ANALYTIC EXTENSIONS OF DIFFERENTIABLE FUNCTIONS DEFINED IN CLOSED SETS*

BY

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I. Differentiable functions in closed sets

1. **Introduction.** Let A be a closed set, bounded or unbounded, in euclidean n-space E, and let $f(x)$ be a function defined and continuous in A. It is well known that this function can be extended so as to be continuous throughout E.‡ If A satisfies certain conditions, the solution of the Dirichlet problem is a function harmonic in $E-A$ and taking on the given boundary values in A. Two questions which arise are the following: Is there always a function differentiable, or perhaps analytic, in $E-A$, and taking on the given values in A? If the given function $f(x)$ is in some sense differentiable in A, can the extension $F(x)$ be made differentiable to the same order throughout E?

These questions are answered in the affirmative in Theorem I. We use a definition of the derivatives of a function in a general set which arises naturally from a consideration of Taylor's formula. In Part II, a differentiable extension of $f(x)$ is found, whether $f(x)$ is differentiable to finite or infinite order. Part III is devoted to some general approximation theorems. It is well known that a continuous function in a bounded closed set can be approximated uniformly (together with any finite number of derivatives) by polynomials; we show that functions defined in *open* sets may be approximated (together with derivatives) by *analytic* functions, the approximation being closer and closer as we approach the boundary of the set. This theorem, together with the results of Part II, furnish an immediate proof of Theorem I. In Part IV we give some extensions of Theorem I; in particular, we show that

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‡ See references in a paper by P. Urysohn, Mathematische Annalen, vol. 94 (1925), p. 293, footnote 51.

A continuous extension the author has not seen in the literature may be given as follows; we assume for simplicity that A is bounded. Let $h(r)$ $(r \geq 0)$ be a continuous and monotone increasing function such that $h(0)=0$, and if x and y are any two points of A whose distance apart is r_{xy}, then $|f(x)-f(y)| \leq h(r_{xy})$. For any points x of E and y of A, set $H(x, y)=f(y)-h(r_{xy})$; then if x is in A, $H(x,y) \leq f(x)$. The continuous extension of $f(x)$ is $F(x)$, which at each point x of E equals the maximum of $H(x, y)$ as y varies over A.

the extension of $f(x)$ may be made analytic at the isolated points of A. Theorem III includes all preceding results but Lemma 7.

2. **Notations.** We shall write all equations involving n variables as if there were but a single variable present. For instance, we write

$$f_0(x) \quad \text{for} \quad f_{0\ldots 0}(x_1, \cdots, x_n),$$

$$D_k f(x') \quad \text{for} \quad \frac{\partial^{k_1 + \cdots + k_n}}{\partial x_1^{k_1} \cdots \partial x_n^{k_n}} f(x_1', \cdots, x_n'),$$

$$\binom{k}{l} \quad \text{for} \quad \binom{k_1}{l_1} \cdots \binom{k_n}{l_n},$$

etc. For any n-fold subscript k, we put

$$\sigma_k = k_1 + \cdots + k_n.$$

Note that $\sigma_{k+l} = \sigma_k + \sigma_l$. r_{xy} will always denote the distance between x and y (unless x and y are complex). As an example, (3.1) below is short for

$$f_{k_1 \cdots k_n}(x_1', \cdots, x_n')$$

$$= \sum_{\substack{l_1 + \cdots + l_n \\ \leq m - (k_1 + \cdots + k_n)}} \frac{f_{k_1 + l_1, \cdots, k_n + l_n}(x_1, \cdots, x_n)}{l_1! \cdots l_n!} (x_1' - x_1)^{l_1} \cdots (x_n' - x_n)^{l_n}$$

$$+ R_{k_1 \cdots k_n}(x_1', \cdots, x_n'; x_1, \cdots, x_n).$$

3. **Differentiable functions in subsets of E.** Let $f(x)$ be defined in the set A, and let m be an integer ≥ 0. We say $f(x) = f_0(x)$ *is of class C^m in A in terms of the functions $f_k(x)$* $(\sigma_k \leq m)$ if the functions $f_k(x)$ are defined in A for all n-fold subscripts k with $\sigma_k \leq m$, and

$$(3.1) \qquad f_k(x') = \sum_{\sigma_l \leq m - \sigma_k} \frac{f_{k+l}(x)}{l!} (x' - x)^l + R_k(x'; x)$$

for each $f_k(x)$ $(\sigma_k \leq m)$, where $R_k(x'; x)$ has the following property. Given any point x^0 of A and any $\epsilon > 0$, there is a $\delta > 0$ such that if x and x' are any two points of A with $r_{xx^0} < \delta$ and $r_{x'x^0} < \delta$, then

$$(3.2) \qquad |R_k(x'; x)| \leq r_{xx'}^{m-\sigma_k} \epsilon.$$

One might define the derivatives of a function at the points of a set B, when the function is defined in a larger set A. We shall not do this here.

If $m = 0$, (3.1) and (3.2) state merely that $f(x)$ is continuous. Note that the conditions are satisfied automatically at all isolated points of A, no matter how the $f_k(x)$ are defined there.

It is easily seen that the $f_k(x)$ are continuous in a neighborhood of each point of A, and are thus bounded there. From this we prove that if $f(x)$ is of class C^m in A in terms of the $f_k(x)$ $(\sigma_k \leqq m)$, then it is of class $C^{m'}$ in A $(m' < m)$ in terms of the $f_k(x)$ $(\sigma_k \leqq m')$.

Any function we shall say is of class C^{-1} in A. $f(x)$ is of class C^∞ in A in terms of the $f_k(x)$ (defined for all k) if it is of class C^m in A in terms of the $f_k(x)$ $(\sigma_k \leqq m)$ for each m.

Suppose $f(x)$ is defined throughout the region R, and is of class C^m in terms of the $f_k(x)$ $(\sigma_k \leqq m)$. Then putting $x = (x_1, \cdots, x_n)$, $x' = (x_1, \cdots, x_h + \Delta x_h, \cdots, x_n)$, (3.1) gives

$$(3.3) \qquad f_{k_1 \cdots k_n}(x') = f_{k_1 \cdots k_n}(x) + f_{k_1, \cdots, k_h+1, \cdots, k_n}(x) \Delta x_h$$

$$+ R^{(h)}_{k_1 \cdots k_n}(x'; x)$$

(provided $\sigma_k < m$), where $R^{(h)}_{k_1 \cdots k_n}(x'; x)/\Delta h \to 0$ as $\Delta h \to 0$, which shows that

$$(3.4) \qquad \frac{\partial}{\partial x_h} f_{k_1 \cdots k_n}(x) = f_{k_1, \cdots, k_h+1, \cdots, k_n}(x) \qquad (\sigma_k < m)$$

in R; thus in this case, $f(x)$ is of class C^m in the ordinary sense, and the $f_k(x)$ are the partial derivatives of $f(x)$. The converse is true, by Taylor's Theorem.

4. The main theorem of the present paper is the following:

THEOREM I. *Let A be a closed subset of E, and let $f(x) = f_0(x)$ be of class C^m (m finite or infinite) in A in terms of the $f_k(x)$ $(\sigma_k \leqq m)$. Then there is a function $F(x)$ of class C^m in E in the ordinary sense, such that*

(1) $F(x) = f(x)$ *in* A,

(2) $D_k F(x) = f_k(x)$ *in* A $(\sigma_k \leqq m)$,

(3) $F(x)$ *is analytic in* $E - A$.†

Of course (2) includes (1).

No such theorem holds if we leave out the uniformity condition on $R_k(x'; x)$, i.e. if we assume merely that for any x and $\epsilon > 0$ there is a $\delta > 0$ such that if $r_{xx'} < \delta$, then $|R_k(x'; x)| < r_{xx'}^{m - \sigma_k} \epsilon$. The following example shows this. Let A be the set of points (using one variable) $x = 0$, $1/2^s$ and $1/2^s + 1/2^{2s}$ $(s = 1, 2, \cdots)$. Set $f(x) = 0$ at $x = 0$ and $1/2^s$, and $f(x) = 1/2^{2s}$ at the remaining points. Set $f_1(x) \equiv 0$ in A. The above condition is satisfied, but there is no extension of $f(x)$ which has a continuous first derivative.

5. The following lemma will be needed; its proof is elementary.

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† It is seen from the proof in §16 that $F(x)$ is analytic in a complex region with the following property. If x is a point of $E - A$ distant 3ρ from A, then the region contains all points within a distance ρ of x.

