A SPECIAL INTEGRAL FUNCTION

BY
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1. Some years ago Collingwood and Valiron† proposed the problem of whether there could exist an integral function whose minimum modulus on every circle \(|z| = r\) is bounded, but possessing no asymptotic paths. By an asymptotic path we mean a continuous path tending to infinity along which the value of the function tends to a limit.

In this paper I show how to construct such a function. It is obtained by considering the well known Weierstrassian non-differentiable function

\[
\sum_{n=0}^{\infty} c^n a^n
\]

where \(c(1 < c < 2)\) and \(a\) (an integer) are suitably chosen. We may observe that, if \(a\) is large enough, the Weierstrassian function possesses no asymptotic paths which tend to the boundary \(|z| = 1\), while, for sufficiently small \(c\), its minimum modulus on circles \(|z| = r < 1\) is bounded, and every point of the unit circle is an essential singularity for the function.

2. Consider the function

\[
F_N(z) = \left( \sum_{n=0}^{N} c^n a^n \right) \exp \left\{ - \left( \frac{z}{1 - a^{-N}} \right)^{N^a N} \right\}
\]

where \(c > 1\) (say \(c = 3/2\)), and \(a\) is large. We first show that the minimum modulus of \(F_N(z)\) on any circle \(|z| = r\) is bounded, independently of \(N\). Clearly, for \(r > 1\), \(F_N(r)\) does not exceed

(1) \(Bc^N r^N \exp (-r - e^{N^a N})\),

where \(B\), here and in the sequel, denotes an absolute positive constant (it may denote a different constant in different contexts). For \(r \geq 1\), \(N \geq 1\), the expression (1) does not exceed a fixed constant, and thus it is sufficient to consider \(F_N(z)\) with \(|z| \leq 1\).

Let \(r\) be a fixed number less than 1. We consider the value of \(F_N(re^{i\theta})\) where \(\theta\) is chosen according to the following rules. We first stipulate that

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$a^N\theta \equiv 0 \pmod{2\pi}$, so that also $2^N a^N \theta \equiv 0 \pmod{2\pi}$, and thus the second factor

$$\exp \left\{ - \left( \frac{z}{1 - a^{-N}} \right)^{N^2} \right\}$$

of $F_N(z)$ is real and less than 1 in modulus. There are now $a$ possible reduced values of $a^{N-1}\theta \pmod{2\pi}$. We choose that one which makes

$$\left| \sum_{n=N-1}^{N} c^{n^2} \exp (ia^n\theta) \right|$$

a minimum. We now choose that one of the reduced values of $A^{N-2}\theta \pmod{2\pi}$ which makes

$$\left| \sum_{n=N-2}^{N} c^{n^2} \exp (ia^n\theta) \right|$$

a minimum, and so on. We show that if $a$ is sufficiently large the resulting value of

$$\left| \sum_{n=0}^{N} c^{n^2} \exp (ia^n\theta) \right|$$

will not exceed a fixed constant independent of $N$. The argument is almost identical with that given in an earlier paper.* We have at the first stage $a$ possible values of

(2) $$\sum_{n=N-1}^{N} c^{n^2} \exp (ia^n\theta)$$

There is one of these for which the angle between the lines joining the point $c^{N^2} \exp (ia^n\theta)$ to the origin and to the point

$$\sum_{n=N-1}^{N} c^{n^2} \exp (ia^n\theta)$$

is less than or equal to $\pi/a$. Then the value of (2) can be seen by elementary geometry to lie between

$$c^{N-1} c^{N-1} \sec (\pi/a) - c^{N-1}$$

which, if $a$ is sufficiently large ($c = 3/2$), is certainly not greater than $c^{N-1} c^{N-1}$. Thus

$$\min_{a^N \theta = 0} \left| \sum_{n=N-1}^{N} c^{n^2} \exp (ia^n\theta) \right| \leq c^{N-1} c^{N-1}.$$  

Having now fixed the reduced value of $a^{N-1}\theta \pmod{2\pi}$, we look for the minimum value of

$$\left| \sum_{n=-N}^{N} c^n r^a \exp(ia^n\theta) \right|.$$  

There is certainly one value for the expression (3), such that the angle between the lines joining the point

$$c^n r^a \exp(ia^n\theta)$$

and the origin to the point

$$c^n r^a \exp(ia^n\theta)$$

to the origin and to the point

$$\sum_{n=N+2}^{N} c^n r^a \exp(ia^n\theta)$$

is less than or equal to $\pi/a$. Thus the value of (3) lies between

$$c^{N-2} r^a \exp(ia^{N-2})$$

and, if $a$ is sufficiently large, it does not exceed $c^{N-2} r^a$. An inductive process will now show that

$$\min_{a^{N}\theta=0} \left| \sum_{n=0}^{N} c^n r^a \exp(ia^n\theta) \right| \leq r \leq 1,$$

and we have shown that, for all values of $r$, the minimum modulus of $F_N(z)$ on $|z|=r$ is less than an absolute constant.

