SUBHARMONIC FUNCTIONS AND MINIMAL SURFACES*

BY

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INTRODUCTION

0.1. Let $f(w)$, given by

$$f(w) = x(u, v) + iy(u, v), \; w = u + iv, \; w \text{ in } D,$$

where D is some domain of definition, be an analytic function of the complex variable w. Then $x(u, v)$, $y(u, v)$ satisfy the Cauchy-Riemann differential equations

$$x_u = y_v, \quad x_v = -y_u, \tag{1}$$

the subscripts denoting differentiation. These equations (1) are not symmetric in x, y, but they imply the symmetric set

$$x_u^2 + y_u^2 = x_v^2 + y_v^2, \; x_u x_v + y_u y_v = 0. \tag{2}$$

Conversely, (2) implies either (1) or

$$y_u = x_v, \; y_v = -x_u. \tag{3}$$

From either (1), (2) or (3) it follows that $x(u, v)$, $y(u, v)$ are harmonic functions:

$$x_{uu} + x_{vv} = 0, \; y_{uu} + y_{vv} = 0.$$

If (1) holds, $y(u, v)$ is said to be the conjugate harmonic function of $x(u, v)$, or if (3) holds, $x(u, v)$ is said to be the conjugate harmonic function of $y(u, v)$; generally, if (2) holds then $x(u, v)$, $y(u, v)$ will be called a *couple of conjugate harmonic functions.*

0.2. Generalizing this situation to the case of three functions $x(u, v)$, $y(u, v)$, $z(u, v)$, (u, v) in D, we shall call $x(u, v)$, $y(u, v)$, $z(u, v)$ a *triple of conjugate harmonic functions* provided the following conditions are satisfied:

$$E = G, \quad F = 0, \tag{i}$$

where

$$E = x_u^2 + y_u^2 + z_u^2, \; F = x_u x_v + y_u y_v + z_u z_v, \; G = x_v^2 + y_v^2 + z_v^2;$$

$$x(u, v), \; y(u, v), \; z(u, v) \tag{ii}$$

are harmonic.

* Presented to the Society, December 29, 1932; received by the editors January 23, 1933.

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It might be noted that if one of the coordinate functions vanishes identically, say $z \equiv 0$, then (ii) is implied by (i); but in general this implication does not hold.

0.3. While this generalization no doubt would be of interest from a purely analytic viewpoint, the following *theorem of Weierstrass* shows that it actually is very important geometrically: A necessary and sufficient condition that a surface given in terms of isothermic parameters (that is, parameters u, v such that $E=G$, $F=0$) be minimal is that the coordinate functions be harmonic.

Thus the theory of minimal surfaces appears as the theory of triples of conjugate harmonic functions, while the theory of couples of conjugate harmonic functions is the theory of analytic functions of a complex variable. As a matter of fact, theorems and methods in theory of functions always have served as tools and models in the theory of minimal surfaces.

0.4. The purpose of the present paper is the development of this analogy in the direction of *the principle of the maximum.* If $f(w)$ is an analytic function in a region R, then $|f(w)|$ takes on its maximum on the boundary of R. Similarly, if $x(u, v)$, $y(u, v)$, $z(u, v)$ form a triple of conjugate harmonic functions in R, then $(x^2+y^2+z^2)^{1/2}$ takes on its maximum on the boundary of R; this is easily shown to be true even if the three harmonic functions are not conjugate. However, the effectiveness of the principle of the maximum in the case of analytic functions depends essentially upon the fact that certain operations (multiplication for instance), if performed on analytic functions, yield analytic functions again. This situation does not seem to admit of any direct generalization to minimal surfaces. It is our purpose to show that despite this lack of direct analogy many important applications of the principle of the maximum can be generalized to minimal surfaces. Our tool is the following simple lemma (see §2):

Three functions $x(u, v)$, $y(u, v)$, $z(u, v)$, continuous in a domain, form there a triple of conjugate harmonic functions if and only if $\log[(x+a)^2+(y+b)^2+(z+c)^2]^{1/2}$ *is subharmonic for every choice of the real constants a, b, c.*

This lemma permits us to apply the theory of subharmonic functions,* so important in theory of functions, to the theory of minimal surfaces. For the convenience of the reader, we give in §1 the necessary definitions and facts concerning subharmonic functions.

* See F. Riesz, *Sur les fonctions subharmoniques et leur rapport à la théorie du potentiel* (in two parts), Acta Mathematica, vol. 48 (1926), pp. 329–343, and vol. 54 (1930), pp. 321–360; P. Montel, *Sur les fonctions convexes et les fonctions sousharmoniques*, Journal de Mathématiques, (9), vol. 7 (1928), pp. 29–60; S. Saks, *Sur une inégalité de la théorie des fonctions*, Acta Szeged, vol. 4 (1928), pp. 51–55, and *On subharmonic functions*, Acta Szeged, vol. 5 (1932), pp. 187–193.

1. Subharmonic functions and functions of class PL

1.1. In this section we present the definition of subharmonic functions and give those results concerning these functions which we shall need in the sequel.

Let $g(u, v)$ be a continuous function of two variables, defined in a domain D (connected open set). Suppose that for each point (u_0, v_0) of D we have

$$g(u_0, v_0) \leqq \frac{1}{2\pi}\int_0^{2\pi} g(u_0 + \rho\cos\phi, v_0 + \rho\sin\phi)d\phi \tag{4}$$

for each sufficiently small value of the radius ρ. Then the function $g(u, v)$ is said to be *subharmonic* in D.*

The definition can be extended to the case of discontinuous functions, but we shall be concerned in this paper only with continuous subharmonic functions.

