theorem I will be found in §5. The Jordan and related theorems follow of course from the above theorems.

2. **Point set background.** Elementary properties of point sets we shall need may be found in Hausdorff, *Mengenlehre*, chapter VI. A continuous curve is a metric space which can be expressed as the continuous image of a closed line segment. An arc is the topological image of a closed line segment; a simple closed curve, the topological image of a circle.

Two fundamental lemmas are the following:

LEMMA A.\* *A compact, connected and locally connected metric space is a continuous curve, and conversely.*

LEMMA B.† *A continuous curve is arcwise connected.*

That is, any two points  $p$  and  $q$  in the set are end points of an arc  $pq$  in the set. Using the definition of a continuous curve, it is easily seen that two continuous curves which have common points form a continuous curve.

From these lemmas we deduce the following known theorems.

LEMMA C. *Any continuum  $C$  of diameter  $< \epsilon$  in a continuous curve  $R$  is contained in a continuous curve  $C'$  in  $R$  of diameter  $< \epsilon$ .*

Say  $\delta(C) = \epsilon - \epsilon'$ .  $R$  being the continuous image of a closed line segment, we can divide this segment into segments so small that the diameter of the image of each is  $< \epsilon'$ . We let  $C'$  be the union of all of these images which have points in common with  $C$ .

LEMMA D. *A continuous curve  $R$  is locally arcwise connected.*

That is, given a point  $p$  and an  $\epsilon > 0$ , there is a  $\delta > 0$  such that if  $q \in V_\delta(p)$ , then there is an arc  $pq$  in  $R$  of diameter  $< \epsilon$ . As  $R$  is locally connected, we can take  $\delta$  so that if  $q \in V_\delta(p)$ , there is a continuum  $C$  in  $R$  of diameter  $< \epsilon$  containing  $p$  and  $q$ . The continuum  $C$  is contained in a continuous curve  $C'$  of diameter  $< \epsilon$ , and  $C'$  is arcwise connected; hence there is an arc  $pq \subset C' \subset R$ , and  $\delta(pq) < \epsilon$ .

$R$  is of course uniformly locally arcwise connected, by the Borel Theorem.

LEMMA E. *A connected open subset  $R'$  of a continuous curve  $R$  is arcwise connected.*

\* See G. T. Whyburn, *Concerning continuous images of the interval*, American Journal of Mathematics, vol. 53 (1931), pp. 670-674.

† See references in R. L. Moore, *Report on continuous curves*, Bulletin of the American Mathematical Society, vol. 29 (1923), p. 293, footnote (†).

If there are two points  $p$  and  $q$  in  $R'$  which are joined by no arc in  $R'$ , let  $A$  contain  $p$  and all points of  $R'$  joined to  $p$  by an arc in  $R'$ , and put  $B = R' - A$ ; then there is no arc joining a point of  $A$  to a point of  $B$  in  $R'$ . As  $R'$  is connected, there is a point  $p'$  in one of these sets, say  $B$ , which is a limit point of points of the other set,  $A$ . As  $R'$  is open in  $R$ ,  $\rho(p', R - R') = \epsilon > 0$ . We can take  $q'$  in  $A$  so close to  $p'$  that there is an arc  $p'q'$  in  $R$  of diameter  $< \epsilon$ . But then  $p'q' \subset R'$ , a contradiction.

Suppose  $R$  is connected, and  $p \in R$  is such a point that  $R - p$  is not connected. Then  $p$  is called a cut point of  $R$ .

LEMMA F. *Let  $R$  be a continuous curve without a cut point. Then for every  $\epsilon > 0$  there is a  $\delta > 0$  such that if  $\rho(q, p) \geq \epsilon$  and  $\rho(q', p) \geq \epsilon$ , then there is an arc  $qq'$  with no points in  $V_\delta(p)$ .*

Suppose the contrary. Then there are three sequences of points  $\{p_n\}$ ,  $\{q_n\}$ ,  $\{q'_n\}$ , approaching points  $p$ ,  $q$ ,  $q'$ , respectively, with  $\rho(q_n, p_n) \geq \epsilon$ ,  $\rho(q'_n, p_n) \geq \epsilon$ , and such that for each  $n$ , any arc  $q_nq'_n$  must contain points in  $V_{\delta_n}(p_n)$ , where  $\lim_{n \rightarrow \infty} \delta_n = 0$ . By Lemma D it is seen that for any  $n$  greater than some  $N$  there are arcs  $q_nq$ ,  $q'_nq'$ , with no points in  $V_{\epsilon/2}(p)$ . It follows that any arc  $qq'$  must pass through  $p$ , contradicting Lemma E (as  $p$  is not a cut point).

3. **Combinatorial background.\*** A  $k$ -simplex, or abstract  $k$ -simplex, is a set of  $k$  elements (say points)  $a_1a_2 \cdots a_k$ . The order in which we write the points is immaterial. For  $k=0, 1$  and  $2$  we use also the terms *vertex*, *segment* and *triangle* respectively. A  $k$ -chain is a set of  $k$ -simplexes, and is written as the sum of these simplexes. The *sum (mod 2)* of several  $k$ -chains is the  $k$ -chain containing those simplexes which occur in an odd number of the  $k$ -chains.

The *boundary*  $K$  of a  $k$ -simplex  $L$ ,  $k > 0$ , is the sum of all  $(k-1)$ -simplexes formed by dropping out one of the vertices of the simplex. We write  $L \rightarrow K$ . A 0-simplex has no boundary. Thus

$$a \rightarrow 0, ab \rightarrow a + b, abc \rightarrow ab + ac + bc.$$

The *boundary* of a  $k$ -chain is the sum (mod 2) of the boundaries of the simplexes of the chain. Thus

$$ab + bc + cd \rightarrow a + d, abc + bcd \rightarrow ab + ac + bd + cd.$$

Evidently the *boundary of a sum of several  $k$ -chains is the sum of the boundaries of the chains*. If a  $k$ -chain has no boundary, it is called a  $k$ -cycle. (Any 0-chain is a 0-cycle.) *The boundary of a  $k$ -chain ( $k > 0$ ) is a  $(k-1)$ -cycle*. This is evi-

\* Compare L. Vietoris, *Über den höheren Zusammenhang kompakter Räume*, *Mathematische Annalen*, vol. 97 (1927), pp. 454-472.

dent if the  $k$ -chain is a  $k$ -simplex. The general case then follows from the last theorem.

