CYCLIC FIELDS OF DEGREE EIGHT*

BY

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1. Introduction. Let $F$ be any non-modular field, $C$ be an algebraic extension of degree $n$ of $F$. Then $C = F(x)$ is the field of all rational functions with coefficients in $F$ of a root $x$ of an equation $\phi(\omega) = 0$ which has coefficients in $F$, degree $n$, and transitive group $G$ for $F$.

The problem of the construction of all equations of degree $n$ and group $G$ is evidently equivalent to the problem of the construction of all corresponding fields $C$. Moreover the construction of a set of canonical equations $\psi(\omega) = 0$ with the property that every $C = F(x)$ of degree $n$ and group $G$ is equal to an $F(y)$ defined by a $\psi(\omega) = 0$ provides a solution of both problems.

One of the most important problems in the algebraic theory of fields is the construction of all cyclic fields of degree $n$ over $F$. This is the case where $G$ consists of the $n$ distinct powers $S_i (i = 0, 1, \cdots, n - 1)$ of a single substitution $S$. In this case $G$ is also the group of all automorphisms of $C$. Moreover this problem has been reduced to the case $n = p^s$, $p$ a prime.

Cyclic fields of degree 2, $2^2$ have been constructed.† In the present paper we shall use purely algebraic methods to construct all cyclic fields of degree $2^8 = 8^1$ over any non-modular field $F$.

2. General theory of cyclic fields. Let $F$ be any non-modular field and let $C$ be a cyclic field of degree $n$ over $F$. Then if

$$n = p_1^{s_1} \cdot p_2^{s_2} \cdots p_r^{s_r},$$

where the $p_i$ are distinct primes, it is well known that $C$ is the direct product

$$C = C^{(1)} \times C^{(2)} \times \cdots \times C^{(t)}$$

of cyclic fields $C^{(i)}$ of degree $p_i^{s_i}$ over $F$. Conversely every direct product (2) is a cyclic field of degree $n$ over $F$. It is thus certain that the problem of constructing all cyclic fields of degree $n$ over $F$ is equivalent to the corresponding problem for the case $n = p^s$.

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† Cf. §2.
‡ Cyclic fields of degree eight have been considered by F. Mertens in the Wiener Sitzungsberichte, vol. 125 (1916), pp. 741–831. But he considered algebraic number fields, used the arithmetic theory of ideals, and did not give very explicit results. His method is not at all applicable to the case we are considering (where $F$ is a general field). Moreover I believe the results obtained here are more explicit and give a more definite construction for $C$ even for the cases considered by Mertens.
Let then $C = C_e$ have degree $n = p^e$, $p$ a prime. It is well known that we may define a chain of fields

$$ C_e > C_{e-1} > \cdots > C_1 > C_0 = F, $$

where $C_e$ is cyclic of degree $p^e$ over $F$, cyclic of degree $p$ over $C_{e-1}$. In fact let $S$ be the automorphism of $C$ generating its group $G$ of automorphisms. Then this group of order $n$ is given by

$$ G_e = \langle I, S, S^2, \cdots, S^{n-1} \rangle, S^n = I. $$

But if $T = S^{p^{e-1}}$ then

$$ H = \langle I, T, T^2, \cdots, T^{p^{e-1}} \rangle $$

is an invariant sub-group of $G$ of index $p^{e-1}$ defining a sub-field $C_{e-1}$ of degree $p^{e-1}$ and with Galois group

$$ G_{e-1} = \langle I, \sigma, \sigma^2, \cdots, \sigma^{m-1} \rangle, m = p^{e-1}, $$

isomorphic with $G_e$ but with $T = S^m$ in $G$ corresponding to the identity of $G_{e-1}$.

We may now consider every $C_e$ as a cyclic field of degree $p$ over a cyclic field $C_{e-1}$ of degree $p^{e-1}$ to obtain some of the properties of $C_e$. But if $C_e$ is cyclic of degree $p$ over $C_{e-1}$ which is cyclic of degree $p^{e-1}$ over $F$, then it is not necessarily true that $C_e$ is cyclic over $F$. Thus we shall also require a consideration of further properties.

We are interested here only in the case $p = 2$. Let $C$ be a cyclic field of degree $n = 2^e$, $e > 1$ over $F$, and let $D$ be its uniquely defined sub-field of degree $m = 2^{e-1}$. Then $C$ is a quadratic field over $D$,

$$ C = D(x), x^2 = a \text{ in } D, $$

where $1, x$ are linearly independent with respect to $D$. The substitution $S$ generating the cyclic group of $C$ has order $n$ and $D$ consists of all quantities $d$ of $C$ such that

$$ ds^m = d \quad (m = 2^{e-1}). $$

For convenience of notation we shall write

$$ c^{sk} = c^{(k)}, $$

so that $c^{(m)} = c$ whenever $c$ is in $D$ but not otherwise. Then

$$ [x^{(m)}]^2 = a^{(m)} = a = x^2, $$

and $x^{(m)} = \pm x$. But $x$ is not in $D$. Hence

$$ x^{(m)} = - x. $$
In particular let \( x' = \alpha + \beta x \) where \( \alpha \) and \( \beta \) are in \( D \). Then \((x')^2 = a' = \alpha^2 + \beta^2 a + 2\alpha\beta x \). But \( a' \) is in \( D \) so that \( 2\alpha\beta = 0 \). If \( \beta = 0 \) then \( x' = \alpha \) is in \( D \) and \((x')^{(n-1)} = x = \alpha^{(n-1)} \) is in \( D \), a contradiction. Hence \( \beta \neq 0 \), \( \alpha = 0 \) and
\[
(11) \quad x' = \beta x, \quad \beta \text{ in } D.
\]

It is obvious that \( D(x) = D(bx) \) for every non-zero \( b \) of \( D \). Hence all of the above properties as well as those we may derive later will hold for any \( bx \) taken as the quantity generating \( C \), a quadratic field over \( D \).

We shall assume first that \( n = 2 \). Then \( D = F \) and, since \( x \) is not in \( D \), the field \( F(x) = D(x) = C \) is a quadratic field over \( F \) generated by \( x \). Let next
\[
m = 2^{e-1} = 2g, \quad g \geq 1,
\]
so that \( n \geq 4 \). Then \( D = K(y), y^2 = \alpha \) in \( K \), is a quadratic field over the field \( K \) of all quantities \( k \) of \( C \) such that
\[
k'^2 = k.
\]

The field \( F(x) \) is a sub-field of \( C = D(x) \). But \( x \) is not in \( D \). Hence \( F(x) \) is a quadratic field over \( F \) generated by \( x \). Let next \( m = 2^{e-1} = 2g \), \( g \geq 1 \), so that \( n = 4 \). Then \( D = K(y), y^2 = \alpha \) in \( K \), is a quadratic field over the field \( K \) of all quantities \( k \) of \( C \) such that
\[
k'^2 = k.
\]

Theorem 1. Let \( C \) be a cyclic field of degree \( n = 2m \) over \( F \), \( C = D(x) \) where \( D \) is a cyclic sub-field of \( C \) of degree \( m = 2^{e-1} \) over \( F \) so that \( x \) may be chosen so that
\[
x^2 = a \quad \text{in } D.
\]

Then \( x' = \beta \cdot x \) where \( \beta \) is in \( D \) and has the property that \( x^{(m)} = -x \). Moreover this latter property implies that \( F(x) = C \), \( F(a) = D \).