1.2. It follows immediately from the definition that a subharmonic function $g(u, v)$ cannot attain its maximum value at any (interior) point of D, unless $g(u, v)$ is identically constant.†

1.3. If a function $g(u, v)$ has continuous partial derivatives of the second order, then a necessary and sufficient condition that $g(u, v)$ be subharmonic is that its Laplacian be $\geqq 0$:

$$\Delta g \equiv g_{uu} + g_{vv} \geqq 0.\ ‡$$

1.4. Let $g(u, v)$ be subharmonic in the ring

$$r_1 < [(u - u_0)^2 + (v - v_0)^2]^{1/2} < r_2,$$

and let $M(r)$ denote the maximum of $g(u, v)$ on

$$(u - u_0)^2 + (v - v_0)^2 = r^2,\ r_1 < r < r_2.$$

Then $M(r)$ is a convex function of $\log r$.§

1.5. Obviously, if $g(u, v)$ and $h(u, v)$ are both subharmonic in D, then $g(u, v) + h(u, v)$ also is subharmonic there.

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* This definition is due to F. Riesz. See Acta Mathematica, loc. cit., first part, p. 331.

† See F. Riesz, Acta Mathematica, loc. cit., first part, p. 331.

‡ See F. Riesz, Acta Mathematica, loc. cit., first part, p. 335.

§ See P. Montel, Journal de Mathématiques, loc. cit., where this fact and similar elementary facts concerning subharmonic functions are presented in a systematic way.

1.6. A function $p(u, v)$, defined in a domain D, will be said to be *of class* PL in D provided the following conditions are satisfied there.

(i) $p(u, v)$ is continuous.

(ii) $$p(u, v) \geqq 0.$$

(iii) $\log p(u, v)$ is subharmonic in the part of D where $p(u, v) > 0$.

1.7. If $p(u, v)$ is of class PL, then $p(u, v)$ is subharmonic. Indeed, at points where $p(u, v) = 0$ the condition (4) of Riesz obviously is satisfied; and elsewhere the fact that $\log p(u, v)$ is subharmonic implies that $p(u, v)$ is subharmonic.*

1.8. Obviously (see §1.5), the product of a finite number of functions of class PL, or any positive power of a function of this class, is again a function of class PL.

1.9. The class PL is invariant under conformal mapping. (The same remark applies to the class of subharmonic functions.) That is, if $p(u, v)$ is of class PL in D and if D is mapped conformally on a (U, V) domain $\overline{D}$, then $p(u, v)$ is transformed into a function $q(U, V)$ which is of class PL in $\overline{D}$.

1.10. A necessary and sufficient condition that a non-negative function $p(u, v)$ be of class PL is that $e^{\alpha u+\beta v}\, p(u, v)$ be subharmonic for every choice of the real constants α, β.† It follows from this (see §1.5) that the sum of a finite number of functions of class PL is again a function of class PL.

1.11. The classical example of a function of class PL is the absolute value of an analytic function $f(w)$ of $w = u + iv$. If $f(w)$ is different from zero in a domain, then $\log |f(w)|$ is harmonic there. Thus $|f(w)|$ is just barely of class PL. As a consequence, a great number of theorems concerned with $|f(w)|$ are a fortiori true for functions of class PL. We now shall state some of these generalized theorems which will be used in the sequel. The proofs run exactly in the same way as for $|f(w)|$; for this reason we shall sketch just a few of the proofs, and otherwise shall give references to typical proofs concerning $|f(w)|$.

1.12. Let $p(u, v)$ be bounded and of class PL in $u^2 + v^2 < 1$. Suppose $p(u, v)$ remains continuous on a certain arc σ of $u^2 + v^2 = 1$, and vanishes there. Then $p(u, v) \equiv 0$.

Proof.‡ Choose the integer n so large that $2\pi/n$ is less than the length of

* See P. Montel, Journal de Mathématiques, loc. cit., p. 39.

† This criterion is due to Montel, Journal de Mathématiques, loc. cit., p. 40, who proved it under the assumption that $p(u, v)$ has continuous partial derivatives of the first and second order. For the case of a merely continuous $p(u, v)$, the theorem has been proved by T. Radó, *Remarque sur les fonctions subharmoniques*, Paris Comptes Rendus, vol. 186 (1928), pp. 346–348.

‡ Cf. Pólya und Szegö, *Aufgaben und Lehrsätze aus der Analysis*, Berlin, J. Springer, 1925, vol. I, p. 139, problem 279.

the arc σ. If we rotate the unit circle about its center through an angle of $2\pi/n$, $p(u, v)$ is transformed into a new function $p_1(u, v)$ of class PL (see §1.9). Let $p_2(u, v), \cdots, p_{n-1}(u, v)$ be the functions of class PL resulting from further successive rotations of the unit circle through the angle $2\pi/n$. Then $\psi(u, v) = pp_1 \cdots p_{n-1}$ is again of class PL (see §1.8), and $\psi(u, v) \to 0$ if (u, v) converges to any point of $u^2+v^2=1$. Since $\psi \geqq 0$, it follows from this (see §1.2) that $\psi(u, v) \equiv 0$. In particular, $\psi(0, 0) = p(0, 0)^n = 0$, that is to say, $p(u, v)$ vanishes at the origin. As any point of $u^2+v^2<1$ can be thrown, by conformal mapping of the unit circle upon itself, into the origin, it follows that $p(u, v) \equiv 0$.