LEMMA G. *If  $K \rightarrow a + b$  is a 1-chain, then there is a chain of segments  $aa_1, a_1a_2, \dots, a_nb$  in  $K$ .*

For otherwise we could divide the segments of  $K$  into two groups  $K_1 \supset a$  and  $K_2 \supset b$ , no two simplexes from different groups having a common vertex. But then  $K_1 \rightarrow a, K_2 \rightarrow b$ , which cannot be, as the boundary of any 1-chain contains an even number of vertices.

A 1-circuit is a 1-cycle of the form  $a_1a_2, a_2a_3, \dots, a_{n-1}a_n, a_na_1$ , the vertices being distinct except as shown.

LEMMA H. *Any 1-cycle  $K$  is a sum of 1-circuits.*

If  $a_1a_n$  is a segment of  $K$ , then  $K + a_1a_n \rightarrow a_1 + a_n$ , as  $K \rightarrow 0$  and  $a_1a_n \rightarrow a_1 + a_n$ . We can thus find a set of distinct segments and vertices  $a_1a_2, \dots, a_{n-1}a_n$  in  $K + a_1a_n$  not containing  $a_1a_n$ . This with  $a_1a_n$  is a 1-circuit  $K_1$ . As  $K_1 \rightarrow 0$ ,  $K + K_1$  is a 1-cycle containing no segments of  $K_1$ , and it contains a 1-circuit  $K_2$ . Continuing, we find  $K = K_1 + K_2 + \dots + K_m$ .

4. A  $k$ -chain  $K$  is said to *lie in a point set  $R$*  if each vertex of  $K$  is in  $R$ . Any vertex now has both a name and a position. Two vertices are distinct if their names are distinct, irrespective of whether they coincide in position or not.  $\epsilon$  being a positive number, a  $k$ -simplex  $K \subset R$  is called an  $(\epsilon, k)$ -simplex in  $R$  if  $\delta(K) < \epsilon$ , i.e. if any two vertices of  $K$  are within  $\epsilon$  of each other. A  $k$ -chain is an  $(\epsilon, k)$ -chain if each of its simplexes is an  $(\epsilon, k)$ -simplex. A  $k$ -cycle  $K$  in  $S$  is said to be  $\epsilon$ -homologous to zero ( $K \epsilon \sim 0$ ) in  $R$  if there is an  $(\epsilon, k+1)$ -chain  $L$  in  $R$  of which  $K$  is the boundary. If  $K_1 \epsilon \sim 0$  and  $K_2 \epsilon \sim 0$ , then  $K_1 + K_2 \epsilon \sim 0$ . We write also  $K_1 \epsilon \sim K_2$  for  $K_1 + K_2 \epsilon \sim 0$ . If  $K_1 \epsilon \sim K_2$  and  $K_2 \epsilon \sim K_3$ , then  $K_1 \epsilon \sim K_3$ .

Suppose the closed set  $R$  contains the simple closed curve  $J$ . If for every  $\epsilon > 0$  there is a  $\delta > 0$  such that any  $(\delta, 1)$ -cycle on  $J$  is  $\epsilon \sim 0$  in  $R$ , then we say that  $J \sim 0$  in  $R$ . If  $J$  is  $\sim 0$  in  $R$  but is not  $\sim 0$  in any proper closed subset of  $R$  containing  $J$ , then we say that  $J$  is *irreducibly  $\sim 0$*  in  $R$ .

LEMMA I. *Given a simple closed curve  $J$ , let us divide it into the arcs\*  $\overline{a_1a_2}, \overline{a_2a_3}, \dots, \overline{a_{n-1}a_n}, \overline{a_na_1}$ , each of diameter  $< \epsilon/2$ . Let  $\delta$  be smaller than the distance between any two of these arcs which have no common points. Then if  $K' = a_1a_2 + a_2a_3 + \dots + a_na_1$  and  $K$  is any  $(\delta, 1)$ -cycle on  $J$ ,  $K$  is either  $\epsilon \sim 0$  or  $\epsilon \sim K'$  on  $J$ .*

By Lemma H,  $K$  is a sum of 1-circuits  $K_1, \dots, K_m$ . If we show that each

\* Here,  $\overline{a_1a_2}$  denotes an arc, and  $a_1a_2$ , a segment.

$K_i$  is  $\epsilon \sim \alpha_i K'$ ,  $\alpha_i = 0$  or  $1$ , it will follow that  $K = \sum K_i \epsilon \sim \sum \alpha_i K' = 0$  or  $K'$  (depending on whether  $\sum \alpha_i$  is even or odd), and the lemma will be proved.

Consider any  $K_i = b_1 b_2 + b_2 b_3 + \dots + b_n b_1$ , say. If a vertex  $b_j$  of  $K_i$  does not lie on any point  $a_k$ , say  $b_j \subset a_k a_{k+1}$ ; add to  $K_i$  the boundary of the  $\epsilon$ -triangles  $b_{j-1} b_j a'_k + b_j b_{j+1} a'_k$ , where  $a'_k$  is a new vertex lying on  $a_k$ . The result is an  $(\epsilon, 1)$ -circuit  $K_i^{(1)} \epsilon \sim K_i$ , the vertex  $b_j$  having been replaced by the vertex  $a'_k$ . Repeat the process till we have an  $(\epsilon, 1)$ -circuit  $K'' = c_1 c_2 + c_2 c_3 + \dots + c_r c_1 \epsilon \sim K_i$ .

Now any two consecutive vertices  $c_j, c_{j+1}$  lie on the same or consecutive vertices of  $K'$ . Suppose  $c_j$  is on  $a_k$  and  $c_{j+2}$  is on  $a_{k+p}$ ,  $p \neq 2$  or  $-2$ . Then add the boundary of  $c_j c_{j+1} c_{j+2}$ , replacing the segments  $c_j c_{j+1} + c_{j+1} c_{j+2}$  by the single segment  $c_j c_{j+2}$ . Continue till we arrive at a (possibly void)  $(\epsilon, 1)$ -circuit  $K^* = d_1 d_2 + \dots + d_r d_1 \epsilon \sim K_i$ . If  $d_1$  lies on  $a_k$ , then  $d_{j+1}$  lies on  $a_{k \pm j}$ , where we put  $n + p = p$ , etc.

