SECTIONS OF POINT SETS*

BY

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1. Introduction

A section of a plane point set $E$ is defined as that subset of $E$ which contains all points of $E$ lying on a line $L$. If $L$ is a horizontal line the section is called a horizontal section and if $L$ is a vertical line, the section is called a vertical section. It is the purpose of this paper to study the relations between $E$ and its horizontal and vertical sections. Kuratowski and Ulam†, Sierpiński‡, and Fubini§, have considered various phases of this problem. Baire||, Hahn¶, Kempisty** and others have considered the closely related problem of finding the relations between a function $f(x, y)$ and the functions obtained by holding $x$ or $y$ constant.

In order to state results in a general manner, $E$ will be regarded as a subset of a combinatorial product space $A \times B$ where $A$ and $B$ are metric spaces and $B$ is separable. Such a space is defined as the collection of all pairs of points $(x, y)$, $x$ being a point of $A$ and $y$ being a point of $B$. The distance between $(x_1, y_1)$ and $(x_2, y_2)$ is here defined to be $\sqrt{(x_1 x_2)^2 + (y_1 y_2)^2}$. The plane is a special case of such a space in which $A$ and $B$ are straight lines, and all the results of this paper apply to the plane and also to an $(m+n)$-dimensional euclidean space considered as the product of an $m$-dimensional and an $n$-dimensional euclidean space.

Because $A \times B$ is analogous to the plane, the subset of points $(x, y)$ such that $x = a$ is called a vertical section of $A \times B$ and is denoted by $a \times B$ or $(x = a)$; similarly the subset of points $(x, y)$ such that $y = b$ is called a horizontal section of $A \times B$ and is denoted by $A \times b$ or $(y = b)$. If $E$ is any subset of $A \times B$ the set $E \cdot (x = a)$ is called a vertical section of $E$ and the set $E \cdot (y = b)$ is called a horizontal section of $E$.

* Presented to the Society, November 25, 1932; received by the editors February 1, 1933.
† Fundamenta Mathematicae, vol. 19, p. 247; see also an article by Kuratowski in vol. 17, p. 275.
‡ Fundamenta Mathematicae, vol. 1, p. 112.
§ Rendiconti della Reale Accademia dei Lincei, (5), vol. 16, I. For a statement of Fubini’s theorem see also Carathéodory, Vorlesungen über Reelle Funktionen, 1927, p. 621.
¶ Mathematische Zeitschrift, 1919, p. 306.
†† If $p$ and $q$ are any two points of a metric space, $(pq)$ denotes the distance between $p$ and $q$. 915
If $E$ is closed, all horizontal and vertical sections of $E$ are closed, and if $E$ is open, its horizontal and vertical sections are open (relative to the sections of $A \times B$ which contain them). A similar proposition is true for sets $F$ or $O$ of type $\alpha$. Converse propositions are not true. The plane set of points $(1/n, 1/n)$, where $n$ takes all integral values, is such that each of its horizontal and vertical sections contains at most one point and is therefore closed. The point $(0, 0)$ is a limit point of the set which is not in the set. Sierpinski* has constructed a plane set every section of which (not merely horizontal and vertical sections) contains at most two points and which is non-measurable in the Lebesgue sense. This example shows that the fact that every horizontal and vertical section of $E$ is of type $\alpha$ is not a sufficient condition that $E$ be of type $\alpha$, and that in order to obtain such a sufficient condition, further restrictions on the sections or on the relations between them must be imposed.

By restricting the vertical sections to a type of set called $I$-set (or the complement of such a set) sufficient conditions may be obtained that a set be of various types. This is done in §3. Necessary and sufficient conditions that sets with restricted vertical sections be of class $\alpha$ are given in §6. Uses of sets called gratings are considered in §7. Theorems are given which show that boundaries of sets with certain kinds of sections lie on sets of lines of the first category. The results are applied in §8 to prove a theorem of Baire concerning functions of two variables continuous in each of them and to obtain a result regarding Kempisty's generalization of this theorem.

2. Horizontal sections of class $M$

The following definitions will be useful.

**Definition 1.** If the inner points of a set are dense on the set, the set is called an $I$-set.

A set may have this property with respect to $A \times B$ or it may be a subset of a section of $A \times B$ and have this property with respect to the section, this latter being the case which will most often arise.

**Definition 2.** Given a point $(a, b)$, the set of points $(a, y)$ where $(by) < r$, $r$ a positive number, is called an open vertical interval of center $(a, b)$ and radius $r$.

A closed vertical interval is defined in the same way except that $(by) \leq r$. Closed and open horizontal intervals may also be defined. A vertical interval might also be defined as $a \times S$ where $S$ is a sphere in $B$ of center $b$ and radius $r$.

* See the previous reference.
Definition 3. If a set \( G \) lies on a horizontal section, \( G(r) \) is the set of points of open vertical intervals of radii \( r \) and centers at the points of \( G \).

Definition 4. If \( M \) is a family* of point sets \( M \) lying on horizontal sections of \( A \times B \), \( M(r) \) is the family of all point sets \( M(r) \) for \( r \) ranging over all positive numbers.

If \( M \) is a family of point sets, \( M_s(M) \) denote, as is conventional, the families composed of all possible sums (products) of an enumerable number of sets of \( M \).

