ADDITION TO THE NOTE† ON SOME FUNCTIONALS‡

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7. We need to recall a few known definitions.

Given an abstract space E (i.e., an arbitrary set of elements), a family $\mathfrak{E}$ of sets in E is said to be additive if it satisfies the following conditions:

(i) The empty set (0) belongs to $\mathfrak{E}$.

(ii) If a set X belongs to $\mathfrak{E}$, its complement CX (with respect to the space E) also belongs to $\mathfrak{E}$.

(iii) If $\{X_n\}$ is a sequence of sets belonging to $\mathfrak{E}$, the set $X=\sum X_n$ also belongs to $\mathfrak{E}$.

If $F(X)$ is a finite real-valued function of sets, defined for all sets of an additive family $\mathfrak{E}$, and if

$$F\left(\sum_n X_n\right) = \sum_n F(X_n) \tag{7.1}$$

for any finite sequence $\{X_n\}$ of sets of $\mathfrak{E}$, of which no two have points in common, then $F(X)$ is called an additive function of sets of $\mathfrak{E}$. If (7.1) holds for any finite or infinite sequence $\{X_n\}$ of sets belonging to $\mathfrak{E}$, of which no two have points in common, then $F(X)$ is said to be a completely additive function of sets of $\mathfrak{E}$.

In this paragraph we assume that $\mathfrak{R}^*$ is an additive family in the space E, and $\mu(X)\geqq 0$ is a completely additive and finite-valued function of sets of $\mathfrak{R}^*$. The sets X belonging to $\mathfrak{R}^*$ are called measurable, $\mu(X)$ being the measure of X. A measurable set X is a singular set if for any measurable subset Y of X either $\mu(Y)=0$ or $\mu(X-Y)=0$.

An additive function $F(X)$ of measurable sets is absolutely continuous if $F(X)=0$ whenever X is of measure zero. This together with the property of being completely additive, is equivalent to the statement that for any $\epsilon>0$ there exists an $\eta>0$ such that $\mu(X)<\eta$ implies $|F(X)|<\epsilon$.

The family $\mathfrak{R}^*$ of measurable sets may be regarded as a metric complete space with the distance defined by §

† This volume, pp. 549–556. In the present addition we extend the results of §2 to completely additive functions of sets in an abstract space. The author is indebted to Professor Tamarkin for criticisms.

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§ This definition corresponds to that of distance in the space R of characteristic functions of §2.

$$d(X_1, X_2) = \mu(X_1 - X_1X_2) + \mu(X_2 - X_1X_2). \tag{7.2}$$

If two measurable sets differ by subsets of measure zero they are regarded as the same elements of the space $\mathfrak{R}^*$. Any completely additive and absolutely continuous function of measurable sets may be regarded as a continuous functional on the metric space $\mathfrak{R}^*$.

LEMMA 1. *If A is a measurable set of positive measure, then, for any positive number ϵ, the set A contains either a singular set of measure $>\epsilon$ or a measurable set of positive measure $\leqq \epsilon$.*

Suppose that A contains neither a singular set of measure $>\epsilon$, nor a measurable set of positive measure $\leqq \epsilon$. Then there will exist a measurable subset A_1 of A such that $0<\mu(A_1)<\mu(A)$. The set $A-A_1$ must be a non-singular set of measure $>\epsilon$, and, by the same argument, $A-A_1$ contains a measurable subset A_2 such that $0<\mu(A_2)<\mu(A-A_1)$. By repeating this process we obtain an infinite sequence of measurable sets $\{A_n\}$ of positive measure, of which no two have points in common. Since the series

$$\sum_{n=1}^{\infty}\mu(A_n) = \mu\left(\sum_{n=1}^{\infty}A_n\right)$$

converges, for n sufficiently large, we have $0<\mu(A_n)<\epsilon$. This, however, contradicts the assumption that A contains no measurable set of measure $\leqq \epsilon$.

LEMMA 2. *Given an arbitrary number $\epsilon>0$, the space E may be expressed as the sum of a finite number of measurable sets $E_1, E_2, \cdots, E_p$ such that $E_iE_j=0$ for $i\neq j$, while each E_i is either a singular set or a set of measure $\leqq \epsilon$.*

We observe that for an arbitrary pair of singular sets, either one of them contains the other, with the possible exception of a set of measure zero, or else their common part is of measure zero. Since $\mu(E)<\infty$, on the basis of this remark we can find a finite sequence of singular sets $E_1, E_2, \cdots, E_n$ of measure $>\epsilon$ such that

$$E_iE_j = 0 \text{ for } i \neq j, \tag{7.3}$$

while the set

$$A = E - \sum_{i=1}^{m}E_i \tag{7.4}$$

contains no singular set of measure $>\epsilon$.