If  $K^*$  contains no segments,  $K_i \epsilon \sim 0$ . Otherwise, following the vertices  $d_1, d_2, \dots, d_r, d_1$  of  $K^*$ , we have gone around  $J$   $p$  times say. Add to  $K^*$  the boundaries of all the  $2r$   $\epsilon$ -triangles of the following sort. If  $d_j$  lies on  $a_k$ , and  $d_{j+1}$  on  $a_{k \pm 1}$ , two of the triangles are  $d_j d_{j+1} a_k$  and  $d_{j+1} a_k a_{k \pm 1}$ . The result is an  $(\epsilon, 1)$ -cycle  $pK' = 0$  or  $K'$ . Thus  $K_i \epsilon \sim 0$  or  $K'$ , and the proof is complete.

An immediate consequence of this lemma is

LEMMA J. *Let the simple closed curve  $J$  lie in the closed set  $R$ . If for every  $\epsilon > 0$  there is a 1-cycle  $K'$  in  $J$  as above described which is  $\epsilon \sim 0$  in  $R$ , then  $J \sim 0$  in  $R$ .*

LEMMA K. *If  $\gamma$  is an arc, then for every  $\epsilon > 0$  there is a  $\delta > 0$  such that any  $(\delta, 1)$ -cycle on  $\gamma$  is  $\epsilon \sim 0$  on  $\gamma$ .*

The proof below holds in fact if  $\gamma$  is a closed  $k$ -cell, any  $k$ . It is sufficient to prove it for the case that  $\gamma$  is a closed line segment, in which case we can take  $\delta = \epsilon/2$ .†

Let  $K$  be a  $(\delta, 1)$ -cycle on  $\gamma$ , let  $a_0 b_0$  be a segment of  $K$ , and say  $\delta(\gamma) = \alpha$ . Choose a fixed point  $p$  in  $\gamma$ , and an integer  $n > \alpha/\delta$ . Let the vertices  $a_1, a_2, \dots, a_{n-1}$  divide the segment  $a_0 p$  into  $n$  equal parts, and similarly for the vertices  $b_1, b_2, \dots, b_{n-1}$ . Add to  $K$  the boundaries of all triangles of the form  $a_i a_{i+1} b_i, a_{i+1} b_i b_{i+1}, a_{n-1} b_{n-1} p$ , and of all similar triangles corresponding to the other segments of  $K$ . The result is  $0$ . As all the triangles employed are  $\epsilon$ -triangles,  $K \epsilon \sim 0$  in  $\gamma$ .

5. Outline of the proof of Theorem I. The proof runs as follows.

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† The essential point in the proof below is that  $\gamma$  is convex: any two points of  $\gamma$  are end points of a line segment in  $\gamma$ . The proof is then easily extended to the case of any set homeomorphic with  $\gamma$ .

(a) In §6 we show how an arc  $\gamma$  can be drawn in  $R$  crossing  $J$ , † avoiding two given closed sets.  $R - \gamma$  is not connected.

(b) In §7 we prove some lemmas. These show (§8) that  $R - \gamma$  contains exactly two components  $A'$  and  $B'$ . If  $A = A' + \gamma$ , then  $A$  and its boundary curve  $J_A$  (which is  $\gamma$  plus a part of  $J$ ) satisfy condition (1) of the theorem; similarly for  $B = B' + \gamma$  and  $J_B$ . Further,  $A$  and  $B$  are continuous curves.

(c) In §9 it is shown that any arc in  $A$  (or  $B$ ) crossing  $J_A$  ( $J_B$ ) divides  $A$  ( $B$ ). Thus  $A$  and  $J_A$  ( $B$  and  $J_B$ ) satisfy all the conditions of the theorem. Hence we can cut up each set just as we cut up  $R$ , and can continue indefinitely.

(d) The object of §10 is to prove that  $R$  may be cut into pieces of arbitrarily small diameter.

(e) The homeomorphism between  $R$  and  $R'$  is now easily established. We cut  $R$  up indefinitely, and cut  $R'$  in a corresponding fashion. Any point  $p$  of  $R$  lies in a descending sequence of pieces; the corresponding sequence in  $R'$  determines a point  $p'$ , which we let correspond to  $p$ .

We turn now to the detailed proof.

6. An arc crossing  $J$ . We prove here

LEMMA L. ‡ Let the simple closed curve  $J$  be  $\sim 0$  in the continuous curve  $R$ . Let  $c$  and  $d$  be two points of  $J$ , dividing  $J$  into the two arcs  $\eta_1$  and  $\eta_2$ . If  $C$  and  $D$  are two closed sets in  $R$  containing  $c$  and  $d$  respectively, and  $C \cdot D = 0$ , then there is an arc  $\gamma$  in  $R$  joining  $\eta_1$  to  $\eta_2$  which has no points in  $C$  or in  $D$ .

Say  $\rho(C, D) = 3\epsilon$ , and put  $C' = W_\epsilon(C)$ ,  $D' = W_\epsilon(D)$ ; then  $\rho(C', D') = \epsilon$ . Take  $\sigma$  so small that any two points in  $R$  within  $\sigma$  of each other are joined by an arc of diameter  $< \epsilon$  (Lemma D). Take  $\delta$  so small that any  $(\delta, 1)$ -cycle on  $J$  is  $\sigma \sim 0$  in  $R$ . Construct the  $(\delta, 1)$ -cycle  $K = cc_1 + c_1c_2 + \cdots + c_md + dd_1 + d_1d_2 + \cdots + d_nc$ ,  $c_i \subset \eta_1$ ,  $d_i \subset \eta_2$ . There is a  $(\sigma, 2)$ -chain

$$L = L_C + L_D \rightarrow K$$

in  $R$ , where we let  $L_C$  contain all those triangles of  $L$  with vertices in  $C'$ , and let  $L_D$  be the rest of  $L$ .