LEMMA 1. *Let $w(z)$ be a continuous function of one variable defined throughout an interval containing z_0, let A^* be a closed set in this interval, and let w_0' be a fixed number. Suppose that for every $\epsilon > 0$ there is a $\delta > 0$ such that*

(1) *if z is in A^* and $|z - z_0| < \delta$, then*

$$\left| w(z) - w(z_0)/(z - z_0) - w_0' \right| < \epsilon;$$

(2) *if z is not in A^* and $|z - z_0| < \delta$, then the derivative $w'(z)$ exists and*

$$\left| w'(z) - w_0' \right| < \epsilon.$$

Then $w(z)$ has a derivative at z_0, and $w'(z_0) = w_0'$.

II. DIFFERENTIABLE EXTENSIONS

6. **The functions $\psi_k(x'; x)$.** We shall make use of functions defined as follows for x in A and x' in E (m finite):

$$(6.1) \qquad \psi_k(x'; x) = \sum_{\sigma_l \leq m - \sigma_k} \frac{f_{k+l}(x)}{l!} (x' - x)^l \qquad (\sigma_k \leq m);$$

$\psi_k(x'; x)$ is the value at x' of the polynomial of degree at most $m - \sigma_k$ which approximates the function $f_k(x)$ to the $(m - \sigma_k)$th order at x. Keeping x fixed, it is a polynomial in x', given by Taylor's formula in terms of its value and derivatives at x. In terms of these functions, (3.1) becomes

$$(6.2) \qquad f_k(x') = \psi_k(x'; x) + R_k(x'; x) \qquad (\sigma_k \leq m).$$

The lth derivative of the function of x' $\psi_k(x'; x)$ at x' is $\psi_{k+l}(x'; x)$; if we express $\psi_k(x''; x)$ by Taylor's formula in terms of its value and derivatives at x', we obtain

$$\psi_k(x''; x) = \sum_l \frac{\psi_{k+l}(x'; x)}{l!} (x'' - x')^l$$

$$= \sum_l \frac{(x'' - x')^l}{l!} \sum_j \frac{f_{k+l+j}(x)}{j!} (x' - x)^j.$$

The definition of $\psi_k(x''; x')$ in conjunction with this identity gives, for any points x and x' in A and x'' in E,

$$\psi_k(x''; x') = \sum_l \frac{f_{k+l}(x')}{l!} (x'' - x')^l$$

$$(6.3) \qquad = \sum_l \frac{(x'' - x')^l}{l!} \left[\sum_j \frac{f_{k+l+j}(x)}{j!} (x' - x)^j + R_{k+l}(x'; x) \right]$$

$$= \psi_k(x''; x) + \sum_l \frac{R_{k+l}(x'; x)}{l!} (x'' - x')^l.$$

7. The function $\Theta(x)$. Let R be the region given by the inequalities $|x_h| < 1$ $(h = 1, \cdots, n)$, let R' be R minus the origin, and let R^* be the boundary of R. Define the functions θ, θ', Θ as follows:

$$(7.1) \qquad \theta(x) = 2(1 - x_1^2) \cdots (1 - x_n^2) - 1 \text{ in } R',$$

$$(7.2) \qquad \theta'(x) = \frac{\theta(x)}{1 - \theta^2(x)} \qquad\qquad \text{in } R',$$

$$(7.3) \qquad \Theta(x) = \begin{cases} e^{\theta'(x)} & \text{in } R', \\ 0 & \text{in } E - R. \end{cases}$$

It is seen that $-1 < \theta(x) < +1$, $\theta(x) \to +1$ as $x \to 0$, and $\theta(x) \to -1$ as $x \to R^*$; hence $\theta'(x) \to +\infty$ as $x \to 0$ and $\theta'(x) \to -\infty$ as $x \to R^*$. Consequently $\Theta(x) \to +\infty$ to infinite order as $x \to 0$ and $\Theta(x) \to 0$ to infinite order as $x \to R^*$; also $\Theta(x)$ is of class C^∞ for $x \neq 0$. If $\Theta'(x) = 1/\Theta(x)$ in R' and $\Theta'(x) = 0$ for $x = 0$, then $\Theta'(x)$ is of class C^∞ in R.

8. The subdivision of $E - A$. Divide E into n-cubes of side 1, and let K_0 be the set of all these cubes whose distances from A are at least $6n^{1/2}$ (if there are any). In general, having constructed the cubes of K_{s-1}, divide each cube which is now present but is not in $K_0 + \cdots + K_{s-1}$ into 2^n cubes of side $1/2^s$, and let K_s be the set of all these cubes whose distances from A are at least $6n^{1/2}/2^s$ (if there are any).

The distance from any cube C of K_s to A is $< 18n^{1/2}/2^s$ $(s \geq 1)$; for it lies in a cube C' of the previous subdivision which does not belong to K_{s-1}, and whose distance from A is therefore $< 6n^{1/2}/2^{s-1}$.

Any cube C of K_s is separated from any cube C' of K_{s+2} by at least four cubes of K_{s+1}. For the distance from C to A is $\geq 12n^{1/2}/2^{s+1}$, the distance from any point of C' to A is $< 9n^{1/2}/2^{s+1}$, and the diameter of any cube of K_{s+1} is $n^{1/2}/2^{s+1}$.

9. The functions $\phi_\nu(x)$. We introduce the following definitions:

$y^1, y^2, \cdots$ is the set of all vertices of cubes of $K_0 + K_1 + \cdots$, arranged in a sequence.

r_ν is the distance from y_ν to A $(\nu = 1, 2, \cdots)$.

x^ν is a fixed point of A whose distance from y^ν is r_ν.

b_ν is the length of side of the largest cube of $K_0 + K_1 + \cdots$ with y^ν as a vertex.

I_ν is the set of points x for which $|x_h - y_h^\nu| \leq b_\nu$ $(h = 1, \cdots, n)$; B_ν is its boundary.

$$\pi_\nu(x) = \Theta\left(\frac{x_1 - y_1^\nu}{b_\nu}, \cdots, \frac{x_n - y_n^\nu}{b_\nu}\right) \quad \text{in} \quad E - y^\nu;$$

$$\pi_\nu'(x) = \Theta'\left(\frac{x_1 - y_1^\nu}{b_\nu}, \cdots, \frac{x_n - y_n^\nu}{b_\nu}\right) \quad \text{in} \quad I_\nu - B_\nu.$$

$$\phi_\nu(x) = \begin{cases} \dfrac{\pi_\nu(x)}{\sum_\lambda \pi_\lambda(x)} & \text{in} \quad E - A, \ x \neq y^1, y^2, \cdots, \\ 1, & x = y^\nu, \\ 0, & x = y^\mu \ (\mu \neq \nu). \end{cases}$$

Suppose y^* is a given point of $E-A$, distant δ_* from A (or from a given point x^0 of A), and suppose y^* lies in the cube C of K_s. Then if I_ν, with center y^ν, has points in common with C, and y^ν is distant d_ν from A (or from x^0),

$$(9.1) \qquad\qquad \delta_*/2 \leqq d_\nu < 2\delta_*.$$

To prove this, say C' is a largest cube with y^ν as a vertex, and C' is in K_t; then $t \geqq s-1$. The diameter of C' is $n^{1/2}/2^t$; hence y^ν is distant at most $n^{1/2}/2^t \leqq 2n^{1/2}/2^s$ from any point of I_ν. As the diameter of C is $n^{1/2}/2^s$, y^ν is distant at most $3n^{1/2}/2^s$ from y^*. But $\delta_* \geqq 6n^{1/2}/2^s$, and the inequalities follow.

Each function $\pi_\nu(x)$ is >0 in $I_\nu - B_\nu - y^\nu$ and only there; it approaches ∞ and 0 to infinite order as x approaches y^ν and B_ν, respectively. Each point x of $E-A$ is interior to some cube I_ν, hence $\pi_\nu(x) > 0$ for some ν, and $\sum \pi_\lambda(x) > 0$ in $E-A$, justifying the definition of $\phi_\nu(x)$. Note that $\phi_\nu(x)$ is $\neq 0$ in $I_\nu - B_\nu$ and only there; also

$$(9.2) \qquad\qquad \sum_\nu \phi_\nu(x) = 1 \quad \text{in} \quad E - A.$$

We shall show that $\phi_\nu(x)$ is of class C^∞ in $E-A$. This is obvious at points $x \neq y^\nu$. Consider a small neighborhood U_λ of y^λ, $\lambda \neq \nu$. $\pi_\lambda'(x)$ is of class C^∞ in U_λ; hence the same is true of $\phi_\nu = \pi_\lambda' \pi_\nu/(1 + \pi_\lambda' \sum_{\mu \neq \lambda} \pi_\mu)$ in U_λ. Similarly $\phi_\nu = 1/(1 + \pi_\nu' \sum_{\mu \neq \nu} \pi_\mu)$ is of class C^∞ in a small neighborhood U_ν of y^ν; the statement follows.