A set of vertical sections \( K \) is said to be everywhere dense if the set of points \([a]\), such that \((x = a)\) is in \( K \), is everywhere dense in \( A \). In a similar manner other point-set properties, for example the property of being in the first or the second category, are ascribed to sets of vertical sections and to sets of horizontal sections as well.

By the projection of a point \((x, y)\) on \((y = b)\) is meant the point \((x, b)\). The projection of a set of points \( E \) on \((y = b)\) is the set of points formed by projecting all the points of \( E \) on \((y = b)\).

Theorem 1. If each vertical section of \( E \) is an open set whose complement is an I-set and horizontal sections of \( E \) belong to \( M \), then \( E \) belongs to the family \([M_s(r)]_r\).

Since \( B \) is separable there exists an enumerable everywhere dense set \((y = b_t)\) of horizontal sections of \( A \times B \). Let \( r_h \) be a sequence of positive numbers approaching 0 as a limit. Let \( K_t \) be the projection of \( E \cdot (y = b_t) \) on \((y = b_t)\). Let \( A_{im} \) be the product of \( E \cdot (y = b_t) \) and all sets \( K_i \) such that \((b_i b_t) < r_m\). The set \( A_{im}(r_m) \) is a subset of \( E \). To prove this, suppose that \( A_{im}(r_m) \) contains a point \((a, b)\) of \( CE \). The point \((a, b_t)\) is in \( A_{im}(r_m) \) and also all points of the open vertical interval of radius \( r_m \) and center \((a, b_t)\) are in \( A_{im}(r_m) \). Since \((a, b)\) lies in this open vertical interval, there is by hypothesis some inner point† \((a, c)\) of \( CE \) in this interval. There is then an \( \epsilon \) such that if \((cy) < \epsilon, (a, y)\) is in \( CE \). Since the \( b_t\)'s are everywhere dense in \( B \) there is some \( b_i \), say \( b_n \), such that \((cb_n) < \epsilon\). The point \((a, b_n)\) is then in \( CE \). This \( b_n \) may be so chosen that \((b_n b_n) < r_m\). The set \( E \cdot (y = b_n) \) does not con-

* It is here supposed that if \( E \) consisting of points \((x, b)\) is in \( M \), the set of points \((x, c)\), where \( x \) has the same range as in \( E \) and \( c \) is any point of \( B \), is also in \( M \). This restriction is made for convenience. It is necessary for the proofs of some of the following theorems in which use is made of the projections of sets from one horizontal section to another.

† A more explicit statement is as follows: If \( V \) is any vertical section of \( A \times B \), \( V \cdot E \) is open in \( V \) and \( V \cdot E \) is an I-set in \( V \), etc. Language similar to that in the hypothesis of Theorem 1 will be used throughout, with the meaning given in this note.

‡ With respect to the section \((x = a)\).
tain \((a, b_i)\) and \(K_i\) will not contain \((a, b_i)\). Therefore \((a, b_i)\) will not be in \(A_i(r_m)\). From this contradiction, it follows that \(A_i(r_m)\) is in \(E\).

To prove that every point of \(E\) is in some \(A_i(r_m)\), let \((c, d)\) be any point of \(E\). There is an \(\epsilon\) such that if \((dy) < \epsilon\), \((c, y)\) is in \(E\). Choose \(r_m\) and \(b_i\) such that \(b_i d < r_m < \epsilon/2\). Every point \((c, y)\) of the open vertical interval of center \((c, b_i)\) and radius \(r_m\) is in \(E\). For since \((b_i y) < \epsilon/2\) and \((b_i d) < \epsilon/2\), it follows from the triangle axiom that \((dy) < \epsilon\). Therefore the set \(K_i\) contains \((a, b_i)\) if \((b_i b_i) < r_m\), and \(A_i(r_m)\) contains all points \((c, y)\) such that \((b_i y) < r_m\). It must then contain \((c, d)\) since \((b_i d) < r_m\).

Thus \(E = \bigcup_i A_i(r_m)\), which proves the theorem.

**Theorem 2.** If vertical sections of \(E\) are closed \(I\)-sets and horizontal sections of \(E\) belong to the family \(\mathcal{N}\), then \(E\) belongs to the family \([\mathcal{N}(r)]\).

Let \((y = b_i)\) be an enumerable everywhere dense set of horizontal sections of \(A \times B\) and let \((r_i)\) be a sequence of positive numbers approaching 0. Let \(K_i\) be the projection of \(E \cdot (y = b_i)\) on \((y = b_i)\). Let \(A_i(r)\) be the sum of \(E \cdot (y = b_i)\) and all sets \(K_i\) such that \((b_i b_i) < r\). The set \(E_m = \bigcup_i A_i(r_m)\) is a member of the family \([\mathcal{N}(r)]\). The set \(E_m\) contains \(E\), for all \(m\). To prove this let \((c, d)\) be any point of \(E\). By hypothesis there is for each \(r_m\) a vertical interval \(V_e\), of radius \(\epsilon\), which contains only points of \(E\) and for every point \((c, y)\) of which \((dy) < r_m\). There is some \((y = b_i)\) which cuts \(V_e\) in a point \((c, b_i)\). The point \((c, b_i)\) is then in \(E\). The set \(A_i(r_m)\) contains \((c, d)\) because \((db_i) < r_m\).