Say

$$L_C \rightarrow K_C = K'_C + K^*,$$

where we let  $K'_C$  contain all those segments of  $K_C$  which are also in  $K$ . As  $L_C \subset V_\sigma(C')$ ,  $K^* \cdot D' = 0$ . Define  $K_{D'}$  by the relation

† That is,  $\gamma$  lies in  $R$ , and has only its end points on  $J$ .

‡ Compare P. Urysohn, *Über Räume mit verschwindender erster Brouwerscher Zahl*, Proceedings, Amsterdam Akademie van Wetenschappen, vol. 31 (1928), pp. 808-810.

$$L_D \rightarrow K_D = K_D' + K^*.$$

Adding these relations gives  $L$  on the left, and hence  $K$  on the right:

$$K = K_C' + K_D'.$$

As all the segments of  $K_C'$  are in  $K$ ,  $K_D'$  must contain just those segments of  $K$  not in  $K_C'$ ; in particular, it contains no segments of  $K^*$ . Hence all the segments of  $K^*$  are present in  $K_D' + K^*$ , the boundary of  $L_D$  (i.e. none have canceled out with segments of  $K_D'$ ). Hence, as  $L_D \cdot C' = 0$ ,

$$K^* \cdot C' = K^* \cdot D' = 0.$$

As  $K_C$  is the boundary of  $L_C$ , it is a 1-cycle; hence

$$K_C + cc_1 = K_C' + K^* + cc_1 \rightarrow c + c_1.$$

By Lemma G,  $K_C + cc_1$  contains a chain of segments joining  $c_1$  to  $c$ . Following this chain, let  $p_s$  be the first vertex in  $\eta_2$ , and  $p_0$ , the last vertex before  $p_s$  in  $\eta_1$ , and say  $p_0p_1, p_1p_2, \dots, p_{s-1}p_s$  are the segments in between. We shall show that these segments are in  $K^*$ . If  $s > 1$  this is obvious, as then  $p_1, \dots, p_{s-1}$  exist and are not on  $J$ . Suppose  $s = 1$  and  $p_0p_1$  is not in  $K^*$ ; then it is in  $K_C' + cc_1$ . It could only be the segment  $cc_1$ . But  $cc_1$  lies in  $K$  and not in  $K_D$ , hence it is in  $K_C$ ; it is not in  $K^*$ , hence it is in  $K_C + K^* = K_C'$ , and therefore not in  $K_C' + cc_1$ . This proves the statement.

Now let  $\overline{p_i p_{i+1}}$  be an arc of diameter  $< \epsilon$  in  $R, i = 0, \dots, s - 1$ . These arcs form a continuous curve, from which we can pick out an arc  $\gamma$  (Lemma B) joining  $\eta_1$  to  $\eta_2$ ; we can take  $\gamma$  so only its end points are on  $J$ . As  $p_i p_{i+1} \subset K^*$  and  $\delta(\overline{p_i p_{i+1}}) < \epsilon, \gamma$  has no points in  $C$  or in  $D$ , and the lemma is proved.

7. We prove three lemmas.

LEMMA M. *If  $J \subset C, J \sim 0$  in  $C + D$ , and  $C \cdot D = \text{an arc } \gamma$ , then  $J \sim 0$  in  $C$ .*

Given an  $\epsilon > 0$ , choose first  $\epsilon_1$  so small that any  $(3\epsilon_1, 1)$ -cycle on  $\gamma$  is  $\epsilon \sim 0$  in  $\gamma$  (Lemma K). Take next  $\epsilon_2 < \epsilon_1$  so that if  $p \in D$  and  $\rho(p, C) < \epsilon_2$ , then  $\rho(p, \gamma) < \epsilon_1$ . (If  $D_1 = D - D \cdot V_{\epsilon_1}(\gamma)$ , take  $\epsilon_2 < \rho(D_1, C)$ .) Take finally  $\delta < \epsilon_2$  so that any  $(\delta, 1)$ -cycle  $K$  on  $J$  is  $\epsilon_2 \sim 0$  in  $C + D$ ; we shall show that  $K \epsilon \sim 0$  in  $C$ .

Let  $L \rightarrow K$  be an  $(\epsilon_2, 2)$ -chain in  $C + D$ . Take any vertex  $p$  of  $L$  in  $D \cdot V_{\epsilon_2}(C) - \gamma$ , and replace it by a vertex  $p' \in \gamma$ , where  $\rho(p, p') < \epsilon_1$ .  $L$  is thus replaced by a  $(3\epsilon_1, 2)$ -chain  $L'$ , in which each triangle lies wholly in either  $C$  or  $D$ . Moreover,  $L' \rightarrow K$ , as no vertices of  $K$  have been moved.

Put  $L' = L_C + L_D$ , where  $L_C$  contains those triangles of  $L'$  in  $C$ . Say

$$L_C \rightarrow K + K^*; \text{ then } L_D \rightarrow K^*.$$

$K^*$  is a  $(3\epsilon_1, 1)$ -cycle lying in  $C \cdot D = \gamma$ ; it bounds an  $(\epsilon, 2)$ -chain  $L^*$  in  $\gamma$ . Hence

$$L_C + L^* \rightarrow (K + K^*) + K^* = K.$$

$L_C + L^*$  is an  $(\epsilon, 2)$ -chain in  $C$ , and the lemma is proved.

LEMMA N. *Let  $A \cdot B = \gamma$ , an arc whose end points are  $a$  and  $b$ . Let the arcs  $\alpha$  and  $\beta$  join  $a$  and  $b$  in  $A$  and  $B$  respectively, neither having any points other than  $a$  and  $b$  in common with  $\gamma$ . If  $\alpha + \beta \sim 0$  in  $A + B$ , then  $\alpha + \gamma \sim 0$  in  $A$ .*

Given an  $\epsilon > 0$ , choose  $\epsilon_1, \epsilon_2$  and  $\delta$  as in the last lemma. Take  $(\delta, 1)$ -chains  $K_\alpha, K_\beta$  and  $K_\gamma$  in  $\alpha, \beta$  and  $\gamma$  respectively, each bounded by  $a + b$ ; by Lemma J, it is sufficient to show that  $K_\alpha + K_\gamma \epsilon \sim 0$  in  $A$ .