10. **The derivatives of the $\phi_\nu(x)$.** Consider two (closed) cubes C and C' of $K_0 + K_1 + \cdots$, and let J and J' be those sets I_ν with points in C and C' respectively. We shall say C and C' *are of the same type* if the sets in J' can be brought into coincidence with the sets in J by a translation and stretching of the axes, that is, if the structure of the subdivision about C' is the same as that about C. There are but a finite number, say d, of possible types of cubes, and for some number c, there are at most c sets I_ν with points in any given cube C.

Take a fixed cube C of K_0 and a fixed k. As each $\phi_\nu(x)$ is of class C^∞, $D_k\phi_\nu(x)$ is bounded in C; there are only a finite number of these functions $\neq 0$ in C, and hence they are uniformly bounded:

$$|D_k\phi_\nu(x)| < N_k(C) \quad \text{in} \quad C \qquad (\nu = 1, 2, \cdots).$$

Consider now any cube C' of any K_s, and let C be a (perhaps hypothetical) cube of K_0 of the same type as C'. If $I_{\lambda_1'}, \cdots, I_{\lambda_t'}$ are the sets I_λ with points in C', let $I_{\lambda_1}, \cdots, I_{\lambda_t}$ be the corresponding sets with points in C; the latter set of sets is carried into the former by a translation of the axes and a stretching by a factor $1/2^s$. Each function ϕ_{λ_q} corresponding to I_{λ_q} goes thereby into the function

$$\phi_{\lambda_q'}(x) = \phi_{\lambda_q}[y^{\lambda_q} + 2^s(x - y^{\lambda_q'})]$$

corresponding to $I_{\lambda_q'}$. Therefore, differentiating σ_k times with respect to x,

$$D_k\phi_{\lambda_q'}(x) = 2^{s\sigma_k}D_k\phi_{\lambda_q}[y^{\lambda_q} + 2^s(x - y^{\lambda_q'})]$$

for x in C', and hence

$$|D_k\phi_\nu(x)| < 2^{s\sigma_k}N_k(C) \quad \text{in} \quad C' \qquad (\nu = 1, 2, \cdots),$$

as $\phi_\nu(x) = 0$ in C' for $\nu \neq \lambda_1', \cdots, \lambda_t'$. Now the constants $N_k(C)$ take on at most d distinct values for a fixed k; if we let N_k be the largest of these, we can state: *Given any n-fold set of numbers k, there is a number N_k such that if C is any cube of K_s, then*

$$(10.1) \qquad\qquad |D_k\phi_\nu(x)| < 2^{s\sigma_k}N_k \quad \text{in} \quad C \qquad (\nu = 1, 2, \cdots).$$

11. A differentiable extension of $f(x)$, m finite. We are now in a position to prove, for m finite,

LEMMA 2. *Under the conditions of Theorem I, there is a function $g(x)$ of class C^∞ in $E-A$, having the properties (1) and (2) of Theorem I.*

For each ν $(\nu = 1, 2, \cdots)$ there are functions $\phi_\nu(x)$ and $\psi(x; x^\nu) = \psi_0(x; x^\nu)$; we put

$$(11.1) \qquad g(x) = \begin{cases} \sum_\nu \phi_\nu(x)\psi(x; x^\nu) & \text{in } E - A, \\ f(x) & \text{in } A. \end{cases}$$

As the $\phi_\nu(x)$ and $\psi(x; x^\nu)$ are of class C^∞ in $E-A$, the same is true of $g(x)$. The function $g(x) = f(x)$ is of class C^m at all inner points of A, by §3. It remains to show that $D_kg(x)$ exists, equals $f_k(x)$, and is continuous, at all boundary points of A, for $\sigma_k \leq m$.

Take a fixed boundary point x^0 of A, and any ϵ, $0 < \epsilon < 1$. Take

$$\eta < \epsilon/\{2c[(m+2)!]^n(108n^{1/2})^mN\} \quad \text{and} \quad \eta < \epsilon/6,$$

where N is the largest of the numbers N_k for $\sigma_k \leq m$. Take $M > |f_k(x)|$ ($\sigma_k \leq m$, x in A and $r_{xx^0} \leq 1$), and take

$$\delta < \epsilon/\{6(m+1)^nM\} \quad \text{and} \quad \delta < 1$$

so small that (3.2) holds at the point x^0 with ϵ replaced by η. Take now any point y^* of $E-A$ within a distance $\delta/4$ of x^0; we shall show that

$$(11.2) \qquad\qquad |D_kg(y^*) - f_k(x^0)| < \epsilon \qquad\qquad (\sigma_k \leq m).$$

Say the distance from y^* to A is $\delta_*/4$ (then $\delta_* < \delta$), and let x^* be a point of A distant $\delta_*/4$ from y^*. Consider the sum in (6.1) with x' and x replaced by x^* and x^0 respectively; as each l_h is $\leq m$, it contains at most $(m+1)^n$ terms. If we take the term with $l_1 = \cdots = l_n = 0$ to the other side, there is in each remaining term a factor $(x_h^* - x_h^0)^{l_h}$ with $l_h > 0$. As each $|x_h^* - x_h^0|$ is $< \delta < 1$, we find

$$|\psi_k(x^*; x^0) - f_k(x^0)| < (m+1)^nM\delta < \epsilon/6.$$

But also $|R_k(x^*; x^0)| < \eta < \epsilon/6$; hence, using (6.2),

$$|f_k(x^*) - f_k(x^0)| < \epsilon/3.$$

Similarly we see that $|\psi_k(y^*; x^*) - f_k(x^*)| < \epsilon/6$; therefore

$$(11.3) \qquad\qquad |\psi_k(y^*; x^*) - f_k(x^0)| < \epsilon/2 \qquad\qquad (\sigma_k \leq m).$$

Say y^* lies in the cube C of K_s, and let $I_{\lambda_1}, \cdots, I_{\lambda_t}$ be those sets I_λ with points in C. Each corresponding point y^{λ_s} is distant $< \delta/2$ from x^0, by (9.1), and hence each corresponding point x^{λ_s} is distant $< \delta$ from x^0. As the same is true of x^*, (3.2) gives

$$(11.4) \qquad\qquad |R_k(x^\nu; x^*)| \leq r_{x^\nu x^*}^{m-\sigma_k}\eta \qquad\qquad (\nu = \lambda_1, \cdots, \lambda_t).$$

Set

$$\zeta_{\nu;k}(x) = \psi_k(x; x^\nu) - \psi_k(x; x^*) \qquad (\nu = \lambda_1, \cdots, \lambda_t);$$

then as $r_{x^\nu x^*} < \delta_*$ and $|x_h - x_h^\nu| < \delta_*$ for x in C, $|(x-x^\nu)^l| < \delta_*^{\sigma_l}$, and (6.3) and (11.4) give

$$(11.5) \qquad\qquad |\zeta_{\nu;k}(x)| < (m+1)^n\delta_*^{m-\sigma_k}\eta \text{ in } C \qquad (\nu = \lambda_1, \cdots, \lambda_t).$$

Using (9.2), we see that

$$(11.6) \qquad\qquad g(x) = \psi(x; x^*) + \sum_{s=1}^t \phi_{\lambda_s}(x)\zeta_{\lambda_s;0}(x) \text{ in } C.$$

As $D_k\psi(x; x^\nu) = \psi_k(x; x^\nu)$ and therefore $D_k\zeta_{\nu;0}(x) = \zeta_{\nu;k}(x)$,

$$D_k g(x) = \psi_k(x; x^*) + \sum_{s=1}^{t} \sum_l \binom{k}{l} D_l\phi_{\lambda_s}(x)\zeta_{\lambda_s;k-l}(x) \quad \text{in } C.$$

(10.1) and (11.5) give, as $t \leq c$ (see §10)· and

$$\binom{k_h}{l_h} \leq m!,$$

(11.7)　　$\left| D_k g(x) - \psi_k(x; x^*) \right| < \sum_l c[(m+1)!]^n 2^{s\sigma_l} N\delta_*{}^{m-\sigma_k+\sigma_l}\eta \quad \text{in } C.$

Now the distance from C to A is $> \delta_*/6$; also, as C is in K_s, this distance is $< 18n^{1/2}/2^s$. Hence $18n^{1/2}/2^s > \delta_*/6$, or, $2^s < 108n^{1/2}/\delta_*$. This gives, as $\sigma_k \leq m$ and $\delta_* < 1$,

(11.8)　　$\left| D_k g(x) - \psi_k(x; x^*) \right| < c[(m+2)!]^n (108n^{1/2})^m N\delta_*{}^{m-\sigma_k}\eta < \epsilon/2$

in C, and in particular, at y^*. This inequality together with (11.3) gives (11.2), as required.