The set \(E\) is therefore included in \(\prod_m E_m\). To prove that \(E = \prod_m E_m\), let \((a, b)\) be any point of \(CE\). There is an \(\epsilon\) such that if \((by) < \epsilon\), \((a, y)\) is in \(CE\). In order that \((a, b)\) be in \(A_i(r_m)\) it is necessary that \((bb_i) < r\). Choose \(r_m < \epsilon/2\). If \((bb_i) < r_m\), and \((b_i y) < r_m\), it follows that \((by) < \epsilon\), and consequently \((a, y)\) is in \(CE\). Therefore if \((bb_i) < r_m\), \(A_i(r_m)\) cannot contain \((a, b)\) because all points on the open vertical interval of center \((a, b_i)\) and radius \(r_m\) are in \(CE\). As has been mentioned, \(A_i(r_m)\) cannot contain \((a, b)\) if \((bb_i) \geq r_m\). Therefore for \(r_m\) chosen as it has been, \((a, b)\) is not in \(E_m\) and consequently not in \(\prod_m E_m\). It follows that \(E = \prod_m E_m\), which completes the proof of the theorem.

3. Horizontal sections of the type \(\alpha\)

**Lemma 1.** Let \(G\) be a set lying on a horizontal section of \(A \times B\). If \(G\) is an \(O_{\alpha}(\alpha \geq 0)\) or an \(F_{\alpha}(\alpha > 0)\), \(G(r)\) is an \(O_{\alpha}\) or an \(F_{\alpha}\). If \(G\) is analytic, \(G(r)\) is analytic.

The first part of the lemma is true for sets \(O_0\) and sets \(O_1\), and can be shown to be true for sets \(O_\alpha\) by transfinite induction. The latter part of the lemma may be readily proved from the definition of an analytic set.

* For a discussion of sets \(F_{\alpha}\) and \(O_\alpha\), see de la Vallée Poussin, *Intégrales de Lebesgue*, p. 132.
Theorem 3. If the horizontal sections of $E$ are $O_a$'s and the vertical sections are closed $I$-sets, then $E$ is an $F_{a+1}$.

This follows from Theorem 2. $\mathcal{M}$ is here the family of $O_a$'s in horizontal sections of $A \times B$. $\mathcal{M}_a$ is the same family and by the preceding lemma $\mathcal{M}_a(r)$ is a family of $O_a$'s in $A \times B$, as is $[\mathcal{M}_a(r)]_0$. Therefore $[\mathcal{M}_a(r)]_0$ is a family of $F_{a+1}$'s in $A \times B$ and $E$ must be an $F_{a+1}$.

By taking complements the following is proved:

Theorem 4. If the horizontal sections of $E$ are $F_a$'s and the vertical sections are open sets whose complements are $I$-sets, then $E$ is an $O_{a+1}$.

That the change in classification mentioned in Theorems 3 and 4 actually may occur, is shown by the following plane set. On the line $y = x$, take a set $E$ which is an $O_{a+1}(\alpha \geq 1)$* and an $O$ of no lower class.† Then $E = \sum A_i$ where $A_i$ is an $F_a$ at most. At each point of $A_i$ erect a vertical interval of length $1/i$, closed at the end touching $y = x$ and open at the other end, and denote the set thus obtained by $H$. Horizontal sections of $H$ are $F_a$'s. This can be seen as follows. If $L$ is any horizontal line and $p$ any point on this line above the line $y = x$, vertical intervals from only a finite number of the sets $A_i$ can cut $L$ to the left of, or at, $p$. Therefore the points of $H$ on $L$ to the left of or at $p$ must form an $F_a$. This is true however close $p$ may be to $y = x$. Let $p_n$ be a sequence of points on $L$ above $y = x$, approaching $y = x$. Let $E_n$ be the points of $(CH) \cdot L$ to the left of or at $p_n$. Then $E_n$ is an $O_a$, and $\sum E_n$ is an $O_a$. Therefore the points of $CH$ on $L$ to the left of $y = x$ form an $O_a$, and consequently the points of $H$ on $L$ form an $F_a$. Since $\alpha \geq 1$, it does not matter whether or not the intersection of $L$ and $y = x$ is in $H$. Although horizontal sections of $H$ are $F_a$'s, the set $H$ itself must be at least an $O_{a+1}$ since $y = x$ cuts it in an $O_{a+1}$.

Denote by $R$ that part of the complement of $H$ which lies on or above $y = x$. Horizontal sections of $R$ are $O_a$'s but $R$ itself is an $F_{a+1}$ at least, since $y = x$ cuts it in an $F_{a+1}$. By Theorem 3, $R$ is an $F_{a+1}$ at most. This example shows that under the hypothesis of Theorem 3, it is impossible to draw a stronger conclusion on the $F$ classification of $E$ than the one there given.

Theorem 5. If the horizontal sections of $E$ are $O_a$'s and the vertical sections are open sets whose complements are $I$-sets, then $E$ is an $O_{a+2}$.

This follows from Theorem 1. The family $\mathcal{M}$ is here the family of sets $O_a$ on horizontal sections of $A \times B$. $\mathcal{M}_a$ is then the family of sets $F_{a+1}$ on hori-

* A single open vertical interval furnishes an example in case $\alpha = 0$.
† For a proof of the existence of functions of all classes (which proves the existence of sets of all classes) see de la Vallée Poussin, *Intégrales de Lebesgue*, p. 145 ff.
orizontal sections. \( \mathcal{M}_u(r) \) contains only sets \( F_{a+1} \) in \( A \times B \) by Lemma 1. Therefore \( \left[ \mathcal{M}_u(r) \right]_e \) contains only sets \( O_{a+1} \).