1.13. Let $p(u, v)$ be bounded and of class PL in $u^2+v^2<1$. Suppose $p(u, v)$ vanishes in a subdomain k of $u^2+v^2<1$. Then $p(u, v) \equiv 0$.

Proof. Consider any fixed point (u_0, v_0) of k. Then given any point (u_1, v_1) in $u^2+v^2<1$ but not in k, there exists a circle passing through (u_0, v_0), tangent to $u^2+v^2=1$ from within, and containing (u_1, v_1) in its interior. The theorem of §1.12 applies to this circle.

1.14. Let $p(u, v)$ be bounded and of class PL in the angle $0 < \text{arc tg } (v/u) < \alpha$. Let $p(u, v)$ remain continuous on the ray $u>0$, $v=0$, and let $p(u, 0) \to 0$ as $u \to +0$. Then in every angle $0 < \text{arc tg } (v/u) < \alpha - \sigma$, where $\sigma > 0$, we have $p(u, v) \to 0$ as $(u, v) \to (0, 0)$ in any manner.*

Of course this theorem is true if the domain of definition is only the sector $0 < \text{arc tg } (v/u) < \alpha$, $0 < u^2+v^2 < r_0^2$; the proof is the same in either case.

1.15. Let $p(u, v)$ be bounded and of class PL in $u^2+v^2<1$. Let (u', v'), (u'', v'') be two distinct points on $u^2+v^2=1$. Let (u_n', v_n'), (u_n'', v_n'') be two sequences in $u^2+v^2<1$, converging to (u', v'), (u'', v'') respectively, and let C_n be a continuous arc, joining (u_n', v_n') and (u_n'', v_n''), and comprised in the ring $1-\epsilon_n < (u^2+v^2)^{1/2} < 1$, where $\epsilon_n > 0$, and $\epsilon_n \to 0$. Denote by η_n the maximum of $p(u, v)$ on C_n and suppose that $\eta_n \to 0$. Then $p(u, v) \equiv 0$.†

1.16. Let $p(u, v)$ be $\leqq 1$ and of class PL in $r^2 = u^2+v^2 < 1$. Let $p(0, 0) = 0$ and suppose that for a certain $\alpha > 0$, $p(u, v)/r^\alpha$ remains bounded in $0 < r < 1$. Then $p(u, v) \leqq r^\alpha$. If the equality holds for any (u, v), $0 < u^2+v^2 < 1$, then it holds identically.‡

Proof. Let $M(r)$ denote the maximum of $p(u, v)$ on $u^2+v^2=r^2$. Then

* This generalizes a theorem of Lindelöf. Cf. Pólya und Szegö, loc. cit., p. 138, problem 277. The proof, given there for the special case when $p(u, v)$ is the absolute value of an analytic function, applies without the change of a word to the general case considered above.

† Cf. L. Bieberbach, *Lehrbuch der Funktionentheorie*, Berlin, B. G. Teubner, 1927, vol. II, pp. 19–21.

‡ This generalizes the Lemma of Schwarz. See C. Carathéodory, *Conformal Representation*, London, Cambridge University Press, 1932, p. 39. The example $p(u, v) = (u^2+v^2)^{1/4}$ shows that the value $\alpha = 1$ which holds for the Lemma of Schwarz does not hold in the general case.

$M(r)/r^\alpha$ is the maximum of $p(u, v)/r^\alpha$ on $u^2+v^2=r^2$. Since $\log r^\alpha$ is harmonic, $p(u, v)/r^\alpha$ is of class PL in $0<u^2+v^2<1$. Therefore (see §1.4), $M(r)/r^\alpha$ is a convex function of $\log r$, $-\infty<\log r<0$. If such a function is bounded from above, then it is a non-decreasing function. Consequently, from

$$\lim_{r\to 1} M(r)/r^\alpha \leqq 1$$

it follows that $p(u, v)/r^\alpha \leqq 1$, $0<r<1$. If the equality holds for any (u, v), $0<u^2+v^2<1$, then (see §1.2) it holds identically.

2. A characterization of minimal surfaces

2.1. If $x(u, v)$, $y(u, v)$, $z(u, v)$ form a triple of conjugate harmonic functions (see §0.2) in a domain D, then we shall say that the equations

$$x = x(u, v),\ y = y(u, v),\ z = z(u, v),\ (u, v) \text{ in } D, \tag{5}$$

give a *minimal surface in typical representation.* In this statement, the term *minimal surface* is used in a more general sense than is customary in differential geometry, where the condition $EG-F^2>0$ is always required. In §4.1, we shall use the term minimal surface in an (apparently) even more general sense.

If the equations (5) give a minimal surface $\mathfrak{M}$ in typical representation, then the function $(x^2+y^2+z^2)^{1/2}$ will be called the *norm* of $\mathfrak{M}$ and will be denoted by $|\mathfrak{M}|$ or $|\mathfrak{M}(u, v)|$ or $|\mathfrak{M}(w)|$, where $w=u+iv$.