The proof can now be completed with the aid of Lemma 1. (11.2) with $k=0$ shows that $g(x)$ is continuous throughout E. Take any number $k = (k_1, \cdots, k_n)$ with $\sigma_k < m$, and put $k' = (k_1, \cdots, k_h+1, \cdots, k_n)$. Assuming that $D_k g(x)$ is continuous in E, we shall show that $D_{k'} g(x)$ exists and is continuous in E. Take any boundary point $x^0 = (x_1^0, \cdots, x_n^0)$ and put $z_0 = x_h^0$, $w(z) = w(x_h) = D_k g(x_1^0, \cdots, x_h, \cdots, x_n^0)$, $w_0' = f_{k'}(x^0)$. Let A^* be the set of points of A for which $x_p = x_p^0$ ($p \neq h$). (3.3) with $x = x^0$ and $\Delta x_h = x_h - x_h^0$, and (11.2) with k replaced by k', show that the conditions of the lemma are fulfilled; hence $\partial w(z_0)/\partial x_h = D_{k'} g(x^0)$ exists and equals $f_{k'}(x^0)$. (11.2) shows that $D_{k'} g(x^0)$ is continuous at x^0. Therefore $g(x)$ is of class C^m in E.

12. **A differentiable extension of $f(x)$, m infinite.** We now prove Lemma 2 for the case $m = \infty$. For any given m, let $\psi_{m;k}(x'; x)$ ($\sigma_k \leq m$) be the function given by the right hand side of (6.1). Choose the axes so that the origin falls on a point of A. Let S_p be the set of all points of E whose distances from the origin are $\leq 2^p$, $p = 1, 2, \cdots$. Let M_p be the maximum of $\left| f_k(x) \right|$ for $\sigma_k \leq p$ and x in $A \cdot S_p$, and let $N^{(p)}$ be the maximum of N_k for $\sigma_k \leq p$. Choose for each positive integer p a number δ_p such that

$$\delta_p < 1/\{2^{2p+1}c[(p+2)!]^n(36n^{1/2})^p N^{(p)} M_{p+1}\}, \quad \delta_p < \delta_{p-1}/2.$$

The extension $g^{(\infty)}(x)$ of $f(x)$ is determined as follows. Given any number ν, determine the number γ_ν so that $\delta_{\gamma_\nu+1} \leq r_\nu < \delta_{\gamma_\nu}$ (see §9); set $\gamma_\nu = 0$ if $r_\nu > \delta_1$.

Put

$$(12.1) \qquad g^{(\infty)}(x) = \begin{cases} \sum_{\nu} \phi_\nu(x)\psi_{\gamma_\nu;0}(x; x^\nu) & \text{in } E - A, \\ f(x) & \text{in } A. \end{cases}$$

Given any fixed k, we shall find an inequality similar to (11.2) for $D_k g^{(\infty)}(x)$. Let $g^{(m)}(x)$ be the extension of $f(x)$ of class C^m given by Lemma 2 ($m = 1, 2, \cdots$). Given any boundary point x^0 of A and any $\epsilon > 0$, choose $p \geqq \sigma_k + 2$ so that x^0 lies in S_p and so that $1/2^p < \epsilon$. Take $\delta < \delta_p$ so that (11.2) with g replaced by $g^{(\sigma_k)}$ will hold for our given k and any y^* of $E - A$ within δ of x^0; we show next that for any such y^*,

$$(12.2) \qquad \left| D_k g^{(\infty)}(y^*) - D_k g^{(\sigma_k)}(y^*) \right| < \epsilon.$$

Choose q so that $\delta_{q+1} \leqq \delta_* < \delta_q$, where δ_* is the distance from y^* to A; then $q \geqq p$. Define C, K_*, $I_{\lambda_1}, \cdots, I_{\lambda_t}$ as in §11. Note that for $\nu = $ any λ_h, $\delta_{\gamma_\nu+1} \leqq r_\nu < 2\delta < 2\delta_p < \delta_{p-1}$, hence $\gamma_\nu + 1 > p - 1$, and thus $\gamma_\nu > p - 2 \geqq \sigma_k$. Set

$$(12.3) \qquad \xi_\nu(x) = \psi_{\gamma_\nu;0}(x; x^\nu) - \psi_{\sigma_k;0}(x; x^\nu) \qquad (\nu = \lambda_1, \cdots, \lambda_t);$$

using (12.1) and (11.1), we see that

$$(12.4) \qquad g^{(\infty)}(x) = g^{(\sigma_k)}(x) + \sum_{u=1}^{t} \phi_{\lambda_u}(x)\xi_{\lambda_u}(x) \quad \text{in } C.$$

Now $D_j \xi_\nu(x) = \psi_{\gamma_\nu;j}(x; x^\nu) - \psi_{\sigma_k;j}(x; x^\nu)$. If we replace k by j in (6.1), then those and only those terms in the sum with $\sigma_l \leqq m - \sigma_j$ occur. Replacing m by γ_ν and σ_k successively and subtracting, we have

$$(12.5) \qquad D_j \xi_\nu(x) = \sum_{\sigma_l = \sigma_k - \sigma_j + 1}^{\gamma_\nu - \sigma_j} \frac{f_{j+l}(x^\nu)}{l!} (x - x^\nu)^l \quad \text{in } C.$$

Now $r_\nu > \delta_*/2$, by (9.1), hence $r_\nu > \delta_{q+2}$, and thus $\gamma_\nu \leqq q + 1$ ($\nu = \lambda_1, \cdots, \lambda_t$); there are therefore less than $(q+2)^n$ terms in the sum, and in each term, $\sigma_j + \sigma_l \leqq q + 1$. It follows that $\left| f_{j+l}(x^\nu) \right| < M_{q+1}$ in each term. Also $\left| x_h - x_{h'}^\nu \right| < 2\delta_* < 2\delta_q$ and $\sigma_l \geqq \sigma_k - \sigma_j + 1$ in each term; hence

$$\left| D_j \xi_\nu(x) \right| < (q+2)^n M_{q+1} 2^{q+1} \delta_*^{\sigma_k - \sigma_j} \delta_q$$

in C. This with (12.4) gives

$$\left| D_k g^{(\infty)}(x) - D_k g^{(\sigma_k)}(x) \right| \leqq \sum_{u=1}^{t} \sum_{j} \binom{k}{j} \left| D_{k-j} \phi_{\lambda_u}(x) \right| \left| D_j \xi_{\lambda_u}(x) \right|$$

$$< c \sum_{j} \binom{k}{j} 2^{s(\sigma_k - \sigma_j)} N^{(\sigma_k)}(q+2)^n M_{q+1} 2^{q+1} \delta_*^{\sigma_k - \sigma_j} \delta_q$$

in C. Now the distance from C to A is $> \delta_*/2$ and is $< 18n^{1/2}/2^\bullet$; hence $2^\bullet < 36n^{1/2}/\delta_*$. Also $\sigma_k < p \leqq q$; therefore

(12.6)
$$\left| D_k g^{(\infty)}(x) - D_k g^{(\sigma_k)}(x) \right| < c[(q+2)!]^n (36n^{1/2})^q N^{(q)} M_{q+1} 2^{q+1} \delta_q$$
$$< 1/2^q < \epsilon$$

in C, and in particular, at y^*, proving (12.2). Using (11.2), we find $\left| D_k g^{(\infty)}(y^*) - f_k(x^0) \right| < 2\epsilon$ for any point y^* of $E - A$ within δ of x^0. Again we can apply Lemma 1 and show that $D_k g^{(\infty)}(x)$ exists and is continuous throughout E. As this is true for every k, the proof is complete.