Suppose that \( F(a) < D \). Then \( a \) is in a proper sub-field of \( D \). But \( D \) is cyclic and its maximal proper sub-field \( K \) contains every proper sub-field of \( D \). Hence \( a \) is in \( K \), \( a^{(e)} = a \), \( [x^{(e)}]^2 = a^{(e)} = x^2 \). Then \( x^2 = \pm x \), \( x^{(m)} = [x^{(e)}]^{(m)} = x \), a contradiction. Hence \( F(a) = D \) and we have

Theorem 2. Let \( x_0 = bx \) where \( b \neq 0 \) is in \( D \). Then \( F(x) = F(x_0) = C \).

The condition \( x' = \beta \cdot x \) imposes two restrictions on \( \beta \). The first is obviously \( x'^2 = \beta^2 \cdot x^2 = (x^2)' = a' = \beta^2 \cdot a \), a necessary and sufficient condition that \( x' \) shall actually equal \( \beta \cdot x \). Next we must have \( x^{(m)} = -x \). But
\[
x'' = (\beta x)' = \beta' \beta x, \quad \ldots, \quad x^{(k)} = [x^{(k-1)}]' = [\beta^{(k-1)} \beta^{(k-2)} \cdots \beta' \beta] x,
\]
and
\[
x^{(m)} = [\beta^{(m-1)} \beta^{(m-2)} \cdots \beta' \beta] x = -x,
\]
so if we write
\[ N_D(\beta) = \beta \beta' \beta'' \cdots \beta^{(m-1)}, \]
then it follows from \( x^{(m)} = -x \) that
\[ (12) \quad N_D(\beta) = -1. \]

Conversely let \( D \) be cyclic of degree \( m = 2^{e-1} \) over \( F \) and let \( a, \beta \) satisfy (12). Then the field \( D(x) \) defined by a root \( x \) of \( x^2 = a \) is a quadratic field over \( D \) if and only if \( a \) is not the square of any quantity of \( D \). But if \( a = c^2, c \) in \( D \), then \( \beta^2 = (a')^2 = (c'c^{-1})^2 \) so that
\[ (13) \quad \beta = \pm (c')(c)^{-1}. \]

But \( m \) is even and
\[ (14) \quad N_D(\beta) = (\pm 1)^m N_D \left( \frac{c'}{c} \right) = (\pm 1)^m = 1, \]
a contradiction of the first equation of (12). Hence \( D(x) \) has degree \( n = 2m \). Also if we define \( x' = \beta \cdot x \) then (12) implies that \( x^{(m)} = -x \) so that we have defined a self correspondence of \( C = D(x) \)
\[ (15) \quad c + dx \longleftrightarrow c' + d'x' = c' + d'\beta x, \]
for every \( c \) and \( d \) of \( D \), \( c + dx \) of \( C \). This correspondence is evidently preserved under addition, subtraction, multiplication and division and is an automorphism of \( C \) if and only if \( x'^2 = a' \) which is satisfied by (12). Hence (15) is an automorphism \( S \) of \( C \) and, since \( S^n \) is an automorphism of \( C \) in which \( x \) corresponds to \( -x \) the order of \( S \) is \( n = 2m \) and \( C \) is a cyclic field. Obviously \( D \) is the set of all quantities of \( C \) unaltered by \( S^n \). By Theorem 1, \( C = F(x) \) and we have proved

**Theorem 3.** Let \( D \) be cyclic of degree \( 2^{e-1} \) over \( F \) with generating automorphism
\[ d \longleftrightarrow d', \]
for every \( d \) of \( D \). Then \( D \) is the unique sub-field of degree \( m \) of a cyclic field \( C \) of degree \( n = 2m \) if and only if there exist quantities \( \beta \neq 0, a \neq 0 \) in \( D \), such that
\[ (16) \quad \beta^2 = \frac{a'}{a}, \quad N_D(\beta) = -1. \]
Moreover every solution of (16) defines a cyclic field of degree \( n \) over \( F \)
\[ (17) \quad C = F(x), \quad x^2 = a \quad \text{in} \; D, \; x \longleftrightarrow x' = \beta \cdot x, \]
as generating automorphism, so that \( D \) is the set of all quantities \( d \) of \( C \) such that \( d^{(m)} = d \).
The case \( m = 1, n = 2 \) is trivial so that we shall assume henceforth that \( n > 2, n = 4g = 2m \). Then (16) implies that

\[
\beta^2 = \frac{a' \beta'' \cdots \beta^{(s-1)}}{a}, \quad \beta'^2 = \frac{a'' \beta' \cdots \beta^{(s-1)}}{a}, \quad \cdots, \quad \beta^{(s-1)} = \frac{a^{(s)} \beta \cdots \beta^{(m-1)}}{a^{(s-1)}}, \quad \cdots, \quad \beta^{(s-1)} = \frac{a^{(s)} \beta \cdots \beta^{(m-1)}}{a^{(s-1)}},
\]

and hence that

\[
[\beta \beta' \cdots \beta^{(s-1)}]^2 = \frac{a^{(s)}}{a}.
\]

Then if

\[
y = a \beta \beta' \cdots \beta^{(s-1)},
\]
equation (19) implies

\[
y^2 = a a^{(s)}.
\]

But \( D \) is a quadratic field \( K(d), d^2 \) in \( K \), over a cyclic field \( K \) of degree \( g \) over \( F \). Moreover

\[
k^{(s)} = k
\]

for every \( k \) of \( K \). Since \( m = 2g, a^{(m)} = a \), we have \( [aa^{(s)}]^{(s)} = a^{(s)}a \) so that \( y^2 \) is in \( K \). Also \( y y^{(s)} = aa^{(s)} [\beta \beta' \cdots \beta^{(s-1)}] [\beta^{(s)} \cdots \beta^{(m-1)}] = a a^{(s)} N_D(\beta) = -aa^{(s)} = -y^2, y^{(s)} = -y \), and \( y \) is not in \( K \). But then \( y \) generates \( D \), a quadratic field over \( K \), and

\[
D = K(y), \quad y^2 = \alpha \text{ in } K.
\]