2.2. *Let*

$$\mathfrak{M}:\quad x = x(u, v),\ y = y(u, v),\ z = z(u, v),\ (u, v) \textit{ in } D, \tag{6}$$

be a minimal surface given in typical representation. Then

$$|\mathfrak{M}| \equiv (x^2 + y^2 + z^2)^{1/2} \tag{7}$$

is of class PL.

It is sufficient to consider points where $|\mathfrak{M}|\neq 0$ (see §1.6). At such points the Laplacian of $\log|\mathfrak{M}|$ is given by $\Delta \log|\mathfrak{M}| = T/|\mathfrak{M}|^4$, with

$$T = (\mathfrak{x}_u^2 + \mathfrak{x}_v^2)\mathfrak{x}^2 - 2[(\mathfrak{x}\mathfrak{x}_u)^2 + (\mathfrak{x}\mathfrak{x}_v)^2],$$

where $\mathfrak{x}$, $\mathfrak{x}_u$, $\mathfrak{x}_v$ denote vectors, namely

$$\mathfrak{x} = (x, y, z),\ \mathfrak{x}_u = (x_u, y_u, z_u),\ \mathfrak{x}_v = (x_v, y_v, z_v),$$

and where the vector products indicated are scalar. The parameters being isothermic, we have

$$\mathfrak{x}_u^2 = \mathfrak{x}_v^2 = \lambda,\ \mathfrak{x}_u\mathfrak{x}_v = 0.$$

Since the partial derivatives of the second order of $\log|\mathfrak{M}|$ are continuous

where $|\mathfrak{M}| \neq 0$, we have only to show (see §1.3) that

$$T \geqq 0. \tag{8}$$

Fix (u_0, v_0); then two cases are possible; either $\lambda = 0$ or $\lambda > 0$. If $\lambda = 0$, then $\mathfrak{x}_u = \mathfrak{x}_v = 0$ and (8) is trivial. If $\lambda > 0$, then the vectors $\mathfrak{x}_u$, $\mathfrak{x}_v$ are both $\neq 0$ and are perpendicular to each other; let ξ denote the unit vector perpendicular to each of them. Then we can write

$$\mathfrak{x} = a\mathfrak{x}_u + b\mathfrak{x}_v + c\xi,$$

where a, b, c are scalars. Therefore

$$\mathfrak{x}^2 = a^2\lambda + b^2\lambda + c^2,$$
$$\mathfrak{x}\mathfrak{x}_u = a\lambda, \ \mathfrak{x}\mathfrak{x}_v = b\lambda,$$
$$T = 2\lambda(a^2\lambda + b^2\lambda + c^2) - 2(a^2\lambda^2 + b^2\lambda^2) = 2\lambda c^2 \geqq 0.$$

2.3. The fact that (7) is of class PL certainly does not characterize minimal surfaces.* However, (6) is still a minimal surface given in typical representation if we shift the xyz-axes. Therefore, for the functions $x(u, v)$, $y(u, v)$, $z(u, v)$ in (6),

$$[(x + a)^2 + (y + b)^2 + (z + c)^2]^{1/2}$$

is of class PL for arbitrary choice of the real constants a, b, c. And, as we now shall show, the converse also is true, so that we have the following

LEMMA. *A necessary and sufficient condition that the continuous functions $x(u, v)$, $y(u, v)$, $z(u, v)$ represent a minimal surface given in typical representation is that $[(x+a)^2+(y+b)^2+(z+c)^2]^{1/2}$ be of class PL for arbitrary choice of the real constants a, b, c.*

2.4. The necessity has been proved above. To prove the sufficiency, observe first that if $(x^2+y^2+z^2)^{1/2}$ is of class PL, then $x^2+y^2+z^2$ also is of class PL (see §1.8). Let then (u_0, v_0) be any fixed point of D, and put $x(u_0, v_0) = x_0$, $y(u_0, v_0) = y_0$, $z(u_0, v_0) = z_0$. Then if C denotes a sufficiently small circle with center at (u_0, v_0) we have

$$(x_0 + a)^2 + (y_0 + b)^2 + (z_0 + c)^2 \leqq \frac{1}{2\pi}\int_C [(x + a)^2 + (y + b)^2 + (z + c)^2]d\phi,$$

whence

* See §2.6.

$$0 \geqq x_0^2 + y_0^2 + z_0^2 - \frac{1}{2\pi}\int_C (x^2 + y^2 + z^2)d\phi - 2\left[a\left(\frac{1}{2\pi}\int_C x d\phi - x_0\right)\right.$$

$$\left. + b\left(\frac{1}{2\pi}\int_C y d\phi - y_0\right) + c\left(\frac{1}{2\pi}\int_C z d\phi - z_0\right)\right]. \tag{9}$$

The point (u_0, v_0) and the circle C being fixed, the right-hand member of this inequality is a linear function of the arbitrary real constants a, b, c. Thus (9) clearly implies that the coefficients of a, b, c vanish. That is to say, $x(u, v)$ for instance has the property that, for every point (u_0, v_0) in D,

$$x(u_0, v_0) = \frac{1}{2\pi}\int_0^{2\pi} x(u_0 + \rho\cos\phi, v_0 + \rho\sin\phi)d\phi,$$

for sufficiently small values of ρ. As is well known, this property characterizes harmonic functions.* Thus it follows that $x(u, v)$, $y(u, v)$, $z(u, v)$ are harmonic functions.