By taking complements there is proved

**Theorem 6.** If the horizontal sections of \( E \) are \( F_a \)'s and the vertical sections are closed \( I \)-sets, then \( E \) is an \( F_{a+1} \).

Whether or not the classification may actually be increased by two under the hypothesis of Theorems 5 and 6 is an open question. That an advance of one may occur is shown by the following plane set. Construct the set \( H \) as in the preceding example, except that the vertical intervals are now to be closed instead of half closed. Horizontal sections of this set are \( F_a \)'s as before and the set itself must be an \( O_{a+1} \) exactly as \( H \) was.

The two following theorems may be proved directly or as a result of the preceding theorems on sets \( O_a \) and \( F_a \).

**Theorem 7.** If horizontal sections of \( E \) are \( A_a \)'s and vertical sections of \( E \) are all closed \( I \)-sets or all open sets whose complements are \( I \)-sets, then \( E \) is an \( A_{a+1} \).

4. Analytic or measurable horizontal sections

If \( \mathcal{M} \) is the family of analytic sets, \( \mathcal{M}_e \) and \( \mathcal{M}_i \) are the same family, from which we have the following theorem.

**Theorem 8.** If horizontal sections of \( E \) are analytic and vertical sections of \( E \) are all closed \( I \)-sets or all open sets whose complements are \( I \)-sets, \( E \) is analytic.

If there is a theory of measure in the space under consideration, as for example in the plane, a theorem similar to Theorem 8 is true for measurable sets.

5. The set \( E_e \)

Let \( E_e \) denote the subset of \( E \) each point of which lies on a closed vertical interval of radius exactly \( e \), which contains only points of \( E \).

For the two theorems of this section the conditions on \( A \) and \( B \) are that they are metric, that every closed sphere in \( B \) is compact and that inner points of a closed sphere in \( B \) are dense on the sphere.

**Theorem 9.** If horizontal and vertical sections of \( E \) are closed, \( E_e \) is closed.

Let \((a_n, b_n)\) be a sequence of points of \( E_e \) converging to a limit point \((a, b)\). Each \((a_n, b_n)\) lies on a closed vertical interval, \( V_e \), containing only points of \( E \), of radius \( e \), and of center \((a_n, c_n)\). An infinite number of points \( b_n \) are such that \((bb_n) < e\), where \( e \) is any positive number. The fact that \((b_n c_n) \leq e\) for all \( n \)

* For a discussion of sets \( A_a \), see de la Vallée Poussin, *Intégrales de Lebesgue*, p. 135.
implies that for an infinite number of points $c_n, (bc_n) < e + \epsilon$, and by hypothesis, these points have some limit point $c$ such that $(bc) \leq e$. Let $V_\epsilon$ be the closed vertical interval of center $(a, c)$ and radius $\epsilon$. Let $(a, y)$ be any point such that $(cy) < \epsilon$, that is, any point on the interior of the vertical interval $V_\epsilon$. It will now be shown that there is an $n$ for which $(a_n, y)$ is in $E$ and $(a_n a) < \eta$, where $\eta$ is any positive number. Consider the $n$'s for which $(a_n a) < \eta$ and select from this group an $n$ such that $(c_n c) < \epsilon - (cy)$, that is, such that $(c_n c) + (cy) < \epsilon$. It follows that $(c_n y) < \epsilon$ and therefore the point $(a_n, y)$ is in $E$. On the horizontal section $(y=y)$ there is thus a sequence of points of $E$ approaching $(a, y)$, and since horizontal sections of $E$ are closed, $(a, y)$ must be in $E$. Therefore, all inner points of the vertical interval $V_\epsilon$ are in $E$. Because of the hypothesis on the space $B$ and the fact that vertical sections of $E$ are closed, it follows that every point of $V_\epsilon$ is in $E$. Since $(bc) \leq e$, the point $(a, b)$ is in $V_\epsilon$, which proves that every limit point of $E_n$ is in $E_\epsilon$.

For Theorem 10, $\epsilon$ is required to have the further property that any point $p$ of $B$ on a closed sphere of radius $> r$ is on a sphere of radius exactly $r$, which is contained in the first sphere.

**Theorem 10.** If horizontal and vertical sections of $E$ are closed and each point of $E$ lies on a closed vertical interval containing only points of $\epsilon$, then $E$ is an $F_\sigma$.

Let $(r_i)$ be a sequence of positive numbers approaching 0. Any point in $E_r$ is in $E_r$, if $r_i \leq r$. Hence $E = \bigcap E_{r_i}$, and since $E_r$ is closed, $E$ is an $F_\sigma$.

6. **Uniformity properties**

**Definition 5.** A point $p$ is said to be a point of uniformity of $E$ if, for some open sphere $S$ in $A \times B$ of center $p$ and for some $\epsilon$, $E \cdot S = E_\epsilon \cdot S$.

**Definition 6.** A point $p$ is said to be a point of uniform separation of $E$ if it is a point of uniformity of $CE$.

The points of uniformity of $E$ form an open set.

For the theorems of this section, $A$ and $B$ are metric separable spaces. In addition it is required that every sphere in $B$ be totally limited* and that $B$ have the property stated just before Theorem 10.

**Theorem 11.** If the vertical sections of $E$ are open, a necessary and sufficient condition that $E$ be an $O_\sigma$ is that the set of points of non-uniform separation of $E$ in $E$ be an $O_\sigma$ and that horizontal sections of $E$ be $O_\sigma$'s.