The field \( D = K(y) \) is a quadratic field over \( K \) which is cyclic of degree \( g \) over \( F \). By Theorem 3 there exist quantities \( \alpha = y^2 \) in \( K \), \( \gamma = y'y^{-1} \) in \( K \), such that

\[
\gamma^2 = \frac{\alpha'}{\alpha}, \quad N_K(\gamma) = \gamma \gamma' \cdots \gamma^{(s-1)} = -1,
\]

and, by this same theorem, \( D = F(y) \). Hence

\[
C = F(x), \quad x^2 = a \text{ in } D, \quad D = F(y), \quad y^2 = \alpha \text{ in } K,
\]

\[
x' = \beta x, \quad y' = \gamma y, \quad a = \frac{y}{\beta \beta' \cdots \beta^{(s-1)}}.
\]

We now wish

\[
\beta^2 = \frac{a'}{a} = \frac{y'}{\beta \beta' \cdots \beta^{(s-1)}} \cdot \frac{\beta \beta' \cdots \beta^{(s-1)}}{y} = \frac{y'}{y} \frac{\beta}{\beta^{(s)}}.
\]
But (27) is equivalent to

\[ \frac{y'}{y} = \beta \beta^{(s)}, \]

that is, \( \gamma = \beta \beta^{(s)} \).

Conversely, let \( y^2 = \alpha \) in \( K \), \( \gamma^2 = \alpha' \alpha^{-1} \), \( N_K(\gamma) = -1 \), so that, by Theorem 3, \( K(y) = F(y) \) is a cyclic field of degree \( 2g \) over \( F \). Let also \( a \) be defined by the third equation of (26), and \( \beta \) be in \( D \) and satisfy

\[ \gamma = \beta \beta^{(s)}. \]

Then

\[ ND(\beta) = [\beta \beta' \cdots \beta^{(s-1)}][\beta \beta' \cdots \beta^{(s-1)}]^{(s)} = [\beta \beta^{(s)}][\beta \beta^{(s)}]' \cdots [\beta \beta^{(s)}]^{(s-1)} \]

\[ = \gamma \gamma' \cdots \gamma^{(s-1)} = N_K(\gamma) = -1. \]

Also

\[ \frac{a'}{a} = \frac{\gamma'}{\beta' \beta'' \cdots \beta^{(s)}} \frac{\beta \beta' \cdots \beta^{(s-1)}}{\gamma} = \frac{\gamma \beta}{\beta^{(s)}} \frac{\gamma}{\beta^{(s)}} \beta = \beta^2, \]

as desired. We are now in a position to prove

**Theorem 4.** Let \( n = 4g = 2^s \) and let \( K \) be a cyclic field of degree \( g \) over \( F \) with automorphism \( k \mapsto k' \) for every \( k \) of \( K \). Then \( K \) is the unique sub-field of degree \( g \) of a cyclic field \( C \) of degree \( n \) over \( F \) if and only if there exist quantities \( \alpha \neq 0 \), \( \gamma \neq 0 \), \( \beta_1 \), \( \beta_2 \) in \( K \) satisfying

\[ \gamma^2 = \frac{\alpha'}{\alpha}, \quad N_K(\gamma) = -1, \quad \gamma = \beta \beta\gamma - \beta^2 \alpha. \]

Every solution of (30) defines a cyclic field \( F(x) = C \) with

\[ \gamma^2 = \alpha, \beta = \beta_1 + \beta_2 y, \quad x^2 = a = \frac{y}{\beta \beta' \cdots \beta^{(s-1)}}, \]

and with generating automorphism \( S \) given by

\[ c = c_1 + c_2 y + (c_3 + c_4 y)x \mapsto c' = c_1' + c_2' \gamma y + (c_3' + c_4' \gamma y)\beta x, \]

for every \( c_1, c_2, c_3, c_4 \) of \( K \), \( c \) of \( C \), so that

\[ x' = \beta x, \quad y' = \gamma y. \]

We obviously also will have
Corollary 1. In Theorem 4 the field $K$ is the field of all quantities of $C$ unaltered by $S^6$, the field $D=F(\gamma)=K(\gamma)$ is the field of all quantities of $C$ unaltered by $S^m$.

For we need only notice in the above that, since $\beta$ is in $D$, $\beta=\beta_1+\beta_2y$ where $\beta_1$ and $\beta_2$ are in $K$. We have then merely replaced the condition $\gamma=\beta^{(q)}$ by the equivalent condition $\gamma=\beta_1^2-\beta_2^2\alpha$ of (30).

We shall now obtain some important restrictions which it is possible to impose on $\beta$. Suppose first that $n=4$, $m=2$. Then $K=F$, $N_K(\gamma)=\gamma=-1$ is in $F$, and (30) becomes merely $\beta_1^2-\beta_2^2\alpha=-1$. If $\beta_1=0$ then $\beta_2\neq 0$, $\alpha=(\beta_2^{-1})^2$ which is impossible if $D$ is a quadratic field over $F$. Hence for this case $\beta_1\neq 0$.

There exists the possibility in the above theorem that $\beta_1\beta_2=0$. We shall be able to restrict $\beta$ so that all fields $C$ are obtained yet $\beta_1\beta_2\neq 0$.

By Theorem 2 if $b=b_1+b_2y$, $b_1,b_2\neq 0$, $b_1$ and $b_2$ in $K$, then $x_0=bx$ also generates $F(x)$ and satisfies

$$x_0^3 = a_0 = \frac{y_0}{\beta_0}\beta_0'\cdots\beta_0^{(q-1)}, \quad x_0' = \beta_0x_0, \quad y_0^3 = \alpha_0, \quad y_0' = \gamma_0y_0,$$

with

$$\gamma_0^3 = \frac{\alpha_0'}{\alpha_0}, \quad N_K(\gamma_0) = -1, \quad \beta_0 = \beta_10 + \beta_20y_0, \quad \beta_10^3 - \beta_20^2\alpha_0 = \gamma_0.$$

But

$$x_0' = (bx)' = b'\beta x = \beta_0x_0 = \beta_0bx,$$

$$\beta_0 = \frac{b'}{b} = \frac{b_1' + b_2'y_0}{b_1 + b_2y}\beta = (b_1' + b_2'y_0)(b_1 - b_2y)e$$

$$= [(b_1'b_1 - b_2'b_2\alpha_0\gamma) + (b_2'y_0b_1 - b_1'b_2)y]e,$$

where

$$e = (b_2^3 - b_2^2\alpha)^{-1}\beta$$

is either in $K$ or a multiple of $\gamma$ by a quantity of $K$ according as $\beta_2=0$ or $\beta_1=0$. But then $\beta_1\beta_2=0$ if and only if

$$b_2'y_0b_1 - b_1'b_2 = 0, \quad b_2'b_1 - b_2'b_2\alpha = 0.$$

Suppose first that $b_2'y_0b_1 - b_1'b_2=0$. Then since $b_1b_2\neq 0$,

$$\gamma = \frac{b_1'b_2}{b_1b_2}, \quad \gamma' = \frac{b_1'^2b_2^2}{b_1'b_2'}, \quad \cdots, \quad \gamma^{(q-1)} = \frac{b_1^{(q)}b_2^{(q-1)}}{b_1^{(q)}b_2^{(q)}}$$

and

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(41) \[-1 = N_K(\gamma) = \frac{b_1 b_2 \cdots b_{l'} b_{l''} \cdots b_{(n-1)n} b_{n-1}}{b_1 b_2 \cdots b_{l'} b_{l''} \cdots b_{(n-1)n} b_{n-1}} = \frac{b_1(b_2 \cdots b_n)}{b_1} = 1,
\]
since \(b_1 = b_1^{(n)}\) and \(b_2 = b_2^{(n)}\) are in \(K\), a contradiction. Hence \(b_2' \gamma b_1 - b_1' b_2 \neq 0\).