2.5. We proceed to show that $E=G$, $F=0$. Let $\mathfrak{x}=(x, y, z)$, and let $\mathfrak{y}=\mathfrak{x}+\mathfrak{a}$, where $\mathfrak{a}$ is an arbitrary constant vector. By assumption, then, $(\mathfrak{y}^2)^{1/2}$ is of class PL so that (see §§1.3 and 2.2)

$$(\mathfrak{y}_u^2 + \mathfrak{y}_v^2)\mathfrak{y}^2 - 2[(\mathfrak{y}\mathfrak{y}_u)^2 + (\mathfrak{y}\mathfrak{y}_v)^2] \geqq 0 \tag{10}$$

at points where $\mathfrak{y}\neq 0$. At points where $\mathfrak{y}=0$, (10) clearly also holds (with the sign of equality).

Consider a definite point (u_0, v_0) in D. Then $\mathfrak{y}_u=\mathfrak{x}_u$, $\mathfrak{y}_v=\mathfrak{x}_v$, regardless of the choice of the constant vector $\mathfrak{a}$. Choose first $\mathfrak{a}=\mathfrak{x}_u(u_0, v_0)-\mathfrak{x}(u_0, v_0)$. Then $\mathfrak{y}(u_0, v_0)=\mathfrak{x}_u(u_0, v_0)$, and (10) gives that

$$EG - E^2 - 2F^2 \geqq 0$$

at the point (u_0, v_0). Choose secondly $\mathfrak{a}=\mathfrak{x}_v(u_0, v_0)-\mathfrak{x}(u_0, v_0)$. (10) gives

$$EG - G^2 - 2F^2 \geqq 0$$

at (u_0, v_0). Addition gives

$$-(E-G)^2 - 4F^2 \geqq 0$$

and consequently $E=G$, $F=0$ at (u_0, v_0). Since (u_0, v_0) was any point in D, the lemma of §2.3 is proved.

2.6. The following remark might help explain the situation.

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* See for instance O. D. Kellogg, *Foundations of Potential Theory*, Berlin, J. Springer, 1929, p. 227.

If $\mathfrak{x}=\mathfrak{x}(u,v)$, (u, v) *in* D, *is a minimal surface in typical representation, then* $E=\mathfrak{x}_u^2$ *is of class PL in* D.*

It is clearly sufficient to consider the case when D is the interior of a circle. Then the components $x(u, v)$, $y(u, v)$, $z(u, v)$, which are harmonic, can be written in the form

$$x = \Re f_1(w),\ y = \Re f_2(w),\ z = \Re f_3(w),$$

where $f_1(w)$, $f_2(w)$, $f_3(w)$ are single-valued analytic functions of $w=u+iv$ in D. We have then

$$x_u - ix_v = f_1',\ y_u - iy_v = f_2',\ z_u - iz_v = f_3',$$

and hence, on account of $E=G$,

$$E = \tfrac{1}{2}(|f_1'|^2 + |f_2'|^2 + |f_3'|^2).$$

Thus E is the sum of three functions of class PL, and consequently (see §1.10) E is also of class PL.

As an example, let us consider the surface of Enneper† (in typical representation)

$$\begin{aligned} x &= 3u + 3uv^2 - u^3, \\ y &= -3v - 3u^2v + v^3, \\ z &= 3u^2 - 3v^2. \end{aligned}$$

Then x_u, y_u, z_u are three harmonic functions, such that the sum of their squares is of class PL. Computation shows that x_u, y_u, z_u are not conjugate. Thus, *in the lemma of* §2.3, *the parameters* a, b, c *are actually necessary, even if the given three functions are known to be harmonic.*

3. Applications

3.1. Let

$$\mathfrak{M}: \quad x = x(u, v),\ y = y(u, v),\ z = z(u, v),\ u + iv = w,\ |w| < 1,$$

be a minimal surface given in typical representation, such that (0, 0) is carried into (0, 0, 0). If $\mathfrak{M}$ is comprised in the unit sphere, $x^2+y^2+z^2\leqq 1$, then

$$|\mathfrak{M}(w)| \leqq |w|,\ 0 < |w| < 1, \tag{11}$$

and

$$E_0^{1/2} \leqq 1, \tag{12}$$

* In a subsequent paper, *Subharmonic functions and surfaces of negative curvature*, in the present number of these Transactions, we point out that if a surface is given in typical representation, then $E=\mathfrak{x}_u^2$ is of class PL if and only if the Gauss curvature of the surface is $\leqq 0$.

† See G. Darboux, *Théorie Générale des Surfaces*, Paris, 1887, vol. I, pp. 372–376.

where $E_0^{1/2}$ denotes the length deformation ratio at the origin. The equalities hold if and only if $\mathfrak{M}$ is a simply-covered circular disc with unit radius.*

Proof. Since $E=G$, $F=0$, we have

$$\lim_{w\to 0} |\mathfrak{M}(w)|/|w| = E_0^{1/2} \tag{13}$$

and therefore $|\mathfrak{M}(w)|/|w|$ remains bounded in $0<|w|<1$. Consequently in $0<|w|<1$ we can apply §1.16, with $\alpha=1$, to the function $p(u, v)=|\mathfrak{M}(w)|$. This gives (11), and then (13) yields (12).