$K_\alpha + K_\beta$  bounds an  $(\epsilon_2, 2)$ -chain  $L$  in  $A + B$ ; we move each vertex of  $L$  in  $B \cdot V_{\epsilon_2}(A) - \gamma$  onto  $\gamma$ , giving a  $(3\epsilon_1, 2)$ -chain  $L' \rightarrow K_\alpha + K_\beta'$ . Say  $L' = L_A + L_B$ , where  $L_A \subset A, L_B \subset B$ . If  $L_A \rightarrow K_\alpha + K^*$ , then  $L_B \rightarrow K_\beta' + K^*$ , and  $K^* \subset \gamma$ .  $K^* + K_\gamma$  is a  $(3\epsilon_1, 1)$ -cycle on  $\gamma$  bounding an  $(\epsilon, 2)$ -chain  $L^*$  in  $\gamma$ . Hence  $L_A + L^* \rightarrow K_\alpha + K_\gamma$  in  $A$ , completing the proof.

LEMMA O. *Let  $\alpha, \beta$  and  $\gamma$  be three arcs such that  $\alpha \cdot \beta = \alpha \cdot \gamma = \beta \cdot \gamma = a + b$ . Say  $\alpha + \gamma \subset A$  and  $\beta + \gamma \subset B$ . If  $\alpha + \gamma \sim 0$  in  $A$  and  $\beta + \gamma \sim 0$  in  $B$ , then  $\alpha + \beta \sim 0$  in  $A + B$ .*

Define  $K_\alpha, K_\beta, K_\gamma$  as before; we need merely show that  $K_\alpha + K_\beta \epsilon \sim 0$  in  $A + B$ . There are  $(\epsilon, 2)$ -chains  $L_A$  and  $L_B$  such that  $L_A \rightarrow K_\alpha + K_\gamma$  in  $A$  and  $L_B \rightarrow K_\beta + K_\gamma$  in  $B$ ; hence  $L_A + L_B \rightarrow K_\alpha + K_\beta$  in  $A + B$ .

8. The set  $R - \gamma$ . Let  $\gamma$  be any arc in  $R$  crossing  $J$ ; say the end points of  $\gamma$  divide  $J$  into the two arcs  $\alpha$  and  $\beta$ . By condition (2) of the theorem,  $R - \gamma$  is not connected. Let  $A'$  and  $B'$  be those components of  $R - \gamma$  containing  $\langle \alpha \rangle \dagger$  and  $\langle \beta \rangle$  respectively. These are not the same component. For if they were, putting  $A = A' + \gamma, D = R - A'$ , we have  $J \subset A, J \sim 0$  in  $R = A + D$ , and  $A \cdot D = \gamma$ ; hence, by Lemma M,  $J \sim 0$  in  $A$ , a proper subset of  $R$ , contrary to condition (1) of the theorem.

The same reasoning shows that  $R$  has no cut point  $p$ ; we need merely replace  $\gamma$  by  $p$  in Lemma M and above.

Put

$$A = A' + \gamma, B = B' + \gamma.$$

If  $D = R - A'$ , then  $A \cdot D = \gamma$  and  $J = \alpha + \beta \sim 0$  in  $R = A + D$ . Hence, by Lemma N,  $\alpha + \gamma \sim 0$  in  $A$ . Similarly,  $\beta + \gamma \sim 0$  in  $B$ . Consequently, by Lemma O,  $J \sim 0$  in  $A + B$ , from which follows that  $A + B = R$ .

Moreover,  $\alpha + \gamma$  is irreducibly  $\sim 0$  in  $A$ . For if  $\alpha + \gamma \sim 0$  in  $A^*, \alpha + \gamma \subset A^* \subset A$ , then, by Lemma O,  $\alpha + \beta \sim 0$  in  $A^* + B$ ; hence  $A^* + B = R$ , which is only possible if  $A^* = A$ . Similarly,  $\beta + \gamma$  is irreducibly  $\sim 0$  in  $B$ .

$\dagger \langle \alpha \rangle$  is  $\alpha$  except for its end points, etc.

Let us show that  $A$  is a continuous curve. It is connected, as  $A'$  is; it is self-compact, being a closed subset of a compact space.  $A$  is locally connected. For if  $p$  and  $q$  are points of  $A$  close enough together, there is an arc  $pq$  in  $R$  of small diameter; if  $pq$  lies partly in  $B'$ , we can replace that part of it by an arc of  $\gamma$  of small diameter. Lemma A now applies. Similarly,  $B$  is a continuous curve.

9. We shall now show that any arc  $\delta$  crossing  $J_A = \alpha + \gamma$  in  $A$  divides  $A$ . The following two lemmas will be useful.

**LEMMA P.** *If  $\eta_1$  and  $\eta_2$  are arcs contained within the arcs  $\gamma$  and  $\beta$  respectively, then there is an arc  $pq$  crossing  $J_B = \beta + \gamma$  in  $B$ , with  $p \subset \eta_1$ ,  $q \subset \eta_2$ .*

This is an immediate consequence of Lemma L, if we take, for the closed sets of that lemma, the closed intervals of  $J_B$  complementary to  $\eta_1$  and  $\eta_2$ .

**LEMMA Q.** *There are no two arcs  $ab$  and  $cd$  in  $R$  without common points, each crossing  $J$ , whose end points are in the order  $acbd$  on  $J$ .*

This follows directly from what we have seen above.

To show that  $\delta$  divides  $A$ , we must consider four cases.

**Case 1.** Both end points of  $\delta$  lie on  $\alpha$ . Suppose  $A - \delta$  is connected; then it is arcwise connected, by Lemma E. Hence there is an arc in  $A - \delta$  joining a point  $p$  of  $\alpha$  lying between the two end points of  $\delta$  and a point  $q$  within  $\gamma$ . If  $\eta_1$  is an arc within  $\gamma$  containing  $q$ , there is an arc  $rs$  in  $B$  joining  $\eta_1$  to a point  $s$  within  $\beta$ , with only its end points  $r$  and  $s$  on  $J_B$ , by Lemma P. The arc  $pqrs$  crosses  $J$  and does not touch  $\delta$ . But the end points of this arc alternate with those of  $\delta$  on  $J$ , contradicting Lemma Q.