We have then proved that if \(b_2 \in F = K\), that is, \(b_2 = 0\), is \(b_2 = d(b_2' \gamma b_1 - b_1' b_2)\neq 0\) as we have shown. It remains only to consider the case \(n > 4, m > 2, b_1 b_1' = \alpha \gamma b_2 b_2'\).

Let \(y_0 = b_2 b_1^{-1} y\). Then \(F(y_0) = F(y)\),
(42) \[y_0 y_0' = b_2 b_1 (b_1 b_1')^{-1} \alpha \gamma = 1, y_0' = (y_0)^{-1}, y_0' = y_0.\]
But the automorphism \(S\) of \(D = F(y_0)\) replacing \(y_0\) by \(y_0'\) has order \(m\). Hence \(m = 2\), a contradiction. We have proved

Theorem 5. Every cyclic field \(F(x)\) of degree \(n = 2^r\) over \(F\) with \(K\) as cyclic sub-field is generated by an \(x\) of Theorem 4 with \(\beta_2 \neq 0\) in (30).

3. Cyclic quartic fields. Let \(n = 4\) so that \(K = F\), \(g = 1\), \(\gamma\) and \(\alpha\) are in \(F\). Then \(N_K(\gamma) = \gamma = -1, \beta_1 \beta_2 - \beta_2 \alpha = -1\) for \(\beta_1 \neq 0, \beta_2 \neq 0\) in \(F\). Put \(\epsilon = \beta_1^{-1}\) and obtain \(-\epsilon^2 = 1 - (\beta_2 \beta_1^{-1})^2 \alpha\), whence if \(u = \beta_2 \epsilon y\) then \(F(u) = F(y)\) and \(u^2 = \beta_2^2 \epsilon^2 y^2 = (\beta_2 \beta_1^{-1})^2 \alpha\),
(43) \[u^2 = 1 + \epsilon^2 = \tau\] in \(F\), \(\beta = \frac{1 + u}{\epsilon}\),
since \(\epsilon = \epsilon(\beta_1 + \beta_2 y) = 1 + u\). Also
(44) \[x^2 = a = \frac{\gamma}{\beta} = \frac{\beta_2 \epsilon y}{\beta_2 (1 + u)} = \frac{u}{\beta_2 (1 + u)} \cdot \frac{1 - u}{1 - u} = \nu(u - \tau),\]
where \(\nu = (-\beta_2 \epsilon^2)^{-1} \neq 0\) is in \(F\). We have therefore proved the well known result

Theorem 6. Every cyclic field \(F(x)\) of degree \(4\) over \(F\) is generated by a quantity \(x\) satisfying
(45) \[x^2 = \nu(u - \tau), \ y = \frac{1 + u}{\epsilon} x, u^2 = \tau = 1 + \epsilon^2,\]
where \(\epsilon\) and \(\nu \neq 0\) are in \(F\) and \(\tau\) is not the square of any quantity of \(F\).

4. Cyclic fields of degree eight. Let now \(n = 8, m = 4, g = 2\). Then \(F(y)\) is a cyclic quartic field. We wish \(\beta = \beta_1 + \beta_2 y\) with \(y^2 = \alpha\) in \(K\), \(\gamma, \beta_1, \beta_2\) in \(K\) and \(\gamma \gamma' = -1\),
(46) \[\beta_1^2 - \beta_2^2 \alpha = \gamma, \beta_1 \beta_2 \neq 0.\]
Let
\[(47) \delta = \beta y = \beta_2 \alpha + \beta_1 y = \delta_1 + y_0,\]
where
\[(48) F(y_0) = F(y), y_0 = \beta_1 y, \delta_1 = \beta_2 \alpha \text{ in } K.\]
Then
\[(49) \beta = y^{-1} \delta = (\beta_1 y_0^{-1}) \delta, (\beta_2 \alpha)^2 - \beta_1^2 \alpha = - \alpha \gamma,\]
so that, since
\[(50) y_0^2 = \alpha_0 = \beta_1^2 \alpha, y_0' = \gamma_0 y_0,\]
we have
\[(51) \beta = (\beta_1 \alpha_0^{-1}) \delta y_0 = (\beta_1 \alpha_0^{-1})(\alpha_0 + \delta_1 y_0).\]
Also \(\beta \beta' \beta'' \beta''' = -1\) and hence
\[(52) a = \frac{y}{\beta \beta'} = - y (\beta \beta')'' = - y \left( \frac{\delta \delta'}{yy'} \right)'' = - \frac{y(\delta \delta')''}{\alpha \gamma}\]
\[= - \frac{y_0}{\beta_1} \left( \beta_1 \beta_1' \right) \frac{(\delta \delta')''}{\alpha \gamma y_0} = \frac{\beta_1' (\delta \delta')''}{\alpha \gamma y_0},\]
where
\[(53) \gamma_0 y_0' = -1, \gamma_0^3 = \frac{\alpha_0'}{\alpha_0}, \delta_1^3 = \delta_0 = - \lambda^{-1} \alpha_0 y_0, \lambda = \beta_1 \beta_1'.\]

Suppose that \(\gamma_0 = \alpha_0\) is in \(F\). Then \(\gamma_0 y_0' = -1\) gives \(\gamma_0^3 = -1, \gamma_0 = i = (1)^{1/2}.\) Also \(\alpha_0' = -\alpha_0\) and if \(K = F(u)\) we may take \(\alpha_0 = u.\) Then the solution of (46) is equivalent to \(\delta_1^2 - \alpha_0 = - \lambda^{-1} \alpha_0 y_0\) where \(\lambda\) is in \(F\) and hence to the solution of \(\delta_1^2 = u(1 - \lambda^{-1} i).\) But if \(\delta_1 = \xi_1 + \xi_2 u\) this implies that \(u^2 = \tau\) in \(F, \xi_1^2 + \xi_2^2 \tau + 2 \xi_1 \xi_2 u = u(1 - \lambda^{-1} i), \xi_1^2 + \xi_2^2 \tau = 0\) and \(\tau = -(\xi_1 \xi_2^{-1})^2 = (i \xi_1 \xi_2^{-1})^2,\) a contradiction of our hypothesis that \(F(u)\) is a quadratic field.