If we define $|\mathfrak{M}(w)|/|w|=E_0^{1/2}$ for $w=0$, then both (11) and (12) are contained in $|\mathfrak{M}(w)|/|w|\leqq 1$, $|w|<1$. If then $|\mathfrak{M}(w)|/|w|=1$ for any w in $|w|<1$, then (see §1.2) the equality is an identity, $|\mathfrak{M}|^2\equiv u^2+v^2$. Differentiation gives

$$\mathfrak{r}\mathfrak{r}_u = u,\ \mathfrak{r}\mathfrak{r}_v = v,$$
$$\mathfrak{r}\mathfrak{r}_{uu} + \mathfrak{r}_u^2 = 1,\ \mathfrak{r}\mathfrak{r}_{vv} + \mathfrak{r}_v^2 = 1,$$

whence addition gives $E=G=1$ throughout. Therefore the area of the minimal surface is

$$A = \iint_{u^2+v^2<1} (EG - F^2)^{1/2} du dv = \pi.$$

It follows from this situation that $\mathfrak{M}$ is a simply-covered circular disc.†

3.2. Let

$$\mathfrak{M}:\quad x = x(u, v),\ y = y(u, v),\ z = z(u, v),\ u^2 + v^2 < 1,$$

be a minimal surface given in typical representation, and let $|\mathfrak{M}|$ be bounded. Suppose $x(u, v)$, $y(u, v)$, $z(u, v)$ remain continuous on a certain arc σ of $u^2+v^2=1$, and $x(u, v)=\text{const.}=x_0$, $y(u, v)=\text{const.}=y_0$, $z(u, v)=\text{const.}=z_0$ there. Then $x(u, v)\equiv x_0$, $y(u, v)\equiv y_0$, $z(u, v)\equiv z_0$.‡

Proof. Apply §1.12 to the function

$$p(u, v) = [(x(u, v) - x_0)^2 + (y(u, v) - y_0)^2 + (z(u, v) - z_0)^2]^{1/2}.$$

3.3. Let

$$\mathfrak{M}:\quad x = x(u, v),\ y = y(u, v),\ z = z(u, v),\ 0 < \operatorname{arc\,tg}(v/u) < \alpha,$$

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* This generalizes the Lemma of Schwarz. Cf. C. Carathéodory, *Conformal Representation*, p. 39.

† See E. F. Beckenbach, *The area and boundary of minimal surfaces*, Annals of Mathematics, (2), vol. 33 (1932), pp. 658–664.

‡ See T. Radó, *Some remarks on the problem of Plateau*, Proceedings of the National Academy of Sciences, vol. 16 (1930), pp. 242–248; J. Douglas, *Solution of the problem of Plateau*, these Transactions, vol. 33 (1931), pp. 262–321.

be a minimal surface given in typical representation, and let $|\mathfrak{M}|$ be bounded. Let further $x(u, v)$, $y(u, v)$, $z(u, v)$ remain continuous on the ray $u>0$, $v=0$, and let $x(u, 0)\to x_0$, $y(u, 0)\to y_0$, $z(u, 0)\to z_0$ as $u\to+0$. Then in every angle

$$0 < \operatorname{arc\,tg} \frac{v}{u} < \alpha - \sigma, \text{ where } \sigma > 0,$$

we have $x(u, v)\to x_0$, $y(u, v)\to y_0$, $z(u, v)\to z_0$ as $(u, v)\to(0, 0)$ in any manner.*

Proof. Apply §1.14 to the function

$$p(u, v) = [(x(u, v) - x_0)^2 + (y(u, v) - y_0)^2 + (z(u, v) - z_0)^2]^{1/2}.$$

As in §1.14, the theorem is true if the domain of definition is only the sector $0<\operatorname{arc\,tg}(v/u)<\alpha$, $0<u^2+v^2<r_0^2$.

3.4. Besides the assumptions of §3.3, suppose $x(u, v)$, $y(u, v)$, $z(u, v)$ remain continuous on the ray $\operatorname{arc\,tg}(v/u)=\alpha$, $u^2+v^2>0$, and let $x(u, v)\to x_1$, $y(u, v)\to y_1$, $z(u, v)\to z_1$ as $(u, v)\to(0, 0)$ along the ray $\operatorname{arc\,tg}(v/u)=\alpha$. Then $x_0=x_1$, $y_0=y_1$, $z_0=z_1$, and $x(u, v)\to x_0=x_1$, $y(u, v)\to y_0=y_1$, $z(u, v)\to z_0=z_1$ as $(u, v)\to(0, 0)$ in any manner in the angle $0<\operatorname{arc\,tg}(v/u)<\alpha$.

Proof. Apply §3.3 to the angles

$$0 < \operatorname{arc\,tg} \frac{v}{u} < \frac{3\alpha}{4} \text{ and } \frac{\alpha}{4} < \operatorname{arc\,tg} \frac{v}{u} < \alpha$$

and compare results. As before, the theorem is still true if the domain of definition is only the sector

$$0 < \operatorname{arc\,tg} \frac{v}{u} < \alpha,\ 0 < u^2 + v^2 < r_0^2 .$$

3.5. The preceding result yields a new proof of the following lemma, used by J. Douglas in his work on the problem of Plateau.†

Let the integrable functions $\xi(\phi)$, $\eta(\phi)$, $\zeta(\phi)$, substituted in the Poisson integral formula, determine the (harmonic) coordinate functions of a minimal surface

$$\mathfrak{M}:\ x = x(u, v),\ y = y(u, v),\ z = z(u, v),\ u^2 + v^2 < 1,$$

in typical representation. Let further $\xi(\phi)$, $\eta(\phi)$, $\zeta(\phi)$ approach definite limit values $\xi_-(\pi)$, $\eta_-(\pi)$, $\zeta_-(\pi)$ and $\xi_+(\pi)$, $\eta_+(\pi)$, $\zeta_+(\pi)$ according as $\phi\to\pi$ in clockwise and counterclockwise senses respectively. Then

$$\xi_-(\pi) = \xi_+(\pi),\ \eta_-(\pi) = \eta_+(\pi),\ \zeta_-(\pi) = \zeta_+(\pi). \tag{14}$$

* This generalizes a theorem of Lindelöf. Cf. Pólya und Szegö, loc. cit.