The necessity will first be demonstrated. If $E$ is an $O_\sigma$ its horizontal sections are $O_\sigma$'s. Let $N$ denote the set of points of non-uniform separation of $E$.

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Since \( N \) is closed, it follows that \( E \cdot N \) is an \( O_a \) if \( \alpha \) is greater than 0. For \( \alpha = 0 \), the necessity is obvious.

The sufficiency will now be demonstrated. Let \( \varphi \) be any point of \( E \) which is a point of uniform separation of \( E \). For some open sphere \( S \) of center \( \varphi \) and some \( \varepsilon, S \cdot (CE) = S \cdot (CE) \). Let \( r \) be a sequence of positive numbers approaching 0 and let \( (y = b_\varphi) \) be an enumerable everywhere dense set of horizontal sections of \( A \times B \). Let \( r_m \) be a fixed element of the sequence \( r \) and let \( \varepsilon \) be the smaller of the two numbers \( \varepsilon/4 \) and \( r_m/4 \). Let \( S_1 \) be a sphere with the same center as that of \( S \) and with radius \( 2\varepsilon \) larger than that of \( S \). Because the projection of \( S_1 \) in \( B \) is a sphere in \( B \) it is totally limited in \( B \) by hypothesis. There exists then a finite set of horizontal sections \( (y = h_i) \) such that every point of \( S_1 \) is a distance less than \( \varepsilon \) from some one of them. Let \( K_i \) be the projection of \( S \cdot E \cdot (y = h_i) \) on \( (y = b_\varphi) \). Denote by \( A_i \) the product of \( S \cdot E \cdot (y = h_i) \) and all \( K_i \)'s such that \( h_i/h_i < b_\varphi \).

In order to show that \( A_i \) is in \( S \cdot E \), let \( (a, y) \) be any point of \( A_i \) and suppose that \( (a, y) \) is in \( C(S \cdot E) \). It is then on a closed vertical interval \( V_\varepsilon \), of radius \( \varepsilon \), containing only points of \( C(S \cdot E) \). By the hypothesis on the space \( B \), the point \( (a, y) \) is also on a vertical interval \( V_\varepsilon \), of radius exactly \( \varepsilon \), \( V_\varepsilon \) being contained in \( V_\varepsilon \). The interval \( V_\varepsilon \) contains only points of \( C(S \cdot E) \). Let \( (a, b) \) be the center of \( V_\varepsilon \). Since \( (a, b) \) must be in \( S_1 \) there is some \( h_n \) such that \( h_n < \varepsilon \). The point \( (a, h_n) \) is therefore in \( C(S \cdot E) \). From the relations \( (b, y) < r_m/2 \) and \( (b, y) < \varepsilon \), it follows that \( (b, y) < r_m/2 \). Since \( (b, y) < \varepsilon \), it follows that \( (h_n, b_\varphi) < r_m + \varepsilon < r_m \). The point \( (a, h_n) \), being in \( C(S \cdot E) \), cannot be in \( S \cdot E \cdot (y = h_n) \). Therefore \( (a, b_\varphi) \) is not in \( K_n \) nor in \( A_i \).

Neither \( (a, b_\varphi) \) nor \( (a, y) \) can then be in \( A_i \). From this contradiction it follows that \( A_i \) is in \( S \cdot E \).

The proof showing that each point of \( S \cdot E \) is in some \( A_i \) is analogous to the proof of a similar proposition given in the demonstration of Theorem 1, and will not be repeated here. Assuming this to be proved, \( S \cdot E = \sum A_i \cdot r_m/2 \). The set \( A_i \) is a finite product of \( O_a \)'s and must be an \( O_a \).

Each \( A_i \) must then be an \( O_a \), and therefore \( S \cdot E \) is an \( O_a \).

Every point of uniform separation of \( E \) is therefore the center of an open sphere \( S \) such that \( S \cdot E \) is an \( O_a \). By Lindelöf's* theorem an enumerable set \( (S_1) \) of such spheres cover the points of uniform separation of \( E \) in \( E \). The set \( N \) of points of non-uniform separation of \( E \) in \( E \) is an \( O_a \) by hypothesis. Therefore \( E = \sum A_i \cdot E + N \) is an \( O_a \).

* This holds since \( A \times B \) is separable when \( A \) and \( B \) are. See the previously cited paper by Kuratowski and Ulam.
Theorem 12. If the vertical sections of $E$ are closed, a necessary and sufficient condition that $E$ be an $F_\alpha(\alpha > 0)$ is that the set of points of non-uniformity of $E$ in $E$ be an $F_\alpha$ and that horizontal sections of $E$ be $F'\$s. The necessity will first be demonstrated. Horizontal sections of $E$ are $F'_\alpha$'s since they are the products of $E$ and horizontal sections of the space $A \times B$. The set $N$ of points of non-uniformity of $E$ is closed and the product of $N$ and $E$ must be an $F_\alpha$.

The sufficiency will now be shown. Let $p$ be any point of uniformity of $E$. It is a point of uniform separation of $CE$ and for some sphere $S$ of center $p$, $(CE) \cdot S$ is an $O_\alpha$ by Theorem 11. As before, an enumerable number of such spheres, $S_i$, cover the points of uniformity of $E$. Since the points of $CE$ in $\sum S_i$ are an $O_\alpha$, the points of $E$ in $\sum S_i$ form an $F_\alpha$. The set $N$ of points of non-uniformity of $E$ in $E$ form an $F_\alpha$ by hypothesis. Since $E = \sum E \cdot S_i + N$, $E$ is an $F_\alpha$.