Hence \(\gamma_0\) is not in \(F\) and the hypothesis \(\beta_2 \neq 0\) of \(\S 3\) is satisfied for \(F(y_0).\) But then
\[(54) y_0^3 = \alpha_0 = \nu (u - \tau), \frac{y_0'}{y_0} = \gamma_0 = \frac{1 + u}{\epsilon}, \quad u^2 = \tau = 1 + \epsilon^2.\]

Also \(-\alpha_0 y_0 = -\nu \epsilon^{-1}(u - \tau)(u + 1) = \nu \epsilon^{-1} u (\tau - 1);\) that is, since \(\tau - 1 = \epsilon^2,\)
\[(55) -\alpha_0 y_0 = \nu \epsilon u.\]
We may now complete our computation (52) of \(a\). We use

\[
(\delta')''y_0 = [(\delta_1 + y_0)(\delta_1' + y_0')]''y_0 = (\delta_1 - y_0)(\delta_1' - y_0)\gamma_0 y_0 = - (\alpha \delta_1' + \delta_1 \alpha y_0) + (\delta_1 \delta_1' + \alpha \gamma_0 y_0) y_0.
\]

Hence

\[
a = \frac{\beta'_1 u}{\nu \tau} \left[ \nu \mu \delta'_1 - \nu(u - \tau)\delta_1 + (\delta_1 \delta_1' - \nu \mu) y_0 \right],
\]

where

\[
\delta_1 = \xi_1 + \xi_2 u, \quad \delta_1' = \xi_1 - \xi_2 u, \quad \beta_1 = \xi_1 + \xi_2 u.
\]

Also (51) gives

\[
\beta = \beta_1 (-\alpha \gamma_0)^{-1} (-\alpha \gamma_0 - \delta_1 \gamma_0 y_0) = \beta_1 (\nu \mu)^{-1} (\nu \mu - \delta_1 \gamma_0 y_0)
\]

and hence

\[
\beta = \frac{\beta_1}{\nu \tau} \left[ \nu \tau - \frac{\delta_1}{\epsilon} (u + \tau) y_0 \right].
\]

We have proved

**Theorem 7.** Every cyclic field \(F(x)\) of degree eight over \(F\) is generated by a quantity \(x\) satisfying

\[
x^2 = a, \quad x' = \beta x,
\]

with \(a\) and \(\beta\) given by (54), (56), (57), (58) such that \(\nu \neq 0\) in \(F\), \(\delta_1 \neq 0\), \(\beta_1 \neq 0\), and if

\[
\lambda = \xi^2 - \xi^2 \tau,
\]

then

\[
\delta_1^2 = a_0 - \lambda^{-1} a_0 \gamma_0 = \nu(u - \tau + \lambda^{-1} \epsilon u).
\]

The quantity \(\delta_1^2 = \xi_1^2 + \xi_2^2 \tau + 2 \xi_1 \xi_2 u\), so that (61) is equivalent to

\[
-\nu \tau = \xi_1^2 + \xi_2^2 \tau, \quad 2 \xi_1 \xi_2 = \nu(1 + \lambda^{-1} \epsilon).
\]

But then

\[-2 \xi_1 \xi_2 \tau = (-\nu \tau)(1 + \lambda^{-1} \epsilon) = (1 + \lambda^{-1} \epsilon)(\xi_1^2 + \xi_2^2 \tau),\]

so that, since \(\nu \neq 0\), equation (61) is equivalent to

\[
\nu = (-\tau)^{-1}(\xi_1^2 + \xi_2^2 \tau) \neq 0, \quad 1 + \lambda^{-1} \epsilon = \frac{-2 \xi_1 \xi_2 \tau}{\xi_1^2 + \xi_2^2 \tau}.
\]

The first equation of (63) will be taken to determine \(\nu\). The second equation becomes

\[
-\epsilon = \lambda \left[ \frac{2 \xi_1 \xi_2 \tau}{\xi_1^2 + \xi_2^2 \tau} \right] = (\xi_1^2 - \xi_2^2 \tau) \left( \frac{(\xi_1^2 + \xi_2^2 \tau + 2 \xi_1 \xi_2 \tau)}{\xi_1^2 + \xi_2^2 \tau} \right),
\]
to be solved for $\xi^2 + \xi^2 \tau \neq 0$. But $\xi^2 + 2 \xi \xi_2 \tau = (\xi_1 + \xi \xi_2 \tau)^2 + (\xi_2 \tau)^2 (1 - \tau) = (\xi_1 + \xi \xi_2 \tau)^2 - (\xi_2 \tau)^2 \tau$. Hence if

$$k = \eta_1 + \eta_2 \nu = \frac{(\xi_3 + \xi \xi_4)(\xi_1 + \xi \xi_2 \tau + \xi_2 \xi_4 \nu)}{\xi_1 + \xi \xi_2 \tau},$$

where $\eta_1$ and $\eta_2$ are then explicitly determined in terms of $\xi_1, \xi_2, \xi_3, \xi_4$, then

$$kk' = \frac{(\xi_1^2 - \xi_2^2 \tau)(\xi_1^2 + \xi_2^2 \tau + 2 \xi \xi_2 \tau)}{(\xi_1^2 + \xi_2^2 \tau)^2} = -\frac{\epsilon}{\xi_1^2 + \xi_2^2 \tau},$$

so that, since $kk' = \eta_1^2 - \eta_2^2 \tau$,

$$-\epsilon = (\xi_1^2 + \xi_2^2 \tau)(\eta_1^2 - \eta_2^2 \tau) \neq 0,$$

where we use $\tau = 1 + \epsilon^2$, $\neq 1$.

Conversely let $\epsilon \neq 0$ satisfy (67) and define $k$ by $k = \eta_1 + \eta_2 \nu$. Define

$$\beta_1 = \frac{(\xi_1^2 + \xi_2^2 \tau)k}{\xi_1 + \xi \xi_2 \tau + \xi_2 \xi_4 \nu} = \xi_3 + \xi \xi_4 \nu, \quad r = \xi_1 + \xi \xi_2 \tau + \xi_2 \xi_4 \nu,$$

where $\beta_1$ exists since $\xi_1^2 + \xi_2^2 \tau \neq 0$ and hence $r \neq 0$. Then we have

$$-\epsilon = kk' (\xi_1^2 + \xi_2^2 \tau) = \frac{\beta_1 \beta_1' \tau \nu'}{\xi_1^2 + \xi_2^2 \tau} = \frac{(\xi_1^2 - \xi_2^2 \tau)(\xi_1^2 + \xi_2^2 \tau + 2 \xi \xi_2 \tau)}{\xi_1^2 + \xi_2^2 \tau},$$

and (64) will be satisfied. Moreover if we define $\nu$ by (63), then (61) will be satisfied. Also $\tau = 1 + \epsilon^2$ must not be the square of any quantity of $F$ if $F(u)$, $u^2 = \tau$, is a quadratic field over $F$ as we are supposing. We have proved

Theorem 8. The solution of (61) is equivalent to the determination of $\nu$ by

$$\nu = (\tau)^{-1}(\xi_1^2 + \xi_2^2 \tau),$$

and the solution of

$$-\epsilon = (\eta_1^2 - \eta_2^2 \tau)(\xi_1^2 + \xi_2^2 \tau)$$

for $\epsilon, \eta_1, \eta_2, \xi_1, \xi_2$ in $F$ and such that $\tau = 1 + \epsilon^2$ is not the square of any quantity of $F$.