13. We prove next a combined extension and approximation theorem.

LEMMA 3. *Let $f(x)$ be of class C^m (m finite) in E, with $D_k f(x) = f_k(x)$ ($\sigma_k \leqq m$) there, and let $f_k(x)$ ($m < \sigma_k \leqq m'$, $m' > m$ finite or infinite) be defined in the closed set A so that $f(x)$ (considered now only in A) is of class $C^{m'}$ there. Then for an arbitrary $\epsilon > 0$ there is a function $g(x)$ which is of class C^m in E, of class $C^{m'}$ in a neighborhood of A, and equals $f(x)$ outside another neighborhood of A, such that*

(13.1)
$$D_k g(x) = f_k(x) \text{ in } A \qquad\qquad (\sigma_k \leqq m'),$$

and

(13.2)
$$\left| D_k g(x) - D_k f(x) \right| < \epsilon \text{ in } E \qquad\qquad (\sigma_k \leqq m).$$

Let $f'(x)$ be the extension of class $C^{m'}$ of the values of $f(x)$ in A given by the last lemma, and put $\zeta(x) = f'(x) - f(x)$; then $\zeta(x)$ is of class C^m in E, and

$$D_k \zeta(x) = 0 \text{ in } A \qquad\qquad (\sigma_k \leqq m).$$

Set $\eta = \epsilon / \{ c[(m+1)!]^n (36n^{1/2})^m N \}$ ($N = \max N_k$ for $\sigma_k \leqq m$). As $\zeta(x)$ is of class C^m and $D_k \zeta(x)$ vanishes in A ($\sigma_k \leqq m$), we can find an open set R containing A so that if y is any point of $R - A$, at a distance δ from A, then

$$\left| D_k \zeta(y) \right| < \eta \delta^{m - \sigma_k} \qquad\qquad (\sigma_k \leqq m).$$

Let $\nu_1, \nu_2, \cdots$ be those numbers such that I_{ν_p} lies wholly in R ($p = 1, 2, \cdots$). We set

(13.3)
$$g(x) = f(x) + \zeta(x) \sum_{p=1}^{\infty} \phi_{\nu_p}(x) \text{ in } E - A,$$

and $g(x) = f(x)$ in A. As $\sum \phi_{\nu_p}(x) = 1$ in an open set surrounding A, $g(x) = f'(x)$ there. As $\sum \phi_{\nu_p}(x) = 0$ in $E - R$, $g(x) = f(x)$ there. The statements about the class of $g(x)$ are true. To show that (13.2) holds, let y be a point of $R - A$, distant δ from A; then, defining $C, K_s, I_{\lambda_1}, \cdots, I_{\lambda_t}$ as in the previous lemma, we have

$$\left| D_k g(y) - D_k f(y) \right| \leqq \sum_p \sum_l \binom{k}{l} \left| D_l \phi_{\nu_p}(y) \right| \left| D_{k-l} \zeta(y) \right|$$

$$< c \sum_l (m!)^n 2^{\sigma_l} N \delta^{m-\sigma_k+\sigma_l} \eta < \epsilon \qquad (\sigma_k \leqq m).$$

14. We close this section with a theorem concerning the isolated points of A. Define α_p as follows:

(14.1)
$$\alpha_p = \begin{cases} m \text{ if } m \text{ is finite} \\ p \text{ if } m \text{ is infinite} \end{cases} \qquad (p = 1, 2, \cdots).$$

LEMMA 4. *Consider the closed set $A = A' + a_1 + a_2 + \cdots$, where $a_1, a_2, \cdots$ are isolated points (then A' is closed), and let m be finite or infinite. Let $f_k(x)$ be defined in A' for $\sigma_k \leqq m$ and at each a_s for all k, so that $f(x)$ is of class C^m in A in terms of the $f_k(x)$ ($\sigma_k \leqq m$). Then there is a function $g'(x)$ of class C^∞ in $E - A'$ and of class C^m in E, such that*

(14.2) $\qquad D_k g'(x) = f_k(x)$ in A' for $\sigma_k \leqq m$ and at each a_r for all k.

Let $g(x)$ be the extension of $f(x)$ of class C^m given by Lemma 2. Let $U_1, U_2, \cdots$ be neighborhoods of $a_1, a_2, \cdots$, chosen so that each is at a positive distance from each other and from A'. If m is finite, we alter $g(x)$ in U_1, next in U_2, etc., by means of the last lemma†, so that the new function $g'(x)$ will take on the required derivatives at $a_1 + a_2 + \cdots$, and so that

(14.3) $\qquad \left| D_k g'(x) - D_k g(x) \right| < 1/p$ in U_p $(\sigma_k \leqq \alpha_p, p = 1, 2, \cdots)$.

(14.2) is an immediate consequence of this inequality and Lemma 1.

III. APPROXIMATION THEOREMS

15. We prove first the following extension of the Weierstrass approximation theorem.‡

LEMMA 5. *Let $g(x)$ be of class C^m in E (m finite), and let S be a bounded closed set in E.§ Then for each $\epsilon > 0$ there exists a function $G(x)$ analytic in E and such that*

(15.1) $\qquad \left| D_k G(x) - D_k g(x) \right| < \epsilon$ in S $(\sigma_k \leqq m)$.

Let R_b be the set of points distant at most b from the origin ($b \geqq 0$). Consider the n-tuple integral

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† We use the last lemma with A replaced by a_r and m' by ∞.

‡ Compare de la Vallée Poussin, *Cours d'Analyse*, vol. II, 2d edition, 1912, pp. 126–137.

§ It is sufficient that $g(x)$ be defined over S, for we can then extend its definition over E, by Lemma 2.

$$(15.2) \quad \Phi(b) = T \int_{R_b} e^{-r_{0y}^2} dy = T \int \cdots \int e^{-(u_1^2 + \cdots + u_n^2)} dy_1 \cdots dy_n,$$

where T is chosen so that $\Phi(\infty) = 1$; then $0 \leq \Phi(b) \leq 1$ for all b. If we replace y by κy and b by κb, we see that

$$(15.3) \quad \Phi(\kappa b) = T\kappa^n \int_{R_b} e^{-\kappa^2 r_{0y}^2} dy.$$

Let $v(x)$ be a function $\equiv 1$ in S, $\equiv 0$ outside some neighborhood of S, and of class C^∞ in E, such that $D_k v(x) = 0$ in S for all k. (Such a function may be found for instance by the aid of Lemma 2.) Put $g'(x) = v(x)g(x)$, and

$$(15.4) \quad G(x) = T\kappa^n \int_E g'(y) e^{-\kappa^2 r_{xy}^2} dy,$$

where κ will be chosen later; $G(x)$ is analytic in E. As r_{xy} is a function of $y - x$ alone, differentiating under the integral sign gives

$$D_k G(x) = T\kappa^n \int_E g'(y) D_k^{(x)} e^{-\kappa^2 r_{xy}^2} dy = (-1)^{\sigma_k} T\kappa^n \int_E g'(y) D_k^{(y)} e^{-\kappa^2 r_{xy}^2} dy,$$

where $D_k^{(x)}$ and $D_k^{(y)}$ denote differentiation with respect to x and y respectively. Integrating by parts σ_k times gives

$$(15.5) \quad D_k G(x) = T\kappa^n \int_E D_k g'(y) e^{-\kappa^2 r_{xy}^2} dy.$$

As $\Phi(\infty) = 1$, we see that

$$(15.6) \quad D_k G(x) - D_k g'(x) = T\kappa^n \int_E [D_k g'(y) - D_k g'(x)] e^{-\kappa^2 r_{xy}^2} dy.$$

Take M so large that

$$(15.7) \qquad |D_k g'(x)| < M \text{ in } E \qquad\qquad (\sigma_k \leq m).$$

The functions $D_k g'(x)$ are uniformly continuous in E; hence there is a $\delta > 0$ such that

$$(15.8) \qquad |D_k g'(y) - D_k g'(x)| < \epsilon/2 \qquad (r_{xy} < \delta, \sigma_k \leq m).$$

Take κ so large that

$$(15.9) \qquad 1 - \Phi(\kappa\delta) < \epsilon/(4M).$$

For a given x, let U consist of all points within δ of x; then if J_1 and J_2 are formed by replacing the domain of integration on the right hand side of

(15.6) by U and $E-U$ respectively, we have, using (15.3),

$$|J_1| < T\kappa^n \int_U \frac{\epsilon}{2} e^{-\kappa^2 r_{zv}^2} dy = \frac{\epsilon}{2} \Phi(\kappa\delta) < \frac{\epsilon}{2},$$

$$|J_2| < T\kappa^n \int_{E-U} 2M e^{-\kappa^2 r_{zv}^2} dy = 2M[1 - \Phi(\kappa\delta)] < \frac{\epsilon}{2},$$

and hence $|D_k G(x) - D_k g'(x)| < \epsilon$ in $E(\sigma_k \leq m)$, which gives (15.1).