**Case 2.**  $\delta$  is an arc  $cd$ , where  $c$  lies within  $\alpha$ ,  $d$  lies within  $\gamma$ . If  $A - \delta$  is connected, let  $pq$  be an arc in this set joining points of  $\alpha$  on opposite sides of  $c$ . If  $\eta_1$  is an arc of  $\gamma$  containing  $d$  but not touching  $pq$ , let the arc  $rs$  join  $\eta_1$  to  $\beta$  in  $B$ ; then the arcs  $pq$  and  $cdrs$  contradict Lemma Q.

**Case 3.** The end points  $c$  and  $d$  of  $\delta$  lie within  $\gamma = ab$ , say in the order  $acdb$ . If  $A - \delta$  is connected, let  $pq$  be an arc in this set joining a point  $p$  within  $\alpha$  to a point  $q$  in  $\gamma$  between  $c$  and  $d$ . If  $\eta_1$  is an arc of  $\gamma$  containing  $q$  but not touching  $\delta$ , let  $r_1s_1$  be an arc in  $B$  joining  $\eta_1$  to a point  $s_1$  within  $\beta$ .

The arcs  $acr_1$  of  $\gamma$  and  $r_1s_1$  form an arc  $acr_1s_1$  crossing  $J$ ; hence

$$R - acr_1s_1 = C_1 + C_2,$$

where  $C_1$  contains the open arc  $\langle as_1 \rangle$  of  $\beta$ , and  $C_2$  contains  $b$  and points connected with  $b$ . As  $r_1s_1$  lies in  $B$ ,  $A' \subset C_1 + C_2$ ; the connected set  $A' + b$  lies thus in  $C_2$ . If  $\eta_2$  is an arc of  $\gamma$  containing  $c$  but not touching  $\eta_1$ , and  $r_2s_2$  is an arc in  $C_1$  joining  $\eta_2$  to a point  $s_2$  of  $\beta$  between  $a$  and  $s_1$ , then  $\eta_2 + r_2s_2$  does not touch  $pqr_1s_1$ , and has only the point  $c$  in common with  $\delta$ .

Similarly, if  $\eta_3$  is an arc of  $\gamma$  containing  $d$  but not touching  $\eta_1$ , there is an arc  $r_3s_3$  in  $R - bdr_1s_1$  such that  $r_3$  lies in  $\eta_3$ ,  $s_3$  lies in  $\beta$  between  $s_1$  and  $b$ , and  $\eta_3 + r_3s_3$  does not touch  $pqr_1s_1$  and has only the point  $d$  in common with  $\delta$ . The arc  $r_3s_3$  does not touch  $r_2s_2$ , as it lies in  $C_2$ . Thus the two arcs  $pqr_1s_1$  and  $s_2r_2cdr_3s_3$  ( $cd = \delta$ ) contradict Lemma Q.

**Case 4.** The same as Case 3, except that  $c = a$  or  $d = b$ , say the latter. Then, in the notation of Case 3, the arcs  $pqr_1s_1$  and  $s_2r_2cb$  ( $cb = \delta$ ) contradict Lemma Q.

This completes the proof that  $A$  and  $J_A$  ( $B$  and  $J_B$ ) satisfy the conditions of Theorem I.

10. **The cutting up of  $R$ .** We are concerned with the following lemma.

**LEMMA R.**  *$R$  may be cut into a finite number of pieces of arbitrarily small diameter.*

Given an  $\epsilon > 0$ , choose  $\delta < \epsilon$  so as to satisfy the requirement in Lemma F. Suppose  $R$  is cut up so that the diameter of the boundary of each piece is  $< \delta$ . Then each piece is of diameter  $< 3\epsilon$ . For otherwise there is a point  $q$  of some piece  $R_i$  at a distance  $\geq \epsilon$  from its boundary  $J_i$ . Let  $p$  be a point of  $J_i$ , and  $q'$ , a point of  $R - R_i$  at a distance  $\geq \epsilon$  from  $p$ . Every arc from  $q$  to  $q'$  must cut the boundary  $J_i$  of  $R_i$  and thus must pass within  $\delta$  of  $p$ , contradicting Lemma F.

The lemma thus follows from

**LEMMA S.** *Given a  $\delta > 0$ ,  $R$  can be cut up so that the diameter of the boundary of each piece is  $< \delta$ .*

Express  $R$  as the union of a finite number of continua:

$$R = K_1 + K_2 + \cdots + K_m, \delta(K_i) < \delta/2.$$

We shall cut up  $R$  in such a manner that no two of these continua  $K_i$  and  $K_j$  have points on the boundary of the same piece of  $R$ , if  $K_i \cdot K_j = 0$ ; the lemma will then follow.

Suppose we have cut  $R$  up a certain amount (perhaps not yet at all), into the pieces  $R_1, R_2, \dots, R_n$ , with boundaries  $J_1, J_2, \dots, J_n$  (we may have  $R$  and  $J$  alone). Of course *each boundary  $J_i$  separates  $R_i$  from the rest of  $R$* . Take any two continua, say  $K_1$  and  $K_2$ , with  $K_1 \cdot K_2 = 0$ , each of which has points on one of these  $J_i$ , say  $J_1$ . We shall cut  $R$  up further so that in the new pieces there is no one (i.e. no *piece*, not merely no *boundary* of a piece) which has any points in common with both  $K_1$  and  $K_2$ ; then on any further cutting up of  $R$ , this will still be true.

Divide the points of  $J_1$  into three sets, as follows. We put a point  $x$  into the first set if it lies in  $K_1$ , or if following  $J_1$  in both directions we reach points

of  $K_1$  before reaching points of  $K_2$ ; we put  $x$  into the second set if the same conditions hold with  $K_1$  and  $K_2$  interchanged; all other points we put into the third set. This set  $L'_3$  consists of open intervals of  $J_1$ , each being bounded by a point of  $K_1$  on one end and a point of  $K_2$  on the other. The points of the first set together with the points  $K_1 \cdot R_1$  form a closed set  $L_1$ , and those of the second set together with  $K_2 \cdot R_1$  form a closed set  $L_2$ . Then  $\rho(L_1, L_2) > 0$  as  $L_1 \cdot L_2 = 0$ , from which follows that there are but a finite number of intervals in  $L'_3$ . As  $K_1$  is connected, each component of  $L_1$  has points on  $J_1$ , and thus on one of the intervals  $L_3$  of  $J_1$  complementary to the intervals of  $L'_3$ . Thus there are a finite number of components  $L_{11}, L_{12}, \dots, L_{1m_1}$  in  $L_1$ . Similarly there are a finite number of components  $L_{21}, L_{22}, \dots, L_{2m_2}$  in  $L_2$ .