† J. Douglas, loc. cit., pp. 304–306.

Proof. It is a well known property of the Poisson integral that, because of the specified nature of the discontinuity of $\xi(\phi)$ at $\phi=\pi$, the function $x(u, v)$ approaches a definite limit if $(u, v)\to(-1, 0)$ along any straight line in $u^2+v^2<1$, this limit being a linear function of the angle from the u-axis to the straight line and varying from $\xi_-(\pi)$ to $\xi_+(\pi)$ as the angle varies from $-\pi/2$ to $\pi/2$. Similar statements hold for $y(u, v)$, $z(u, v)$. But if we join two such straight lines by a circular arc lying in $u^2+v^2<1$, we obtain a sector for which §3.4 applies; consequently, $(x, y, z)\to$ a definite (x_0, y_0, z_0) which does not vary with the angle. That is, the linear functions mentioned above are constants, whence (14).

3.6. Let

$$\mathfrak{M}: \quad x = x(u, v),\ y = y(u, v),\ z = z(u, v),\ (u, v) \text{ interior to } R,$$

where R is a Jordan region,* be a minimal surface given in typical representation, and let $|\mathfrak{M}|$ be bounded. Let further $x(u, v)$, $y(u, v)$, $z(u, v)$ remain continuous on the boundary of R except possibly at a single point (u_0, v_0), and let $(x, y, z)\to(x_0, y_0, z_0)$ and $(x, y, z)\to(x_1, y_1, z_1)$ as (u, v) converges on the boundary to (u_0, v_0) from one side and the other respectively. Then $(x_0, y_0, z_0)=(x_1, y_1, z_1)$ and $x(u, v)\to x_0=x_1$, $y(u, v)\to y_0=y_1$, $z(u, v)\to z_0=z_1$ as $(u, v)\to(u_0, v_0)$ in any manner in R.

The proof follows immediately from §3.4 by conformal mapping. It can be obtained also by following step by step the proof, for the absolute value of an analytic function of a complex variable, based on the rotation-method.†

4. On conformal maps of minimal surfaces

4.1. The most general definition (actually used in the literature) of a minimal surface is as follows.‡

A set of equations

$$x = \xi(\alpha, \beta),\ y = \eta(\alpha, \beta),\ z = \zeta(\alpha, \beta),\ (\alpha, \beta) \text{ in } R, \tag{15}$$

where R denotes a Jordan region,* defines a *continuous surface S of the topological type of the circular disc*, if $\xi(\alpha, \beta)$, $\eta(\alpha, \beta)$, $\zeta(\alpha, \beta)$ are continuous in R.

The surface (15) is a minimal surface if the following condition is satisfied. Given any point (α_0, β_0) interior to R, there exists a vicinity V_0 of (α_0, β_0) and a topological transformation $\bar\alpha=\bar\alpha(\alpha, \beta)$, $\bar\beta=\bar\beta(\alpha, \beta)$ of V_0, such that $\xi(\alpha, \beta)$, $\eta(\alpha, \beta)$, $\zeta(\alpha, \beta)$ are transformed into functions $\bar\xi(\bar\alpha, \bar\beta)$, $\bar\eta(\bar\alpha, \bar\beta)$, $\bar\zeta(\bar\alpha, \bar\beta)$ which form a triple of conjugate harmonic functions in the image $\overline{V}_0$ of V (see §0.2). Such parameters $\bar\alpha$, $\bar\beta$ are called *local typical parameters*.

* That is, the set of points in and on a Jordan curve.

† Cf. C. Carathéodory, *Conformal Representation*, pp. 21–24.

‡ See T. Radó, *Contributions to the theory of minimal surfaces*, Acta Szeged, vol. 9 (1932), p. 9.

4.2. According to the fundamental theorem in the theory of uniformization,† a minimal surface in the general sense defined above admits also of *typical parameters in the large*, in the following sense. If

$$S:\quad x = \xi(\alpha, \beta),\ y = \eta(\alpha, \beta),\ z = \zeta(\alpha, \beta),\ (\alpha, \beta) \text{ in } R, \tag{16}$$

is a minimal surface, in the sense of §4.1, then there exists a topological transformation

$$T:\begin{cases} u = u(\alpha, \beta), & v = v(\alpha, \beta), \quad (\alpha, \beta) \text{ interior to } R, \\ \alpha = \alpha(u, v), & \beta = \beta(u, v), \quad u^2 + v^2 < 1, \end{cases} \tag{17}$$

of the interior of R into $u^2+v^2<1$, such that $\xi(\alpha, \beta)$, $\eta(\alpha, \beta)$, $\zeta(\alpha, \beta)$ are carried into three functions

$$\begin{aligned} &x(u, v) = \xi(\alpha(u, v), \beta(u, v)),\ y(u, v) = \eta(\alpha(u, v), \beta(u, v)), \\ &z(u, v) = \zeta(\alpha(u, v), \beta(u, v)) \end{aligned} \tag{18}$$

which form a triple of conjugate harmonic functions in $u^2+v^2<1$. Our purpose in this section is to study the situation on the boundary.