5. The formulas for $\alpha_0, \gamma_0, \alpha, \beta$. We have seen how every cyclic field $F(x)$ of degree eight over $F$ is generated by a quantity $x$ such that $x^2 = a, x' = \beta x$ where $a$ and $\beta$ are given by (56), (58), (54), (57) as soon as $\nu, \epsilon, \tau = 1 + \epsilon^2, \beta_1 = \xi_4 + \xi \xi_4 \nu$, $\delta_1 = \xi_1 + \xi_2 \nu$ have been determined to satisfy (61). We have also shown that the solution of (61) is equivalent to (70) and the solution of the equation (71) with variables in $F$. Hence we have merely to solve (71), obtaining formulas with parameters for $\epsilon, \eta_1, \eta_2, \xi_1, \xi_2$, obtain formulas for
\[ \xi_2 \text{ and } \xi_3 \text{ by the use of (68), and by the substitution of values so obtained in (54), (56), (57), (58) obtain explicit } a, \beta, \alpha_0, \gamma_0. \text{ But the formulas so obtained would be undesirable because of complexity. Hence we shall confine our further work to a consideration of the only remaining non-trivial part of our problem, the solution of (71). Explicit fields of degree eight may then be obtained by carrying out the above work of substitution for every special case.} \\

6. The case \( i \) in \( F \). Suppose that \( F \) contains a quantity \( i \) such that \( i^2 = -1 \). Then if \( \tau = 1 + \epsilon^2, \epsilon \) in \( F \), we wish to solve \(-\epsilon = (\xi_2 + \xi_2 \tau)(\eta_2 - \eta_2 \tau)\) for \( \xi_1, \xi_2, \eta_1, \eta_2 \) in \( F \) and \( \tau \) not the square of any quantity of \( F \). Let

\[ (72) \quad k_1 = \xi_1 + \xi_2 iu, \quad k_2 = \eta_1 + \eta_2 u, \]

so that \( k_1 \) is in \( F(u) \), \( u^2 = 1 + \epsilon^2 \), \( k_2 \) is in \( F(u) \). Then if

\[ (73) \quad k_3 = k_1 k_2 = \lambda + \mu u, \quad \lambda, \mu \text{ in } F, \]

we have

\[ (74) \quad \lambda^2 - \mu^2 \tau = -\epsilon, \]

since if \( k_3' = \lambda - \mu u \) then \( k_3 k_3' = k_1 k_1' \cdot k_2 k_2' = [\xi_2\tau - (\xi_2 + \xi_2 \tau) \cdot (\eta_2\tau - \eta_2\tau)] = (\xi_2 + \xi_2 \tau)(\eta_2 - \eta_2 \tau) = -\epsilon. \]

Conversely let \( \lambda, \mu \) be a solution of (74). Then if \( k_3 \) is defined by (73) we have

\[ (75) \quad k_1 = (\xi_1 + \xi_2 iu) = \frac{k_3}{k_2} = \frac{\lambda + \mu u}{\eta_1 + \eta_2 u} = \frac{(\lambda \eta_1 - \mu \eta_2 \tau)}{\eta_1^2 - \eta_2^2 \tau} + \frac{\mu \eta_1 - \lambda \eta_2}{\eta_1^2 - \eta_2^2 \tau} u, \]

so that

\[ (76) \quad \xi_1 = \frac{\lambda \eta_1 - \mu \eta_2 \tau}{\eta_1^2 - \eta_2^2 \tau}, \quad \xi_2 = \frac{(\mu \eta_1 - \lambda \eta_2)}{\eta_1^2 - \eta_2^2 \tau} (-i), \]

where \( \eta_1 \) and \( \eta_2 \) not both zero range independently over all quantities of \( F \) so that \( \eta_1^2 - \eta_2^2 \tau \neq 0 \). We have therefore

**Theorem 9.** Let \( i \) be in \( F \), \( i^2 = -1 \), and \( \lambda, \mu, \epsilon \) range over all solutions of

\[ (76) \quad \lambda^2 - \mu^2 \tau = -\epsilon \]

in \( F \) such that \( 1 + \epsilon^2 = \tau \) is not the square of any quantity of \( F \). Then every cyclic field of degree eight over \( F \) is given by (70), (68), (65), (59), (54), (56), (57), (58) for every \( \eta_1, \eta_2 \) not both zero and in \( F \).

We therefore have only to solve (76). Suppose first that \( \mu = 0 \). Then \( \epsilon = -\lambda^2 \) and we have proved
Theorem 10. Let $\lambda$ range over all quantities of $F$ such that $1+\lambda^4$ is not the square of any quantity of $F$. Then (76) is satisfied by $\mu=0$, $\epsilon=-\lambda^2$, and defines corresponding cyclic fields.

Next let $\mu\neq0$. Define

\[ \mu^{-1} = 2\sigma, \lambda \mu^{-1} = \rho, \]

so that

\[ -\epsilon \mu^{-2} = -4\sigma^2 \epsilon = \rho^2 - \tau = \rho^2 - (1 + \epsilon^2), \]

\[ (\epsilon - 2\sigma^2)^2 - \rho^2 = 4\sigma^4 - 1, \]

and

\[ (\epsilon - 2\sigma^2 - \rho)(\epsilon - 2\sigma^2 + \rho) = 4\sigma^4 - 1. \]

Here again we must separate our work into two special cases.