The theorem is not true for $\alpha = 0$.

7. Gratings and categoricity

In this section $A$ and $B$ are to be metric, separable, locally compact* spaces. In such spaces the complement of a set of the first category is of the second category, and open sets are of the second category. These propositions may be proved by a method similar to the method used for proving them in euclidean space. It is necessary to make use of the fact that every monotonic decreasing sequence of non-null compact spheres has a non-null product.$^\dagger$

Definition 7. If a horizontal section $(y = b)$ contains a set $H$ of the second category [in $(y = b)$], such that $H(r)$ is in $E$ for some $r$, $(y = b)$ is said to have property $C$ with respect to $E$.

Definition 8. If a horizontal section $(y = b)$ contains a set $H$, in and everywhere dense in a set $O$ [in and open in $(y = b)$] such that $H(r)$ is in $E$ for some $r$, $(y = b)$ is said to have property $D$ with respect to $E$.

The set $H(r)$ of Definition 8 is called a grating. A point $p$ is said to be within the grating if $p$ is in $O(r)$. A point $p$ is on the grating if it is in $H(r)$. Property $C$ implies property $D$ since a set of the second category must contain a subset everywhere dense in some open set.

Lemma 2. If $K$, a set of vertical sections, is of the second category and if each $K \in K$ contains a vertical interval including only points of $E$, then some horizontal section has properties $C$ and $D$ with respect to $E$; furthermore the center of one of the vertical intervals is on the grating (of property $D$).

* For a definition of this term see Fréchet, Les Espaces Abstraits, p. 223.
† See Banach, Théorie des Opérations Linéaires, pp. 13 and 14.
‡ If $p$ is on a grating it is within the grating.
Let $K_{a\varepsilon}$ be the set of vertical sections containing vertical intervals of $E$ of radii $>3\varepsilon$, where $\varepsilon$ has been so chosen that $K_{a\varepsilon}$ is of the second category. Let $V_{a\varepsilon}$ denote an individual one of these vertical intervals of radius $>3\varepsilon$ and let $\sum V_{a\varepsilon}$ denote the points in all such intervals. From each $V_{a\varepsilon}$ form a vertical interval $V_\varepsilon$ of radius exactly $\varepsilon$ with the same center as $V_{a\varepsilon}$. Each interval $V_\varepsilon$ consists entirely of points of $E$. Let $(b_i)$ be an enumerable set of points everywhere dense in $B$, and let $B_i$ be a sphere in $B$ of center $b_i$ and radius $2\varepsilon$. Let $A_i$ be the subset of $A$ such that for each $a \in A_i$, there is some $V_\varepsilon$ and a corresponding $V_{a\varepsilon}$ for which $V_\varepsilon < a \times B_i < V_{a\varepsilon}$.

It will now be shown that for each $V_\varepsilon$ and corresponding $V_{a\varepsilon}$, of center $(a, b)$, there is an $i$ such that $V_\varepsilon < a \times B_i < V_{a\varepsilon}$, and consequently that $\sum A_i$ is the set in which the sections of $K_{a\varepsilon}$ cut $A$. In order to do this, choose $i$ such that $(b_i, b_i) < \varepsilon$. For each point $(a, y)$ of $V_{a\varepsilon}$, $(b_i, y) < \varepsilon$. By the triangle axiom, $(b_i, y) < 2\varepsilon$. This shows that the vertical interval $a \times B_i$ of center $(a, b_i)$ and radius $2\varepsilon$ includes $V_\varepsilon$. It will now be shown that $a \times B_i < V_{a\varepsilon}$. If $(a, y)$ is any point of $a \times B_i$, $(b_i, y) < 2\varepsilon$. Since $(b_i, b_i) < \varepsilon$, it follows from the triangle axiom that $(b_i, y) < 3\varepsilon$ which is the condition that $(a, y)$ be in $V_{a\varepsilon}$.

Since $\sum A_i$ is the set in which the sections of $K_{a\varepsilon}$ cut $A$, it follows that $\sum A_i$ is of the second category and, consequently, that some particular $A_i$, say $A_n$, is of the second category. The section $(y = b_n)$ has property $C$ with respect to $E$, for the set $A_n \times b_n$ is of the second category in $(y = b_n)$, and each point of $A_n \times b_n$ is the center of a vertical interval of radius $2\varepsilon$ which includes only points of $E$. The section $(y = b_n)$ must then also have property $D$ with respect to $E$. Since each of these vertical intervals contains an interval $V_\varepsilon$, it must contain the center of this interval $V_\varepsilon$ which is the center of the corresponding original interval $V_{a\varepsilon}$. Therefore the center of one of the original intervals is on each vertical interval of the grating.

It is evident that when $A$ and $B$ have similar properties, the parts played by horizontal and vertical sections in any theorem may be interchanged.

**Definition 9.** A point is said to be of the second category with respect to $E$ if every neighborhood of the point contains a subset of $E$ of the second category.

The set of all points of the second category with respect to $E$ is denoted by $E_{se}$. The set $E_{se}$ is closed.