We shall now cut  $R_1$  into a number of pieces, in each of which either  $K_1$  has no points or  $K_2$  has no points. Suppose  $L'_{31}, \dots, L'_{3m'_3}$  and  $L_{31}, \dots, L_{3m_3}$  are the intervals of  $L'_3$  and  $L_3$  respectively, and say they lie in the order  $L_{31}, L'_{31}, L_{32}, L'_{32}, \dots, L_{3m_3}, L'_{3m'_3}$  on  $J_1$ . If we go around  $J_1$ , the intervals of  $L_3$  lie alternately in  $L_1$  and  $L_2$ . Starting at  $L_{31}$ , which lies in  $L_{11}$  say, go around  $J_1$  till we reach another interval  $L_{3k}$  in  $L_{11}$  (we may have gotten back to  $L_{31}$ ). Put  $L_{32}, L_{3, k-1}$  and all of  $J_1$  between these into a set  $M'_2$  (which may be  $L_{32}$  alone), and put  $L_{3k}, L_{31}$ , and all of  $J_1$  between these on the other side from  $L_{32}$  into a set  $M'_1$  (which may be  $L_{31}$  alone).  $L'_{31}$  and  $L'_{3, k-1}$  are the two intervals of  $J_1$  complementary to  $M'_1$  and  $M'_2$ .

No set  $L_{1i}$  or  $L_{2j}$  has points in both  $M'_1$  and  $M'_2$ . This follows for  $L_{11}$  by construction. If it were false for some other set, say  $L_{1s}$ , then  $L_{1s}$  would have points on two intervals  $L_{3p}$  and  $L_{3q}$  separated by  $L_{31}$  and  $L_{3k}$  on  $J_1$ . Now  $L_{11} \cdot L_{1s} = 0$ , hence  $\rho(L_{11}, L_{1s}) > 0$ . As  $R_1$  is a continuous curve, there are continuous curves  $L_{11}^*$  and  $L_{1s}^*$  in  $R_1$  containing  $L_{11}$  and  $L_{1s}$  and such that  $L_{11}^* \cdot L_{1s}^* = 0$  (see Lemma C). These sets are arcwise connected, and we can draw arcs contradicting Lemma Q.

Let  $M_1$  be  $M'_1$  plus all components  $L_{1i}$  and  $L_{2j}$  containing points of  $M'_1$ , and define  $M_2$  similarly. Then  $M_1$  and  $M_2$  are closed,  $M_1 \cdot M_2 = 0$ , and  $M_1 + M_2 \supset L_1 + L_2$ . By Lemma L we can draw an arc  $\gamma_1$  from  $L'_{31}$  to  $L'_{3, k-1}$  which has no points in  $M_1$  or in  $M_2$ .  $R_1$  is thus cut into two pieces, in each of which there is at least one component  $L_{1i}$  or  $L_{2j}$ ; for one contains  $L_{11}$ , and the other contains that  $L_{2j}$  containing  $L_{32}$ . Thus in each piece there are less than  $m_1 + m_2$  components, the number in  $R_1$ .

If one of the resulting pieces contains more than one component, we cut it up, etc. Finally each new piece of  $R_1$  has points of only one component, and thus  $K_1$  and  $K_2$  are separated in  $R_1$ . We now separate  $K_1$  and  $K_2$  in each other piece  $R_i$  of  $R$  also. This is possible, for if  $K_i$  ( $i = 1, 2$ ) has points in any  $R_k$ , it also has points on  $J_k$ .

If now there are any other two of the continua  $K_i$  and  $K_j$ ,  $K_i \cdot K_j = 0$ , each of which has points on some new  $J_k$ , we cut  $R$  further till this is no longer true, etc. This completes the proof.

11. The homeomorphism. Cut  $R$  into pieces of diameter  $<$  some  $\sigma$ . We make corresponding cuts in  $R'$  as follows. The first arc  $\gamma$  drawn in  $R$  cuts  $R$  into the two pieces  $R_1$  and  $R_2$  with boundaries  $J_1$  and  $J_2$  say. Draw any arc  $\gamma'$  crossing  $J'$  in  $R'$ , cutting  $R'$  into the pieces  $R'_1$  and  $R'_2$  with boundaries  $J'_1$  and  $J'_2$ . We note that  $J'_1 + J'_2$  is homeomorphic with  $J_1 + J_2$ , with  $J'_k$  corresponding to  $J_k$ ,  $k = 1, 2$ . Say  $\gamma_1$  is an arc in  $R_1$ , cutting  $R_1$  into pieces  $R_{11}$  and  $R_{12}$  with boundaries  $J_{11}$  and  $J_{12}$ . If  $a_1$  and  $b_1$  are the end points of  $\gamma_1$ , let  $a_1^*$  and  $a_2^*$  be the corresponding points of  $J'_1$  in the above homeomorphism. Draw an arc  $\gamma'_1$  crossing  $J'_1$  in  $R'_1$ , with end points  $a'_1$  and  $b'_1$  close to  $a_1^*$  and  $b_1^*$  respectively (Lemma P);  $R'_1$  is divided thereby into the pieces  $R'_{11}$  and  $R'_{12}$  with boundaries  $J'_{11}$  and  $J'_{12}$ . Moreover,  $J'_{11} + J'_{12} + J'_2$  is homeomorphic with  $J_{11} + J_{12} + J_2$ , with boundaries with the same subscripts corresponding.