Let X be any measurable set and let $\lambda(X)$ denote the least upper bound of the measures of all measurable subsets Y of X such that $\mu(Y)\leqq\epsilon$. It follows from Lemma 1 that $0<\lambda(X)\leqq\epsilon$ for any measurable set $X\subset A$ of positive measure. Hence, by induction, we can determine a sequence $\{X_i\}$ of measurable subsets of A such that

$$X_iX_j = 0 \text{ for } i \neq j, \tag{7.5}$$

$$\epsilon \geqq \mu(X_{n+1}) \geqq \tfrac{1}{2}\lambda\left(A - \sum_{i=1}^{n} X_i\right) \qquad (n = 1, 2, \cdots). \tag{7.6}$$

Upon putting

$$X_0 = A - \sum_{i=1}^{\infty} X_i$$

from (7.6) we have

$$\lambda(X_0) \leqq \lambda\left(A - \sum_{i=1}^{n} X_i\right) \leqq 2\mu(X_{n+1}) \qquad (n = 1, 2, \cdots). \tag{7.7}$$

Since, by (7.5),

$$\sum_{i=1}^{\infty} \mu(X_i) \leqq \mu(A) < \infty, \tag{7.8}$$

the series (7.8) converges and $\lim_n \mu(X_n) = 0$. Thus we infer from (7.7) that $\lambda(X_0) = 0$, whence also $\mu(X_0) = 0$. Let now h be a positive integer such that

$$\mu\left(\sum_{n=h+1}^{\infty} X_n\right) = \sum_{n=h+1}^{\infty} \mu(X_n) \leqq \epsilon, \tag{7.9}$$

and let

$$E_{m+1} = X_1, \cdots, E_{m+h} = X_h, E_{m+h+1} = X_0 + \sum_{n=h+1}^{\infty} X_n.$$

These sets, by (7.6) and (7.9), are of measure $\leqq \epsilon$, and by (7.5) no two of them have points in common. Hence the sequence $E_1, E_2, \cdots, E_{m+h+1}$ satisfies the conditions of Lemma 2.

8. We now are able to generalize Theorems 1 and 2 of §2.

THEOREM 5. *Let $\{F_n(X)\}$ be a sequence of completely additive and absolutely continuous functions of measurable sets. If this sequence converges for any set belonging to a class of the second category in the space $\mathfrak{R}^*$, then the functions $F_n(X)$ are equally absolutely continuous*† *and the sequence $\{F_n(X)\}$ converges for any measurable set $X \subset E - (E_1 + E_2 + \cdots + E_m)$ where $\{E_i\}$ is a finite sequence of singular sets.*

Consequently, if $\{F_n(X)\}$ converges for any measurable set X, the limit function is again a completely additive and absolutely continuous function of measurable sets in E.

† That is, to every $\epsilon > 0$ there corresponds an $\eta > 0$ which depends only on ϵ, such that $|F_n(X)| \leqq \epsilon$ for $n = 1, 2, \cdots$ and for any set X of measure $\leqq \eta$.

The fact that the functions $F_n(X)$ are equally absolutely continuous can be established in exactly the same fashion as in Theorem 1, §2, if we interpret the functions $F_n(X)$ as continuous functionals in the metric complete space $\mathfrak{R}^*$. Now, since by assumption the sequence $\{F_n(X)\}$ converges for any X belonging to a set of the second category in $\mathfrak{R}^*$, there exists in $\mathfrak{R}^*$ a sphere, say $\mathfrak{K}(A_0; r)$, such that $\{F_n(X)\}$ converges for each X of a set everywhere dense in $\mathfrak{K}(A_0; r)$. But the functionals $F_n(X)$ are equally continuous in $\mathfrak{R}^*$, hence the sequence $\{F_n(X)\}$ converges everywhere in the sphere $\mathfrak{K}(A_0; r)$.

Now let

$$E = \sum_{i=1}^{p} E_i$$

be a representation of the space E mentioned in Lemma 2. We may assume that the sets $E_1, E_2, \cdots, E_m$ are singular while the sets $E_{m+1}, \cdots, E_p$ are of measure $\leqq r$.

Let X be an arbitrary measurable set contained in $\sum_{m+1}^{p} E_i$. Then

$$X = \sum_{i=m+1}^{p} XE_i. \tag{8.1}$$

Each set XE_i, $i=m+1, \cdots, p$, is of measure $\leqq r$. Consequently the sets A_0+XE_i and $A_0-A_0XE_i$, $i=m+1, \cdots, p$, are elements of the sphere $\mathfrak{K}(A_0;r)$ and both sequences $\{F_n(A_0+XE_i)\}$, $\{F_n(A_0-A_0XE_i)\}$ converge. Thus the sequence

$$F_n(XE_i) = F_n(A_0 + XE_i) - F_n(A_0 - A_0XE_i)$$

also converges for $i=m+1, \cdots, p$. Hence, by (8.1), the sequence $\{F_n(X)\}$ converges for any measurable set X contained in $E-(E_1+\cdots+E_m)$ where $E_1, \cdots, E_m$ are singular sets.