The derivative $F_N'(z)$ of $F_N(z)$ is

$$\exp\left\{ -\left(\frac{z}{1-a^{-N}}\right)^{N} \right\} \left\{ \sum_{n=0}^{N} (ac)^n z^a\right\} \left( \sum_{n=0}^{N} c^n z^a \right)$$

Now suppose that $a$ is sufficiently large, and that $1-3a^{-N} \leq r \leq 1-2a^{-N}$. Then the expression (4) is majorized* by the single term

$$a^N c^N z^a r^{-N-1}.$$  

Indeed the single term (5) exceeds in modulus

$$a^N c^N (1-3a^{-N}) a^{-N-1} \geq \frac{1}{32} a^N c^N, \ a \geq 6,$$

* See, e.g., G. H. Hardy, Weierstrass' non-differentiable function, these Transactions, vol. 17 (1916), pp. 301–332.
while the difference between the terms (4) and (5) is not greater in modulus than
\[
\exp \left( \frac{1 - 2a^{-N}}{1 - a^{-N}} \right)^N \left\{ \sum_{n=0}^{N-1} (ac)^n + 2 \cdot 2^N a^N \left( \frac{1 - 2a^{-N}}{1 - a^{-N}} \right)^N \left( \sum_{n=0}^{N} c^n \right) \right\} \\
+ a^N c^N \left\{ \exp \left( \frac{1 - 2a^{-N}}{1 - a^{-N}} \right)^N - 1 \right\} \leq 10^{-8} a^N c^N
\]
if \( a \) and \( N \) are sufficiently large.

3. We now write
\[
F(z) = \sum_{k=1}^{\infty} f_k(u_k), \quad u_k = \left( \frac{z}{R_k} \right)^{a_k},
\]
and set, for abbreviation,
\[
f_k(u) = F_k(u), \quad \alpha_k = a_k, \quad \beta_k = a^{N_k + \lambda_k},
\]
where \( \lambda_1 = 0, \lambda_{k+1} = 2(N_k + \lambda_k) \), while \( N_k, R_k (k = 1, 2, \ldots) \) remain to be chosen. We write \( N_1 = R_1 = 1 \) and give an inductive method for choosing \( N_k, R_k \) for \( k > 1 \). Suppose that we have already chosen \( N_1, \ldots, N_{k-1}, R_1, \ldots, R_{k-1} \). Since first \( F_N(z) = O(|z|) \) for small \( z \) uniformly in \( N \), we may choose \( R_k \) so large that
\[
|z| \leq R_k, \quad n = 1, 2, \ldots, k - 1,
\]
whatever the value of \( N_k \) may be. Next since, for \( |z| \leq R/2 \) we have, uniformly in \( N_k \) and \( R \),
\[
\frac{d}{dz} f_k \left[ \left( \frac{z}{R} \right)^{a_k} \right] = O( |z|^{-a_k} R^{-a_k} ),
\]
we may also assume that \( R_k \) is so large that, whatever the value of \( N_k \) may be, we have
\[
|z| \leq R_{k-1}, \quad n = 1, 2, \ldots, k - 1.
\]
This finally fixes \( R_k \). We now choose \( N_k > k \) so great that
\[
\beta_k c^{N_k R_k^{-1}} \geq 10^6 \max_{|z| \leq R_k} \left| \frac{d}{dz} \sum_{m=1}^{k-1} f_m(u_m) \right|.
\]
We next observe that, for \( -(2a)^{-N_k} \pi/4 \leq \theta \leq (2a)^{-N_k} \pi/4, \quad r \geq 1, \)
\[ |f_k(z)| \leq |Bc^{N_k}\exp\left(\frac{r}{1-a^{-N_k}}\right)|z|^N_k \exp\left\{\frac{r}{1-a^{-N_k}}\right\} \]

\[ = Bc^{N_k}\exp\left\{\frac{r}{1-a^{-N_k}}\right\} \]

For \(-a^{-k+1}\pi \leq \theta \leq a^{-k+1}\pi\), the argument of \(u_k\) is \(\alpha_k\theta\), and thus in modulus does not exceed

\[ a^k - a^{-k+1}\pi = a^{-2N_k-k}\pi \leq (2a)^{-N_k}\pi/4; \]

whence, on the range \(|z| \geq R_k\), \(-a^{-k+1}\pi \leq \theta \leq a^{-k+1}\pi\),

\[ \max |f_k(u_k)| \leq Bc^{N_k}\exp\left\{-(Be)^{N_k}\right\}. \]

We may thus increase \(N_k\) if necessary so as to ensure that, on the same range,

\[ \max |f_k(u_k)| \leq 2^{-k}. \]

This finally fixes \(N_k\).