4.3. Using the same notations as in the preceding paragraph, §4.2, *suppose that the functions* $\xi(\alpha, \beta)$, $\eta(\alpha, \beta)$, $\zeta(\alpha, \beta)$ *in* (16) *do not all three reduce to constants on any arc of the boundary of* R.

Then the transformation (17) *remains continuous and one-to-one on the boundaries. As a consequence, the functions* $x(u, v)$, $y(u, v)$, $z(u, v)$ *in* (18) *remain continuous on* $u^2+v^2=1$.

4.4. The preceding assertion will be established if we disprove the following two possibilities.

(i) Suppose there exist in the interior of R two sequences (α_n', β_n'), (α_n'', β_n'') converging to the same point (α_0, β_0) on the boundary of R, such that the corresponding sequences (u_n', v_n'), (u_n'', v_n'') converge to two distinct points (u_0', v_0'), (u_0'', v_0'') on $u^2+v^2=1$. Denote then by l_n an arc in the interior of R, connecting (α_n', β_n') and (α_n'', β_n''), such that l_n converges to (α_0, β_0); and denote by C_n the image of l_n in $u^2+v^2<1$. Then the theorem of §1.15 applies to the function

$$p(u, v) = [(x(u, v) - \xi(\alpha_0, \beta_0))^2 + (y(u, v) - \eta(\alpha_0, \beta_0))^2 + (z(u, v) - \zeta(\alpha_0, \beta_0))^2]^{1/2},$$

and it follows that $p(u, v)$ vanishes identically. Hence $x(u, v)$, $y(u, v)$, $z(u, v)$ and consequently $\xi(\alpha, \beta)$, $\eta(\alpha, \beta)$, $\zeta(\alpha, \beta)$ all reduce to constants. This contradicts the assumption stated in §4.3.

(ii) Denote by (α_1, β_1), (α_2, β_2) any two distinct points on the boundary of

† See C. Carathéodory, *Conformal Representation*, chapter VII, and also the bibliographical notes given there on p. 105.

R, and by C a Jordan arc in the interior of R connecting (α_1, β_1) and (α_2, β_2). On account of the preceding result, the image C^* of C is a Jordan arc in $u^2+v^2<1$ with definite end points on $u^2+v^2=1$. We have to disprove the possibility that these end points coincide. Suppose they do coincide. Then C^* is actually a closed Jordan curve, which has a unique point (u_0, v_0) in common with $u^2+v^2=1$. Denote by D^* the interior of C^*. Then $x(u, v)$, $y(u, v)$, $z(u, v)$ satisfy in D^* the assumptions of §3.6. Hence $x(u, v)$, $y(u, v)$, $z(u, v)$ converge to definite limits x_0, y_0, z_0 if (u, v) converges to (u_0, v_0) from within D^*.

D^* is the image of a domain D in R which is bounded by C and by a certain arc σ of the boundary of R. If (α, β) converges, from within D, to any point of σ, then (u, v) converges to (u_0, v_0) from within D^*. Hence

$$\xi(\alpha, \beta) = x(u, v) \to x_0, \eta(\alpha, \beta) = y(u, v) \to y_0, \zeta(\alpha, \beta) = z(u, v) \to z_0.$$

That is to say, $\xi(\alpha, \beta)$, $\eta(\alpha, \beta)$, $\zeta(\alpha, \beta)$ all three reduce to constants on σ, in contradiction with the assumption made in §4.3.

4.5. We mention the following two special cases of the theorem of §4.3. Suppose that $\xi(\alpha, \beta)\equiv\alpha$, $\eta(\alpha, \beta)\equiv\beta$, $\zeta(\alpha, \beta)\equiv 0$ in the Jordan region R. Then the assumptions of §§4.2 and 4.3 obviously are satisfied and the theorem of §4.3 reduces to the so-called *Osgood-Carathéodory theorem*: If the interior of a Jordan region R is mapped in a one-to-one and conformal way upon $u^2+v^2<1$, the map remains continuous and one-to-one on the boundary of R.†

4.6. Suppose next that the equations (16) carry the boundary of R in a topological way into a Jordan curve Γ. In this case we say that the surface S is bounded by Γ. The theorem of §4.3 implies then the following result. *A minimal surface S (in the general sense of* §4.1*), bounded by a Jordan curve Γ, admits of a representation*

$$x = x(u, v),\ y = y(u, v),\ z = z(u, v),\ u^2 + v^2 \leqq 1, \tag{19}$$

with the following properties:

(i) *$x(u, v)$, $y(u, v)$, $z(u, v)$ form a triple of conjugate harmonic functions in $u^2+v^2<1$*;

(ii) *$x(u, v)$, $y(u, v)$, $z(u, v)$ are continuous in $u^2+v^2\leqq 1$, and the equations* (19) *carry $u^2+v^2=1$ in a topological way into the Jordan curve Γ.*

By way of explanation, let us recall that a Jordan curve might bound several minimal surfaces, as follows from classical examples. The preceding result expresses a property common to *all* these minimal surfaces.

† See C. Carathéodory, *Conformal Representation*, chapter VI.

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