Suppose first that $\epsilon - 2\sigma^2 - \rho = 0$. Then $4\sigma^4 = 1$, $(2\sigma^2)^2 = 1$, so that $2\sigma^2 = \pm 1$. Moreover if $2\sigma^2 = 1$ then $(2\sigma)^2 = 2$, $2\sigma = \mu^{-1} = \pm 2^{1/2}$ so that, since $\lambda = \rho \mu$, we have $\rho = \epsilon - 2\sigma^2 = \epsilon - 1$,

\[ \mu = \pm \frac{2^{1/2}}{2}, \quad \lambda = (\epsilon - 1) \left( \pm \frac{2^{1/2}}{2} \right), \]

and $\epsilon$ ranges over all quantities of $F$ such that $1 + \epsilon^2$ is not the square of any quantity of $F$. Moreover if $2\sigma^2 = -1$ then $\mu^{-1} = \pm 2^{1/2}i$, $\rho = \epsilon - 2\sigma^2 = \epsilon + 1$, $\lambda = \rho \mu$,

\[ \mu = \pm \frac{2^{1/2}i}{2}, \quad \lambda = (\epsilon + 1) \left( \pm \frac{2^{1/2}i}{2} \right). \]

We have therefore proved

Theorem 11. Let $\epsilon$ range over all quantities of $F$ such that $1 + \epsilon^2$ is not the square of any quantity of $F$. Then if $i$ is in $F$, $i^2 = -1$, and $\lambda$, $\mu$ are given by either (80) or (81), so that $2^{1/2}i$ is in $F$, the condition $\lambda^2 - \mu^2 \tau = -\epsilon$ is satisfied, and Theorem 9 defines a set of corresponding cyclic fields of degree eight over $F$.

Suppose finally that $\epsilon - 2\sigma^2 - \rho = \pi \neq 0$. Then $\epsilon - 2\sigma^2 + \rho = (4\sigma^4 - 1)\pi^{-1}$ and $2(\epsilon - 2\sigma^2) = \pi + (4\sigma^4 - 1)\pi^{-1}$ while $2\rho = (4\sigma^4 - 1)\pi^{-1} - \pi$. Also $\lambda = \rho \mu$,

\[ \epsilon = \frac{(\pi + 2\sigma^2)^2 - 1}{2\pi}, \quad \lambda = \frac{4\sigma^4 - \pi^2 - 1}{4\sigma}, \quad \mu = (2\sigma)^{-1}, \]

and we have proved
Theorem 12. Let $F$ contain a quantity $i$ such that $i^2 = -1$. Then every cyclic field of degree eight over $F$ is given by Theorem 9 with $\lambda, \mu, \epsilon$ determined by either Theorem 10 or 11 or by (82) as $\pi \neq 0$ in $F$, $\sigma \neq 0$ in $F$ range over all quantities of $F$ such that $\tau = 1 + \epsilon^2$ is not the square of any quantity of $F$.

7. The case $\tau = \epsilon t$, $t$ in $F$. Let $\tau = -t^2$ where $t$ is in $F$. Then $F$ contains no quantity $i$ such that $i^2 = -1$ since otherwise $\tau = (it)^2$ contrary to the fundamental assumption of our work, namely that $F(u), u^2 = \tau$, shall be a quadratic field over $F$. We wish to solve

\begin{equation}
-\epsilon = [\xi t^2 - (\xi t)^2] [\eta t^2 - \eta^2 \tau],
\end{equation}

that is, since $\eta^2 - \eta^2 \tau \neq 0$,

\begin{equation}
\xi t^2 - (\xi t)^2 = \frac{-\epsilon}{\eta t^2 - \eta^2 \tau} = R \neq 0, \quad -1 = \epsilon^2 + t^2.
\end{equation}

Since $\epsilon \neq 0$ we evidently have $\xi_1 - \xi_2 t = \pi \neq 0$. Then $\xi_1 + \xi_2 t = \pi R^{-1}$ so that

\begin{equation}
\xi_1 = \frac{\pi^2 + R}{2\pi}, \quad \xi_2 = \frac{R - \pi^2}{2\pi}, \quad R = \frac{-\epsilon}{\eta t^2 - \eta^2 \tau},
\end{equation}

and we have proved

Theorem 13. Let $\epsilon$ and $t$ range over all quantities of $F$ such that $-1 = \epsilon^2 + t^2$ and $1 + \epsilon^2 = \tau$ is not the square of any quantity of $F$. Then $i$ is not in $F$, $i^2 = -1$, and every cyclic field of degree eight over $F$ is given by (68), (65), (59), (54), (56), (57), (58), (85) when $\eta_1$ and $\eta_2$ not both zero, $\pi \neq 0$ range independently over all quantities of $F$.

8. The case $\tau \neq \epsilon t$, $i$ not in $F$. Let $-1$ be not the square of any quantity of $F$ and let $K = F(i)$, $i^2 = -1$, so that $F(i)$ is a quadratic field over $K$. Our only remaining case is the case $-\tau \neq t^2$ for any $t$ of $F$. This is sufficient to secure the fact that $K(u), u^2 = \tau$, is a quadratic field over $K$, that is, $F(i, u)$ is a quartic field over $F$.

For otherwise let $\tau = z^2$, $z = z_1 + z_2 i$ where $z_1$ and $z_2$ are in $F$. Then $\tau = z_1^2 - z_2^2 + 2z_1z_2 i$ so that $z_1z_2 = 0$. But $\tau \neq z_2 i$ in $F$, by hypothesis. Hence $z \neq z_1, z_2 \neq 0$ and $z_1 = 0$. Then $\tau = (z_2 i)^2 = -z_2^2$ contrary to hypothesis. We have therefore proved that $\tau$ is not the square of any quantity of $K$, $K(u)$ is a quadratic field over $K$.

We shall now prove

Lemma. Let $\lambda$ and $u$ be in $K = F(i)$ so that we may write $\lambda = \lambda_1 + \lambda_2 i$, $u = \mu_1 + \mu_2 i$ with $\lambda_1, \lambda_2, \mu_1, \mu_2$ in $F$. Let

\begin{equation}
\lambda^2 - \mu^2 \tau = -\epsilon,
\end{equation}

* This is of course not the field $K$ of preceding sections.
where \( \epsilon \neq 0 \) is in \( F \), \( \tau = 1 + \epsilon^2 \). Then

(87) \[ \lambda_1 \lambda_2 = \tau \mu_1 \mu_2, \]

and there exist quantities \( \eta_1, \eta_2 \) in \( F \) and not both zero such that

(88) \[ \lambda_1 \eta_2 = \mu_1 \eta_1, \lambda_2 \eta_1 = \mu_2 \eta_2, \eta_1^2 - \eta_2^2 \tau \neq 0. \]