4. We observe first that, in virtue of (6), \(F(z)\) is in fact an integral function. Next (6) and (9) give us

\[ \max \left\{ \sum_{i=1}^{k-1} + \sum_{i=k+1}^{\infty} |f_i(u_i)| \right\} \leq B, \]

\[ R_k \leq |z| \leq R_{k+1}, -a^{-k+1}\pi \leq \theta \leq a^{-k+1}\pi. \]

Also \(F_{N_{k+1}}\) is so constructed that for fixed \(r\), satisfying \(R_k \leq r \leq R_{k+1}\),

\[ \min |f_{k+1}(u_{k+1})| \leq B, \]

\[ -a^{-k+1}\pi \leq \theta \leq a^{-k+1}\pi. \]

The equations (10) and (11) show that if \(r\) is fixed with \(R_k \leq r \leq R_{k+1}\), then

\[ \min |F(z)| \leq B, \]

where \(B\) is independent of \(r, k\). Thus, the minimum modulus of \(F(z)\) on circles \(|z| = r\) is bounded.

5. We have now to show that \(F(z)\) has no asymptotic path. To do this we show that in certain regions the differential coefficient of \(F(z)\) is not only large but so large that there can be no continuous path passing through all these regions on which \(F(z)\) is bounded.

Consider \(F'(z)\) in the annulus

\[ 1 - 3a^{-N_k} \leq u_k \leq 1 - 2a^{-N_k}. \]
We have
\[
\frac{d}{dz} f_k(u_k) = \frac{\alpha_k}{z} \frac{d}{du_k} f_k(u_k) = \frac{\alpha_k}{z} u_k(ac)^N_k \left( \frac{z}{R_k} \right)^{\beta_k - \alpha_k} (1 + \epsilon),
\]
where \(|\epsilon| \leq 10^{-k}\), in virtue of the remarks at the end of §2. Now, in the annulus considered, when \(a > 6\),
\[
|u_k(ac)^N_k \left( \frac{z}{R_k} \right)^{\beta_k - \alpha_k}| = \beta_k c^N_k R_k^{-1} \left| \frac{z}{R_k} \right|^{\beta_k - 1} 
\geq \beta_k c^N_k R_k^{-1} (1 - 3a^{-N_k})^N_k \geq 10^{-2} \beta_k c^N_k R_k^{-1}.
\]
Also, by (7) and (8), in the annulus considered,
\[
\left| \frac{d}{dz} \left( \sum_{m=1}^{k-1} \sum_{m-k+1}^\infty f_m(u_m) \right) \right| \leq 10^{-k} \beta_k c^N_k R_k^{-1} + 10^{-k},
\]
and thus
\[
F'(z) = \beta_k c^N_k R_k^{-1} \beta_k z \beta_k - 1 (1 + \epsilon'),
\]
where \(|\epsilon'| \leq 3 \cdot 10^{-4} \leq 10^{-2} \pi^{-1} \).

Now let \(\zeta\) and \(\zeta'\) be two points of the annulus (12) and let
\[
\left| \frac{\zeta'}{\zeta} - 1 \right| = 10^{-1} \beta_k^{-1}.
\]
Then (13) shows that
\[
F(\zeta') - F(\zeta) = \int_{\zeta}^{\zeta'} F'(z) dz
\]
\[
\int_{\zeta}^{\zeta'} \beta_k c^N_k R_k^{-1} \beta_k z \beta_k - 1 (1 + \epsilon') dz
\]
\[
= c^N_k \beta_k R_k^{-1} \left\{ \left( \frac{\zeta'}{\zeta} \right) \beta_k - 1 \right\} + R,
\]
where
\[
R \leq 10^{-3} \left| \zeta' - \zeta \right| \beta_k c^N_k R_k^{-1} \beta_k - 1 \leq 10^{-4} c^N_k,
\]
for we may certainly find a path joining \(\zeta\) and \(\zeta'\) of length not exceeding \(|\zeta' - \zeta|\pi\), entirely interior to the annulus (12). The first term of (14) is
\[
c^N_k \beta_k R_k^{-1} \beta_k \left\{ \beta_k (\zeta' - \zeta)/\zeta \right\} (1 + \epsilon''),
\]
where
Since, finally, in virtue of (12), if $a$ is large enough,

$$
| c^{\mathfrak{N}_a^{\beta_\beta} R_{\beta_\beta} - \alpha_\beta (\xi - \xi')/\xi | \geq 10^{-a} c^{N_\beta},
$$

it follows from (14), (15), (16) that

$$
F(\xi') - F(\xi) \geq 2 \cdot 10^{-a} c^{N_\beta}.
$$

Since, for sufficiently large $a$, the breadth of the strip (12) exceeds

$$
10^{-1} R_{\beta_\beta} - 1,
$$

(17) shows that there can be no continuous path, crossing the strip, for which the minimum modulus of $F(z)$ is less than $10^{-a} c^{N_\beta}$ which is arbitrarily large with $k$. Thus there can be no asymptotic path tending to infinity.

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