$G(x)$ may of course be replaced by a polynomial if desired.

16. The above lemma can be generalized as follows.

LEMMA 6. *Let R be an open set and let R_1, R_2, $\cdots$ be bounded open sets (some of which may be void) whose sum is R, such that each $\overline{R}_p = R_p$ plus boundary is in R_{p+1}. Then if $g(x)$ is defined and of class C^m (m finite or infinite) in R, and $\epsilon_1 \geq \epsilon_2 \geq \cdots$ are given positive numbers, there is an analytic function $G(x)$ defined in R such that*

(16.1) $|D_k G(x) - D_k g(x)| < \epsilon_p$ *in* $R - R_p$ $(\sigma_k \leq \alpha_p, p = 1, 2, \cdots)$.

α_p is defined in (14.1). Note that, if R_1, $\cdots$, R_q are void, then

(16.2) $|D_k G(x) - D_k g(x)| < \epsilon_q$ in R $(\sigma_k \leq \alpha_q)$.

Consider the closed set $\overline{R}_{p-1} + (\overline{R}_{p+1} - R_p) + (E - R_{p+2}) = Q_p' + Q_p + Q_p''$; if in Lemma 2 we replace A by this set and $f(x)$ by a function $\equiv 1$ in Q_p and $\equiv 0$ in $Q_p' + Q_p''$, we find a function $u_p(x)$ for each p, of class C^∞ in E, such that

(16.3) $u_p(x) = \begin{cases} 1 \text{ in } Q_p, \\ 0 \text{ in } Q_p' + Q_p''; \end{cases}$ $D_k u_p(x) = 0$ in $Q_p' + Q_p + Q_p''(\sigma_k > 0)$.

(If R_{p+1} is void, we put $u_p(x) \equiv 0$; if R_{p+1} is not void but R_{p-1} is void, we have $u_p(x) = 0$ in Q_p'' and $= 1$ in $\overline{R}_{p+1}$.) Let $Z_p \geq 1$ be such a number that

(16.4) $|D_k u_p(x)| < Z_p$ in E $(\sigma_k \leq \alpha_p, p = 1, 2, \cdots)$.

We define successively analytic functions $G_1(x)$, $G_2(x)$, $\cdots$, by the following formula:

(16.5) $G_p(x) = T\kappa_p^n \int_E u_p(y)[g(y) - \{G_1(y) + \cdots + G_{p-1}(y)\}]e^{-\kappa_p^2 r_{zv}^2} dy.$

(For $p = 1$, the factor in brackets is simply $g(y)$.) κ_p is chosen so that, if we set

(16.6) $H_p(x) = u_p(x)[g(x) - \{G_1(x) + \cdots + G_{p-1}(x)\}],$

then

(16.7) $\quad \left| D_k G_p(x) - D_k H_p(x) \right| < \beta_p' = \epsilon_{p+1}/\{2^{p+2}[(\alpha_{p+1}+1)!]^n Z_{p+1}\}$

$$\text{in } \overline{R}_{p+1} \qquad (\sigma_k \leqq \alpha_{p+1})$$

(see Lemma 5); we shall restrict κ_p further later. Remembering the definition of $u_p(x)$, we see that (16.7) with (16.6) gives

(16.8) $\quad \left| D_k g(x) - D_k\{G_1(x) + \cdots + G_p(x)\} \right| < \beta_p' < \epsilon_p/2 \text{ in } Q_p(\sigma_k \leqq \alpha_{p+1}).$

Differentiating $H_p(x)$ and using (16.4) and (16.8) with p replaced by $p-1$, we see that (compare the derivation of (11.7))

$$\left| D_k H_p(x) \right| < [(\alpha_p+1)!]^n Z_p \beta_{p-1}' = \epsilon_p/2^{p+1} \text{ in } Q_{p-1} \qquad (\sigma_k \leqq \alpha_p).$$

As $u_p(x)$ and its derivatives are 0 in R_{p-1}, this holds in R_{p-1} also; hence, using (16.7), we have

(16.9) $$\left| D_k G_p(x) \right| < \epsilon_p/2^p \text{ in } \overline{R}_p \qquad (\sigma_k \leqq \alpha_p).$$

We set now

(16.10) $$G(x) = G_1(x) + G_2(x) + \cdots ;$$

this is the desired approximation to $g(x)$. To prove this, we see first from (16.9) that $D_k[G_1(x) + \cdots + G_p(x)]$ converges uniformly in any bounded closed subset of R $(\sigma_k \leqq m)$; hence $G(x)$ is defined in R, and

(16.11) $$D_k G(x) = D_k G_1(x) + D_k G_2(x) + \cdots \text{ in } R \qquad (\sigma_k \leqq m).$$

Next (16.9) shows that

(16.12) $\quad \begin{aligned} \left| D_k G_{p+1}(x) + D_k G_{p+2}(x) + \cdots \right| &< \epsilon_{p+1}/2^{p+1} + \epsilon_{p+2}/2^{p+2} + \cdots \\ &\leqq \epsilon_p(1/2^2 + 1/2^3 + \cdots) = \epsilon_p/2 \text{ in } \overline{R}_{p+1} \qquad (\sigma_k \leqq \alpha_{p+1}); \end{aligned}$

this with (16.8) gives $\left| D_k G(x) - D_k g(x) \right| < \epsilon_p$ in $Q_p(\sigma_k \leqq \alpha_p)$, proving (16.1).

It remains to be shown that $G(x)$ is analytic in R. To this end we extend the definition of each $G_p(x)$ to complex values of $x = (x_1' + ix_1'', \cdots, x_n' + ix_n'')$, using (16.5) still. Consider the analytic function of x

$$r_{xy}^2 = \sum(y_h - x_h)^2 = \sum[(y_h' - x_h') + i(y_h'' - x_h'')]^2;$$

as $y_h'' = 0$ in (16.5), the domain of integration being real,

$$\Re(r_{xy}^2) = \sum[(y_h' - x_h')^2 - x_h''^2].$$

Take any point x^0 of R and let U be the complex region of radius ρ about x^0, where ρ is so small that the real points in the complex region of radius 3ρ about x^0 lie in some R_q; we take q so that $3\rho^2 > 1/2^q$. Now if $p > q$, x is in U, and y is in $R - R_{p-1}$, then $\sum x_h''^2 < \rho^2$ and $\sum(y_h' - x_h')^2 \geqq 4\rho^2$, and hence

$$\Re(r_{xy}^2) > 3\rho^2.$$

Also $H_p(y)$ vanishes in R_q and in $E-R_{p+2}$ for $p>q$; therefore if M'_p is the maximum of $|H_p(y)|$ (note that $H_p(y)$ is determined before we determine κ_p) and V_p is the volume of R_p ($p=1, 2, \cdots$),

$$(16.13) \qquad |G_p(x' + ix'')| < T\kappa_p^n \int_{R_{p+2}-R_{p-1}} M'_p e^{-3\kappa_p^2 \rho^2} dy$$

$$< T\kappa_p^n \, e^{-\kappa_p^2/2^p} M'_p V_{p+2}$$

for x in U and $p>q$. Hence if we choose κ_p successively for $p=1, 2, \cdots$, so that this quantity is $<1/2^p$ (and so that (16.7) holds), then the series in (16.10), when defined for complex values of x, converges uniformly in a complex neighborhood of any point of R. Therefore the function $G(x)$ is analytic in R, completing the proof.

17. The numbers κ_p as chosen above depend not only on the functions $u_p(x)$ but also on the function $g(x)$. Under certain restrictions, we can take them independent of $g(x)$, as follows.

LEMMA 7. *Let the open sets R_1, R_2, $\cdots$, the numbers ϵ_1, ϵ_2, $\cdots$, and the functions $u_1(x)$, $u_2(x)$, $\cdots$ be given as in Lemma 6; let $\Delta_1(r)$, $\Delta_2(r)$, $\cdots$ be a sequence of positive continuous functions defined for $r>0$, such that $\Delta_p(r)\to 0$ as $r\to 0$ and $\Delta_{p+1}(r)\geq\Delta_p(r)$; let a be a point of R, and M a positive number. Then there is a sequence of numbers κ_1, κ_2, $\cdots$, with the following property. If $g(x)$ is any function of class C^m defined in R such that $|g(a)|\leq M$ and*

$$(17.1) \qquad |D_k g(x') - D_k g(x)| < \Delta_p(r_{xx'}) \text{ in } \overline{R}_p \qquad (\sigma_k \leq \alpha_p, p = 1, 2, \cdots),$$

and if $G(x)$ is defined in terms of $g(x)$ as in the previous lemma, using the above numbers κ_p, then $G(x)$ is analytic in R and (16.1) holds.