In general, suppose  $R_{i_1 i_2 \dots i_m}$  is a piece that is present after  $R$  is cut a certain amount, and say the arc  $\gamma_{i_1 \dots i_m}$  divides this set into the pieces  $R_{i_1 \dots i_{m-1}}$  and  $R_{i_1 \dots i_{m+2}}$ , with boundaries  $J_{i_1 \dots i_{m-1}}$  and  $J_{i_1 \dots i_{m+2}}$ . If  $a_{i_1 \dots i_m}$  and  $b_{i_1 \dots i_m}$  are the end points of  $\gamma_{i_1 \dots i_m}$ , let  $a_{i_1 \dots i_m}^*$  and  $b_{i_1 \dots i_m}^*$  be the corresponding points on  $J'_{i_1 \dots i_m}$  in the homeomorphism we have already. Draw an arc  $\gamma'_{i_1 \dots i_m}$  crossing  $J'_{i_1 \dots i_m}$ , with end points  $a'_{i_1 \dots i_m}$  and  $b'_{i_1 \dots i_m}$  close to the above points, dividing  $R'_{i_1 \dots i_m}$  into the pieces  $R'_{i_1 \dots i_{m-1}}$  and  $R'_{i_1 \dots i_{m+2}}$ , with boundaries  $J'_{i_1 \dots i_{m-1}}$  and  $J'_{i_1 \dots i_{m+2}}$ . The set of boundaries with primes is now homeomorphic with the set of boundaries without primes, boundaries with the same subscripts corresponding. We note that *if  $R_{i_1 \dots i_m}$  and  $R_{j_1 \dots j_m}$  have common points, then  $R'_{i_1 \dots i_m}$  and  $R'_{j_1 \dots j_m}$  have common points, and conversely.*

Having cut  $R$  into pieces of diameter  $< \sigma$  and having cut  $R'$  in a corresponding fashion, we now cut each piece of  $R'$  into pieces of diameter  $< \sigma/2$  and cut each piece of  $R$  in a corresponding fashion. Next we cut each resulting piece of  $R$  into pieces of diameter  $< \sigma/4$ , etc. Now for any  $\epsilon > 0$  there is an  $m$  such that

$$\delta(R_{i_1 \dots i_m}) < \epsilon, \quad \delta(R'_{i_1 \dots i_m}) < \epsilon,$$

for any  $m$ -fold subscript.

We now establish the homeomorphism between  $R$  and  $R'$ . Let  $p$  be any point of  $R$ . It lies in either  $R_1$  or  $R_2$  (perhaps in both), say in  $R_{i_1}$ . Then it lies in either  $R_{i_1 1}$  or  $R_{i_1 2}$  (perhaps in both), say in  $R_{i_1 i_2}$ , etc. Thus we have a sequence of pieces

$$R \supset R_{i_1} \supset R_{i_1 i_2} \supset \cdots \supset p.$$

The corresponding pieces in  $R'$  have a single limit point:

$$R' \supset R'_{i_1} \supset R'_{i_1 i_2} \supset \cdots \supset p'.$$

This point  $p'$  we let correspond to  $p$ .

If there are different sequences of pieces in  $R$  containing  $p$ , we have different sequences in  $R'$  defining points  $p'$ . However, all these points  $p'$  are the same. For if  $R, R_{i_1}, R_{i_1 i_2}, \cdots$ , and  $R, R_{j_1}, R_{j_1 j_2}, \cdots$ , are two sequences containing  $p$ , then each piece  $R_{i_1 \dots i_m}$  has points in common with  $R_{j_1 \dots j_m}$ , namely, the point  $p$ ; hence, as we saw above,  $R_{i_1'} \dots i_m$  and  $R_{j_1'} \dots j_m$  have common points. Thus the corresponding sequences in  $R'$  close down on a single point. Similarly, to each point  $p'$  in  $R'$  corresponds a single point  $p$  in  $R$ .

Finally, the correspondence is continuous. For take a point  $p$  in  $R$  and an  $\epsilon > 0$ . Let  $p'$  be the corresponding point in  $R'$ , and choose an  $m$  so that  $\delta(R_{i_1'} \dots i_m) < \epsilon$  for all  $m$ -fold subscripts. Consider all the  $R_{i_1 \dots i_m}$  with  $m$ -fold subscripts which contain  $p$ ; these include all points of  $R$  in some  $V_\delta(p)$ . Then if  $q \in V_\delta(p)$ , the corresponding point  $q'$  is in  $V_\epsilon(p')$ , and the continuity is established. This completes the proof of Theorem I.

12. **Proof of Theorem II.** Let  $I$  be a circle in the plane, and let  $S$  be  $I$  plus its interior.  $S$  is self-compact, connected and locally connected, and is thus a continuous curve. That  $I \sim 0$  in  $S$  follows from Lemma K.†

To show that  $I$  is irreducibly  $\sim 0$  in  $S$ , suppose that  $I \sim 0$  in  $S'$ , a proper closed subset of  $S$ ; we can suppose that  $S'$  is a continuous curve. Let  $p$  be a point of  $S$  not in  $S'$ , and let  $V_\delta(p)$  have no points in  $S'$ . Let  $ab$  be a segment of a straight line passing through  $p$  with its ends on  $I$ . Let  $a_1 b_1$  and  $a_2 b_2$  be parallel segments enclosing  $ab$ , and lying at a distance  $\delta$  from  $ab$ . Then in that portion of  $S'$  between  $a_1 b_1$  and  $a_2 b_2$ , the (short) arcs  $a_1 a_2$  and  $b_1 b_2$  are not connected. But if  $C$  and  $D$  are those parts of  $S'$  outside  $a_1 b_1$  and  $a_2 b_2$ , by Lemma L we can draw an arc joining  $a_1 a_2$  to  $b_1 b_2$  in  $S' - (C + D)$ , a contradiction.

Finally, that an arc crossing  $I$  in  $S$  divides  $S$  is a special (and easily proved) case of the Jordan theorem. This completes the proof.

13. **The Jordan theorem.** Let  $J$  be a simple closed curve in the plane. Let  $I$  be a circle containing  $J$  in its interior. Draw two non-intersecting line segments from  $I$  to  $J$ .  $S = I$  plus its interior is thus cut into three closed 2-cells, one of which, say  $R$ , has the boundary  $J$ . Then  $R - J$  is the inside of  $J$ . The points of  $J$  are obviously accessible from either side.

† For  $S$  is a closed 2-cell.