THEOREM 6. *If $\{F_n(X)\}$ is a sequence of completely additive and absolutely continuous functions of measurable sets and if*

$$\overline{\lim_n}\, |F_n(X)| < \infty \tag{8.2}$$

for any set X belonging to a class of the second category in the space $\mathfrak{R}^*$, then there exists a fixed constant M such that

$$|F_n(X)| < M \tag{8.3}$$

for any measurable set $X \subset E-(E_1+\cdots+E_m)$ where $\{E_i\}$ is a finite sequence of singular sets in E.

Consequently, if the inequality (8.2) *holds for every measurable set X, there exists a constant M such that* (8.3) *holds for all measurable sets X in E.*

Let $\mathfrak{R}_k^*$ be the aggregate of sets X such that

$$|F_n(X)| \leqq k \qquad (n = 1, 2, \cdots).$$

By assumption the class $\sum_1^\infty \mathfrak{R}_k^*$ is of the second category in the space $\mathfrak{R}^*$. By the continuity of the functionals $F_n(X)$, the sets $\mathfrak{R}_k^*$ are closed (in the space $\mathfrak{R}^*$). Hence, for some value $k = k_0$, $\mathfrak{R}_{k_0}^*$ contains a sphere, say $\mathfrak{K}(A_0; r)$.

We now introduce the same representation of the space

$$E = \sum_{i=1}^{p} E_i$$

as in the proof of Theorem 5. Let X be an arbitrary measurable set. Since, for $i = m+1, \cdots, p$, the sets XE_i are of measure $\leqq r$, the sets $A_0 + XE_i$ and $A_0 - A_0XE_i$ belong to the sphere $\mathfrak{K}(A_0; r)$. Thus

$$|F_n(A_0 + XE_i)| \leqq k_0, \quad |F_n(A_0 - A_0XE_i)| \leqq k_0,$$

and

$$|F_n(XE_i)| = |F_n(A_0 + XE_i) - F_n(A_0 - A_0XE_i)| \leqq 2k_0.$$

Hence, for any measurable set $X \subset E - (E_1 + \cdots + E_m)$ we have

$$|F_n(X)| = \left|F_n\left(\sum_{i=m+1}^{p} XE_i\right)\right| \leqq 2(p-m)k_0,$$

which completes the proof of Theorem 6.

9. Theorems 5 and 6 contain the corresponding two theorems which have been stated recently by Nikodym.†

I. If $\mathfrak{E}$ is an additive family of sets in an abstract space E, and if the sequence $\{F_n(X)\}$ of completely additive functions of sets of $\mathfrak{E}$ converges for every set X of $\mathfrak{E}$, then the limit function is also a completely additive function of sets of $\mathfrak{E}$.

II. If $\mathfrak{E}$ is an additive family of sets in E and if the sequence $\{F_n(X)\}$ of completely additive functions of sets of $\mathfrak{E}$ is bounded for every set X of $\mathfrak{E}$, then there exists a constant M such that $|F_n(X)| \leqq M$ for $n = 1, 2, \cdots$ and for all $X \subset \mathfrak{E}$.

In order to reduce these theorems to Theorems 5 and 6 respectively we merely have to introduce a measure $\mu(X)$ for the family $\mathfrak{E}$, with respect to which the functions $F_n(X)$ would be absolutely continuous. This can be achieved by putting, for each set $X \subset \mathfrak{E}$,

† O. Nikodym, *Sur les suites des fonctions parfaitement additives d'ensembles abstraits*, Comptes Rendus, vol. 192 (1931), pp. 727–728. The proofs of the results stated in that note will appear in the Monatshefte für Mathematik und Physik.

$$\mu(X) = \sum_{n=1}^{\infty} \frac{V_n(X)}{2^n[V_n(E)+1]}, \tag{9.1}$$

where $V_n(X)$ denotes the absolute variation of $F_n(X)$ on the set X. Since each $V_n(X)$ is a non-negative and completely additive function† of sets of $\mathfrak{E}$ the series (9.1) converges and $\mu(X) \geqq 0$ is a completely additive and finite-valued function of sets of $\mathfrak{E}$. Hence $\mu(X)$ may be taken as a measure in E and, since $F_n(X)=0$, $n=1, 2, \cdots$, for every set X of $\mathfrak{E}$ such that $\mu(X)=0$, the functions $F_n(X)$ are absolutely continuous with respect to this measure. Thus the theorems of Nikodym are reduced to our Theorems 5 and 6.

† See for instance H. Hahn, *Theorie der reellen Funktionen*, 1921, Chapter VI. The absolute variation is called there (p. 400) "absolute Summe."

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