As the u's and their derivatives are uniformly continuous in E, there are functions $\Gamma_p(x)$ of the same sort as the Δ's above such that

$$(17.2) \qquad |D_k u_p(x') - D_k u_p(x)| < \Gamma_p(r_{xx'}) \text{ in } E \ (\sigma_k \leq \alpha_{p+1}, p = 1, 2, \cdots).$$

The conditions on $g(x)$ imply that for some M''_1, $|D_k g(x)|<M''_1$ in $\overline{R}_3$ $(\sigma_k \leq \alpha_2)$.[†] Say

$$|D_k u_p(x)| < Z'_p \text{ in } E \qquad (\sigma_k \leq \alpha_{p+1}, p = 1, 2, \cdots).$$

Then as $u_1(x) = 0$ in $R-R_3$, we have

$$|D_j u_1(x') D_{lg}(x') - D_j u_1(x) D_{lg}(x)| \leq \Gamma_1(r_{xx'}) M''_1 + Z'_1 \Delta_3(r_{xx'})$$

[†] If d_3 is the diameter of $\overline{R}_3$, then $|g(x)|<M+\Delta_1(d_3)$ in $\overline{R}_3$. Now take any $k'=(k_1, \cdots, k_h-1, \cdots, k_n)$ and $k=(k_1, \cdots, k_n)(0<\sigma_k\leq\alpha_2)$. Let $x'x''$ be a line segment parallel to the x_h-axis and lying wholly in $\overline{R}_3$; set $r=|x_h''-x_h'|$. As $|D_{k'}g(x'')-D_{k'}g(x')|<\Delta_1(r)$, the law of the mean gives, for some point x^* of $x'x''$, $|D_k g(x^*)|<\Delta_1(r)/r$. Hence $|D_k g(x)|<\Delta_1(r)/r+\Delta_2(d_3)$ in $\overline{R}_3$ $(0<\sigma_k\leq\alpha_2)$.

for $\sigma_j \leqq \alpha_2$ and $\sigma_l \leqq \alpha_2$ and any x and x' in E. Hence if we put

$$\Delta_1^*(r) = [(\alpha_2 + 1)!]^n [\Gamma_1(r) M_1'' + Z_1' \Delta_3(r)],$$

we shall have, on differentiating $H_1(x) = u_1(x) g(x)$,

(17.3) $$|D_k H_1(x') - D_k H_1(x)| \leqq \Delta_1^*(r_{xx'}) \text{ in } E \qquad (\sigma_k \leqq \alpha_2).$$

Also $|D_k H_1(x)| < [(\alpha_2+1)!]^n Z_1' M_1''$ in $\bar{R}_3$ and $=0$ in $E - R_3$ $(\sigma_k \leqq \alpha_2)$; thus inequalities corresponding to (15.7) and (15.8) hold for $H_1(x)$. Hence if we take $\delta_1 > 0$ so that

$$\Delta_1^*(r) < \beta_1'/2 \qquad (r < \delta_1),$$

and take κ_1 so that

$$1 - \Phi(\kappa_1 \delta_1) < \beta_1'/\{4[(\alpha_2 + 1)!]^n Z_1' M_1''\},$$

then if we form $G_1(x)$ for any admissible $g(x)$ by means of (16.5), (16.7) will hold with $p = 1$; we take κ_1 large enough so that the right hand side of (16.13) with $p = 1$ will be $< 1/2$.

If we differentiate (16.5) with $p = 1$ σ_k times $(\sigma_k \leqq m)$, we derive an equation similar to (15.5); forming this for $x = x$ and $x = x'$ and subtracting, we find (changing y to $y + x' - x$ in one equation)

(17.4) $$D_k G_1(x') - D_k G_1(x) = T\kappa_1^n \int_E [D_k H_1(y + x' - x) - D_k H_1(y)] e^{-\kappa_1^2 r_{xy}^2} dy.$$

This with (17.3), (15.3), and the definition of $\Phi(\infty)$ gives

(17.5) $$|D_k G_1(x') - D_k G_1(x)| \leqq \Delta_1^*(r_{xx'}) \text{ in } E \qquad (\sigma_k \leqq \alpha_2).$$

Assume now we have defined functions $\Delta_p^*(r)$ and have chosen numbers κ_p so that

(17.6) $$|D_k G_p(x') - D_k G_p(x)| \leqq \Delta_p^*(r_{xx'}) \text{ in } E \qquad (\sigma_k \leqq \alpha_{p+1}),$$

so that (16.7) holds, and so that the quantity in (16.13) is $< 1/2^p$, for $p < q$. Then for any admissible $g(x)$, $g(x) - \{G_1(x) + \cdots + G_{q-1}(x)\}$ satisfies the same kind of conditions as $g(x)$; hence, just as before, we find a function $\Delta_q^*(r)$ so that an inequality similar to (17.3) holds for $D_k H_q(x)$ in E $(\sigma_k \leqq \alpha_{q+1})$. Also $H_q(x)$ is bounded properly; hence we can choose κ_q so that (16.7) holds for any admissible $g(x)$ with p replaced by q, and so that (16.13) with $p = q$ is $< 1/2^q$. From this we show, as before, that (17.6) holds with p replaced by q. We can thus continue finding functions $\Delta_p^*(r)$ and numbers κ_p indefinitely. We put finally $G(x) = G_1(x) + G_2(x) + \cdots$, and show, just as in Lemma 6, that $G(x)$ has the required properties. This ends the proof.

IV. Analytic extensions

18. Proof of Theorem I. Let $g(x)$ be the extension of $f(x)$ of class C^m given by Lemma 2. Set $R = E - A$ and define $R_1, R_2, \cdots, \alpha_1, \alpha_2, \cdots$, and numbers $\epsilon_1, \epsilon_2, \cdots$, approaching zero as in §16. Define $G(x)$ in $E - A$ as in Lemma 6, and set $F(x) = G(x)$ in $E - A$, $F(x) = f(x)$ in A. That $F(x)$ is of class C^m in E and property (2) holds follows from (16.1) and Lemma 1, just as in §11; the other facts are obvious.

19. The functions $\omega_{\nu k}(x)$. In the next sections we shall discuss the analyticity of the extension of $f(x)$ at the isolated points of A. Let R be an open set, let $a_1, a_2, \cdots$ be points of R having no limit point in R, let $m_1, m_2, \cdots$ be corresponding integers ≥ 0, and let m be an integer ≥ -1 or ∞. We assume that if $a_{\nu_1}, a_{\nu_2}, \cdots$ is any sequence of points a_ν approaching the boundary of R, then

$$(19.1) \qquad \liminf_{s \to \infty} m_{\nu_s} \geq m.$$

Choose about each a_ν a neighborhood U_ν lying, with its boundary, in R, so that no two have common points. Define the numbers $\rho(\nu; k)$ so that when $(\nu; k)$ runs through the values $(1; k)$, $\sigma_k \leq m_1$; $(2; k)$, $\sigma_k \leq m_2$; etc.; then $\rho(\nu; k)$ runs through the values $1, 2, 3, \cdots$. Let $\rho'(\nu; k)$ equal one plus the largest of the numbers $m_1, \cdots, m_\nu, \rho(\nu; k)$.