For \( -\epsilon = \lambda_2^2 - \mu_2^2 \tau = [\lambda_2^2 - \lambda_1^2 + \tau (\mu_2^2 - \mu_1^2) ] + 2(\lambda_1 \lambda_2 - \mu_1 \mu_2 \tau) i \). Since \(-\epsilon\) is in \( F \) and \( i \) is not in \( F \) we have \( \lambda_1 \lambda_2 - \mu_1 \mu_2 \tau = 0 \) as desired. If \( \lambda_1 \neq 0 \), (88) is satisfied by \( \eta_2 = (\lambda_1^{-1} \mu_1) \eta_1 \) for every \( \eta_1 \neq 0 \) of \( F \) and

\[ \lambda_2 \eta_1 - \eta_2 \mu_2 \tau = [\lambda_2 - (\lambda_1^{-1} \mu_1 \mu_2 \tau)] \eta_1 = \lambda_1^{-1} \eta_1 [\lambda_1 \lambda_2 - \mu_1 \mu_2 \tau] = 0 \]

so that (88) is completely satisfied. If \( \lambda_1 = 0, \mu_2 \neq 0 \), then \( \mu_2 \mu_2 \tau = \lambda_1 \lambda_2 = 0 \) so that \( \mu_1 = 0 \) and (88) is satisfied. Then (88) is satisfied for every \( \eta_1 \neq 0 \) in \( F \) when we take \( \eta_2 = (\mu_2)^{-1} \eta_2 \lambda_2 \). Hence finally let \( \lambda_1 = \mu_2 = 0 \). Then (88) is merely \( \mu_1 \eta_1 = \lambda_2 \eta_1 = 0 \) which is satisfied for any \( \eta_1 \neq 0 \) in \( F \) and by \( \eta_1 = 0 \). Also \( \epsilon \neq 0 \) so that, by (86), \( \lambda = \lambda_2 i \) and \( \mu = \mu_1 \) are not both zero, so that necessarily \( \eta_1 = 0 \).

Consider now the problem of determining a general solution of (71). Suppose we have a solution and then put

(89) \[ k_1 = \xi_1 + (\xi_2 i) u, k_2 = \eta_1 + \eta_2 u, k_3 = \lambda + \mu u = k_1 k_2. \]

Equation (89) implies \( k_3 k_3' = \lambda^2 - \mu^2 \tau = -\epsilon = k_1 k_1' k_2 k_2' = (\xi_1^2 + \xi_2^2 \tau)(\eta_1^2 - \eta_2^2 \tau) \) and (86) is satisfied where

(90) \[ \lambda = \xi_1 \eta_1 + \xi_2 \eta_2 i, \mu = \xi_1 \eta_2 + \xi_2 \eta_1 i. \]

But \( \epsilon \) is in \( F \) and, by the above lemma, \( \lambda_1 \lambda_2 = \tau \mu_1 \mu_2 \). Also \( \lambda_1 = \xi_1 \eta_1, \lambda_2 = \xi_2 \eta_2 i \), so that \( \lambda_1 \eta_2 - \mu_1 \eta_1 = \xi_1 (\eta_1 \eta_2 - \eta_2 \eta_1) = 0, \lambda_2 \eta_1 - \lambda_2 \eta_2 = \xi_2 \eta_1 \eta_2 - \eta_1 \eta_2 i = 0, \) and (88) is satisfied. Hence every solution of (71) defines a solution of (86) in \( K \) for which (87) and (88) are satisfied.

Conversely let (86) be satisfied. By the above lemma, (87), (88) are satisfied. Let \( \eta_1, \eta_2 \) range over all solutions in \( F \) of (88), not both zero, and define \( k_1, k_2, k_3 \) by (89) so that if

\[ k_1 = \frac{k_3}{k_2} = \frac{\lambda_1 + \mu_1 u}{\eta_1 + \eta_2 u} + \frac{\lambda_2 + \mu_2 u}{\eta_1 + \eta_2 u} i, \]

then

\[ \xi_1 = \frac{(\lambda_1 \eta_1 - \mu_1 \eta_2 \tau) + (\mu_1 \eta_1 - \lambda_1 \eta_2) u}{\eta_1^2 - \eta_2^2 \tau} = \frac{\lambda \eta_1 - \mu \eta_2 \tau}{\eta_1^2 - \eta_2^2 \tau} \]

is in \( F \) by (88). Also

\[ \xi_2 u = \frac{\lambda_2 + \mu_2 u}{\eta_1 + \eta_2 u} = \frac{(\lambda_2 \eta_1 - \mu_2 \eta_2 \tau) + (\mu_2 \eta_1 - \lambda_2 \eta_2) u}{\eta_1^2 - \eta_2^2 \tau} = \frac{\mu \eta_1 - \lambda \eta_2 \tau}{\eta_1^2 - \eta_2^2 \tau} u, \]
and \( \xi_2 \) is in \( F \) by (88). Hence (86) determines a set of solutions of (71) and we have proved

**Theorem 14.** Let \( F \) contain no quantity \( i \) such that \( i^2 = -1 \) and let \( \epsilon \neq 0, \lambda, \mu \) range over all quantities of \( K = F(i) \) such that \( \lambda^2 - \mu^2 \tau = -\epsilon, \epsilon \) is in \( F \), and \( \tau = 1 + \epsilon^2, \pm \tau \) is not the square of any quantity of \( F \). Then if we determine all quantities \( \eta_1, \eta_2 \) satisfying (88) and define \( C = F(\xi) \) by (55)-(59), (65), (68) we obtain all cyclic fields \( C \) of degree eight over \( F \).

We therefore need only solve (86). This has already been accomplished in §6. Hence we have, without further proof,

**Theorem 15.** Let \( t \) range over all quantities of \( F \) such that \( \pm (1 + t^4) \) is not the square of any quantity of \( F \). Then if \( \epsilon = -\lambda^2, \lambda = t \) or \( ti, \mu = 0 \), we obtain a solution of \( \lambda^2 - \mu^2 \tau = -\epsilon \) and hence a set of cyclic fields of degree eight over \( F \) by the use of Theorem 14.

Next utilize the proof of Theorem 11. If \( 2\mu = \pm 2^{1/2}, \) then either \( \mu \) is in \( F \) and \( 2^{1/2} \) is in \( F \) or \( 2\mu = \pm ti, -t = 2^{1/2}i \) is in \( F \), \((-2)^{1/2} \) is in \( F \). Similarly if \( 2\mu = \pm 2^{1/2}i \) then again \( 2\mu = \pm t, ti \) and either \( 2^{1/2} \) is in \( F \) or \((-2)^{1/2} \) is in \( F \).

**Theorem 16.** Let \( \epsilon \) range over all quantities of \( F \) such that \( \pm (1 + \epsilon^2) = \pm \tau \) is not the square of any quantity of \( F \) and let either \( 2^{1/2} \) or \((-2)^{1/2} \) be in \( F \) but \( i = (-1)^{1/2} \) be not in \( F \). Then if either (80) or (81) is satisfied and \( \lambda \) and \( \mu \) so defined in \( K = F(i) \) we obtain a set of cyclic fields of degree eight over \( F \) by the use of Theorem 14.

We finally use Theorem 12 to state immediately

**Theorem 17.** Let \( F \) contain no quantity \( i, i^2 = -1 \). Then every cyclic field of degree eight over \( F \) is a cyclic field of Theorems 13, 15, or 16 or is given by Theorems 9, 14, with (82) satisfied as \( \pi \neq 0, \sigma \neq 0 \) range over all quantities of \( F(i) \) such that \( \epsilon \) is in \( F \) and \( \pm (1 + \epsilon)^2 \) is not the square of any quantity of \( F \).

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