A necessary and sufficient condition that $E$ be of the second category is that $E_{se}$ be of the second category. This implies that if $E$ is of the second category, it must be of the second category at a set everywhere dense in an open set, and since $E_{se}$ is closed it must be of the second category at each point of an open set.†

† See Banach, *Théorie des Opérations Linéaires*, p. 13 and the reference there given.
Theorem 13. If vertical sections of $R$ are I-sets and horizontal sections of $T$ are I-sets, and $R \cdot T = 0$, then $R' \cdot T + R \cdot T'$ is of the first category in $A \times B$.

It is sufficient to show that $R \cdot T'$ is of the first category in $A \times B$. Assume that $R \cdot T'$ is of the second category in $A \times B$. It must then be of the second category at every point of a set $O$, open in $A \times B$, which implies that $R$ and $T$ are both dense in $O$. It follows from the hypothesis, that on each vertical section containing a point of $R \cdot T' \cdot O$, there is a vertical interval $V$ containing only points of $R$ and such that $V$ is in $O$. Since the vertical sections containing points of $R \cdot T' \cdot O$ form a set of the second category, Lemma 2 may be applied. By this lemma, there is a horizontal section $L$ containing a set $H$ everywhere dense in $O^*$ ($O^*$ a set open in $L$) such that $H(r)$ is in $R$. The set $H(r)$ is also in $O$ since the intervals $V$ from which it is constructed are in $O$. Suppose there is a point $p$ of $T$ within $H(r)$ and let the horizontal section containing $p$ be $L^*$. The section $L^*$ must contain an inner point (with respect to $L^*$) of $T$ which lies in $O^*(r)$. But this is impossible because $L^* \cdot H(r)$ is dense in $L^* \cdot O^*(r)$ and $H(r)$ contains only points of $R$. There can be, then, no points of $T$ within $H(r)$, but this is a contradiction since $T$ must be dense in $O$. Therefore $R \cdot T'$ is of the first category in $A \times B$.

Corollary 1. If vertical sections of $E$ are I-sets and horizontal sections of $CE$ are I-sets, then $E' \cdot (CE) + E \cdot (CE)'$ is of the first category in $A \times B$.

This follows immediately from the theorem and the fact that $E \cdot (CE) = 0$.

Corollary 2. If vertical sections of $E$ are I-sets and horizontal sections of $CE$ are I-sets, there is an inner point of either $E$ or $CE$ in every set $O$, open in $A \times B$.

This follows from Corollary 1. Points not belonging to $E' \cdot (CE) + E \cdot (CE)'$ are inner points either of $E$ or of $CE$ and this set is everywhere dense in $A \times B$.

Theorem 14. If horizontal and vertical sections of $E$ are I-sets and horizontal sections of $CE$ are I-sets, then $E$ is an I-set in $A \times B$.

Let $p$ be any point of $E$ lying on a horizontal section $L$ and let $O$ be any open set in $A \times B$ containing $p$. It is necessary to show that $O$ contains an inner point of $E$. By hypothesis $L \cdot E$ contains a set $O^*$ open in $L$. The set $O^*$ is of the second category in $L$, and each vertical section $K$ cutting $O^*$ must contain a vertical interval $V$ including only points of $E$ and lying in $O$. From these $V$'s, there may be formed a grating $H(r)$ containing only points of $E$ and contained in $O$. No point of $CE$ can be within this grating because hori-

† If this were not true, $R \cdot T' \cdot O$ would be of the first category in $A \times B$. 

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horizontal sections of $CE$ are $I$-sets. The argument is similar to the one in Theorem 13 and will not be repeated.

Kuratowski and Ulam have a theorem similar to the following:

**Theorem 15.** If $E$ is a set whose horizontal sections are $I$-sets, and $O$ is an open set in $A \times B$, a necessary and sufficient condition that $E \cdot O$ be dense in $O$, is that the vertical sections $L$ for which $L \cdot E \cdot O$ is not dense in $L \cdot O$ form a set $\mathcal{L}$ of the first category.

The sufficiency of the condition follows from the fact that if vertical sections $K$, such that $K \cdot E \cdot O$ is dense in $K \cdot O$, form a set complementary to a set of the first category, they are everywhere dense and therefore the points in $K \cdot E \cdot O$, considering all $K$, must be dense in $O$.

In order to prove the necessity, let $E \cdot O$ be dense in $O$ and suppose the set $\mathcal{L}$ to be of the second category. On each $L$ there is a vertical interval which is in $O$ and which contains no points of $E$, that is, it contains only points of $CE$. By Lemma 2, $CE$ must contain a grating which is in $O$. But this grating can have within it no point of $E$ since horizontal sections of $E$ are $I$-sets. This contradicts the hypothesis that $E$ is dense in $O$, and the theorem is proved.

**Theorem 16.** If vertical sections of $R$ are open, horizontal sections of $T$ are $I$-sets and $RT = 0$, then $R \cdot T'$ lies on a set $K$ of vertical sections, which is of the first category.

Suppose that $K$ is of the second category. Each $K \in K$ contains a vertical interval with a point of $R \cdot T'$ as center and containing only points of $R$. By Lemma 2, there is a grating, composed of points of $R$, containing a point of $R \cdot T'$ on its interior. This is impossible because horizontal sections of $T$ are $I$-sets.

**Corollary 3.** If vertical sections of $E$ are closed and horizontal sections of $E$ are $I$-sets, then $(CE) \cdot E'$ lies on a set $K$, of vertical sections, which is of the first category.

This corollary may be proved by replacing $R$ and $T$ by $CE$ and $E$ in Theorem 16.

If there exists a set $K$ of vertical sections, and a set $\mathcal{L}$ of horizontal sections, so that every point of $E$ lies either on a member of $K$ or a member of $\mathcal{L}$, the set $E$ is said to lie on the set $K$ plus the set $\mathcal{L}$. This language is used to distinguish this case from the case in which every point of $E$ lies both on a member of $K$ and on a member of $\mathcal{L}$; in this latter case $E$ is said to lie on the set $K$ and on the set $\mathcal{L}$.
Corollary 4. If vertical sections of \( R \) are open, horizontal sections of \( T \) are open and \( R \cdot T = 0 \), then \( R' \cdot T + R \cdot T' \) lies on a set \( \mathcal{K} \) of vertical sections plus a set \( \mathcal{L} \) of horizontal sections, both \( \mathcal{K} \) and \( \mathcal{L} \) being of the first category.

This is a slightly stronger conclusion than that of Theorem 13, made possible by the stronger hypothesis given here.

Corollary 5. If vertical and horizontal sections of both \( R \) and \( T \) are open, and \( R \cdot T = 0 \), then \( R' \cdot T + R \cdot T' \) lies on a set \( \mathcal{K} \) of vertical sections and a set \( \mathcal{L} \) of horizontal sections, both \( \mathcal{K} \) and \( \mathcal{L} \) being of the first category.

In Corollary 5, the projection of \( R' \cdot T + R \cdot T' \) on any horizontal or vertical section must be of the first category in the section. This is not necessarily true in Corollary 4.

It will be assumed in the following theorem that the space \( A \) is dense in itself in order that the set \( \mathcal{E} \) there considered may be perfect. It will also be assumed that \( A \) and \( B \) have the properties necessary to apply Theorem 10.

A point of closure of a set \( E \) is a point in some neighborhood of which \( E \) is closed.

Theorem 17. If horizontal and vertical sections of \( E \) are closed and each point of \( E \) lies on a closed horizontal interval containing only points of \( E \), points of closure of \( E \) in \( E \) are dense on \( E \).

By Theorem 10, \( E \) is an \( F_\sigma \). It must then be an \( F_\sigma \) in \( \mathcal{E} = E + E' \). It is necessary to prove that \( \mathcal{E} = E \) is nowhere dense in \( \mathcal{E} \) or, in other words, that limit points of \( E \) not in \( E \) are not dense on \( E \). If it will be shown that \( \mathcal{E} = E \) is of the first category. By Corollary 3, \( \mathcal{E} - E \) (which is the same as \( (CE) \cdot E' \)) lies on a set \( \mathcal{K} \) of vertical sections, of the first category. Let \( A_0 \) be the set of points in which the sections of \( \mathcal{K} \) cut \( A \). The set \( A_0 = \sum A_i \) where each \( A_i \) is nowhere dense in \( A \). Let \( R_i \) be the points of \( \mathcal{E} - E \) lying on those sections, of the set \( \mathcal{K} \) which cut \( A \) in \( A_i \). If \( R_i \) were dense on some portion of \( E \), it would have to have as a limit point every point of some horizontal interval. This follows from the hypothesis that every point of \( E \) lies on a horizontal interval. This is impossible since the projection, \( A_i \), of \( R_i \) on \( A \), would then be everywhere dense in some open set in \( A \). Therefore \( R_i \) is nowhere dense in \( E \), and \( \mathcal{E} = E_\sigma \) is of the first category in \( E \). It follows that \( \mathcal{E} = E \) is nowhere dense in \( \mathcal{E} \), for \( \mathcal{E} = E \) is a \( G_\delta \) and if a \( G_\delta \) is of the first category, it is nowhere dense. 

† In this particular case \( \mathcal{E} = E' \).
‡ For example see Blue, Mathematische Annalen, vol. 102, p. 627, in the proof of Theorem 1.
8. Applications

The spaces $A$ and $B$ are restricted here in the same manner as in §7. An interesting application of Corollary 5 is in the proof of the following result of Baire†:

If $f(x, y)$ is a real-valued function defined on the space $A \times B$ and is continuous in each of the variables separately, points of discontinuity of $f(x, y)$ lie on a set of vertical sections and a set of horizontal sections, both sets of sections being of the first category.

Let $(r_i)$ be the set of rational numbers. Let $R_i$ be the points of $A \times B$ at which $f(x, y) > r_i$ and let $T_i$ be the set at which $f(x, y) < r_i$. From the properties of continuous functions, $R_i$ and $T_i$ have open horizontal and vertical sections and are disjoined. It follows that $R_i \cdot T_i' + R_i' \cdot T_i'$ lies on a set of horizontal and a set of vertical sections of the first category, and the same is true of $\sum_{i>j}(R_i' \cdot T_j + R_i \cdot T_j')$, the sum being taken only over pairs of $i$ and $j$ for which $r_i < r_j$. The points of this sum are the points of discontinuity of $f(x, y)$.

Applying Corollary 4 in the same way to a function upper semi-continuous in one variable and lower semi-continuous in the other, it may be shown that the set of discontinuities, $E$, of such a function lies on a set of horizontal sections plus a set of vertical sections of the first category. The set $E$ is of the first category, but not all sets of the first category lie on a set of horizontal plus a set of vertical sections of the first category. This conclusion therefore contains a result not given by Kempisty.‡

† Acta Mathematica, 1899, p. 94.