Take any positive integer s, and consider all neighborhoods U_ν such that $\rho'(\nu k) \geq s$ for some k $(\sigma_k \leq m_\nu)$; let R_s be the set of all points of R whose distances from these neighborhoods and from the boundary of R are $> 1/s$, and whose distances from the origin are $< s$. Then R_s is a bounded open set, $\overline{R}_s$ lies in R_{s+1} $(s = 1, 2, \cdots)$, $R_1 + R_2 + \cdots = R$, and U_ν lies in $R - R_{\rho'(\nu k)}$ $(\sigma_k \leq m_\nu)$. By Lemma 2, there are functions $\omega_{\nu k}(x)$ of class C^∞ in E, defined for $\sigma_k \leq m_\nu$, $\nu = 1, 2, \cdots$, such that

$$(19.2) \qquad D_l \omega_{\nu k}(a_\nu) = \begin{cases} 1, & l = k \\ 0, & l \neq k \end{cases} \quad (\sigma_l \leq m_\nu); \quad \omega_{\nu k}(x) = 0 \text{ in } E - U_\nu.$$

Choose for each ν a positive number $\beta_\nu < 1/\nu$ so that $\beta_\nu \geq \beta_{\nu+1}$, and

$$(19.3) \qquad \beta_\nu |D_l \omega_{\nu k}(x)| < 1/\nu[(m_\nu + 1)^n] \text{ in } E$$
$$(\sigma_k \leq m_\nu, \ \sigma_l \leq m_\nu, \ \nu = 1, 2, \cdots).$$

Now let $f_{\nu k}$ be any set of numbers, defined for $\sigma_k \leq m_\nu$, $\nu = 1, 2, \cdots$, satisfying the condition

$$(19.4) \qquad |f_{\nu k}| \leq \beta_\nu \qquad (\sigma_k \leq m_\nu, \ \nu = 1, 2, \cdots).$$

Set

(19.5) $c_{\rho(\nu k)} = f_{\nu k}, \quad \omega'_{\rho(\nu k)}(x) = \omega_{\nu k}(x) \quad (\sigma_k \leq m_\nu, \nu = 1, 2, \cdots)$.

Take any $s = \rho(\nu k)$. As $\omega'_s(x) = \omega_{\nu k}(x) = 0$ in $R - U_\nu$, and $R_s = R_{\rho(\nu k)}$ is in $R_{\rho'(\nu k)}$ which has no points in common with U_ν,

(19.6) $\omega'_s(x) = 0$ in R_s $(s = 1, 2, \cdots)$.

20. The transformation L. Define functions $u_1(x), u_2(x), \cdots$ as in Lemma 6. Consider any function

(20.1) $g(x) = \lambda_1 \omega'_1(x) + \lambda_2 \omega'_2(x) + \cdots$ $(|\lambda_s| \leq 1, s = 1, 2, \cdots)$;

such functions and λ's we shall call *admissible*. Set

(20.2) $\epsilon_s = \beta_s / 2^{s+1}$ $(s = 1, 2, \cdots)$.

There are, obviously, functions $\Delta_1(r), \Delta_2(r), \cdots$, so that (17.1) holds for any such $g(x)$; hence, by Lemma 7, we can define numbers $\kappa_1, \kappa_2, \cdots$, so that if $G(x)$ is defined in terms of $g(x)$ as in Lemma 6, then $G(x)$ is analytic in R and (16.1) holds. *In using Lemma 6, we replace α_p by p.*

We note here a certain property of $G(x)$: *If $g(x)$ is admissible and*

(20.3)   *if $g(x) = 0$ in R_s, then $|D_k G(x)| < \epsilon_{s-1}/2^{s-2}$ in $\overline{R}_{s-1}$* $(\sigma_k \leq s - 1)$.

As $u_p(x) = 0$ in $R - R_s$ $(p \leq s - 2)$, $u_p(x)g(x) = 0$ in E $(p \leq s - 2)$. Using (16.5), we see in succession that $G_1(x) \equiv 0, \cdots, G_{s-2}(x) \equiv 0$. This with (16.9) and (16.11) gives

$$|D_k G(x)| \leq \sum_{p=s-1}^{\infty} |D_k G_p(x)| < \sum_{p=s-1}^{\infty} \epsilon_p / 2^p \leq \epsilon_{s-1}/2^{s-2}$$

in $\overline{R}_{s-1}$ $(\sigma_k \leq s - 1)$, as required.

Given any admissible $g(x)$, let $Lg(x)$ be the corresponding function $G(x)$. It follows easily from the definition of $G(x)$ that L is linear:

(20.4) $L[\lambda_1 g_1(x) + \lambda_2 g_2(x)] = \lambda_1 L g_1(x) + \lambda_2 L g_2(x)$.

We show now that for admissible numbers λ,

(20.5) $$L \sum_{s=1}^{\infty} \lambda_s \omega'_s(x) = \sum_{s=1}^{\infty} \lambda_s L \omega'_s(x).$$

To prove this, take any point x^0 of R, in the set R_q, and any $\epsilon > 0$. Take $q' \geq q$ so that $1/2^{q'-s} < \epsilon$. (19.6) and (20.3), for $s = q'+1, q'+2, \cdots$, give, as $\lambda_s \omega'_s(x)$ is admissible,

$$\left| \sum_{s=q'+1}^{\infty} \lambda_s L \omega'_s(x) \right| = \left| \sum_{s=q'+1}^{\infty} L \lambda_s \omega'_s(x) \right| < \sum_{s=q'+1}^{\infty} 1/2^{s-2} = 1/2^{q'-2} < \epsilon/2$$

in $R_{q'}$, and in particular, at x^0. As

$$\sum_{s=q'+1}^{\infty} \lambda_s \omega_s'(x) = 0 \text{ in } R_{q'+1}$$

and is admissible,

$$\left| L \sum_{s=q'+1}^{\infty} \lambda_s \omega_s'(x^0) \right| < 1/2^{q'-1} < \epsilon/2.$$

Moreover

$$L \sum_{s=1}^{\infty} \lambda_s \omega_s'(x) = \sum_{s=1}^{q'} \lambda_s L \omega_s'(x) + L \sum_{s=q'+1}^{\infty} \lambda_s \omega_s'(x);$$

hence

$$\left| L \sum_{s=1}^{\infty} \lambda_s \omega_s'(x^0) - \sum_{s=1}^{\infty} \lambda_s L \omega_s'(x^0) \right|$$

$$\leq \left| L \sum_{s=q'+1}^{\infty} \lambda_s \omega_s'(x^0) \right| + \left| \sum_{s=q'+1}^{\infty} \lambda_s L \omega_s'(x^0) \right| < \epsilon,$$

which proves (20.5).

We prove two inequalities. Take any $(\nu; k)$, $(\mu; l)$ $(\sigma_k \leq m_\nu, \sigma_l \leq m_\mu)$; then

(20.6) $\left| D_k L \omega_{\mu l}(a_\nu) - D_k \omega_{\mu l}(a_\nu) \right| < \epsilon_{\rho(\nu k)},$

(20.7) $\left| D_k L \omega_{\mu l}(a_\nu) - D_k \omega_{\mu l}(a_\nu) \right| < \epsilon_{\rho(\mu l)}.$

The first follows from (16.1) when we note that a_ν is in $R - R_{\rho'(\nu k)}$, and $\epsilon_{\rho'(\nu k)} < \epsilon_{\rho(\nu k)}$, and $\sigma_k \leq m_\nu < \rho'(\nu k)$ (recall that α_p was replaced by p in using Lemma 6). We now prove the second. As $\omega_{\mu l}(x) = 0$ in $R_{\rho'(\mu l)}$ and $\rho'(\mu l) \geq \rho(\mu l) + 1$, (20.3) gives

(a) $\left| D_{k'} L \omega_{\mu l}(x) - D_{k'} \omega_{\mu l}(x) \right| < \epsilon_{\rho(\mu l)} \text{ in } \overline{R}_{\rho'(\mu l)-1}$ $(\sigma_{k'} \leq \rho'(\mu l) - 1).$

Also (16.1) gives

(b) $\left| D_{k'} L \omega_{\mu l}(x) - D_{k'} \omega_{\mu l}(x) \right| < \epsilon_{\rho(\mu l)} \text{ in } R - R_p$ $(\sigma_{k'} \leq p, p \geq \rho'(\mu l) - 1).$

Say a_ν is in $R_{p+1} - R_p$. As a_ν is not in $R_{\rho'(\nu k)}$, $\rho'(\nu k) \leq p$, and $\sigma_k \leq m_\nu \leq \rho'(\nu k) - 1 \leq p - 1$. If $p \geq \rho'(\mu l) - 1$, (20.7) follows directly from (b). If $p < \rho'(\mu l) - 1$, then a_ν is in $R_{\rho'(\mu l)-1}$, and $\sigma_k \leq p - 1 < \rho'(\mu l) - 1$, and (a) applies.

21. **An infinite system of linear equations.** We prove here

LEMMA 8. *Suppose η_s and c_s $(s = 1, 2, \cdots)$, and γ_{st} $(s, t = 1, 2, \cdots)$, are given, so that $1 \geq \eta_s \geq \eta_{s+1} > 0$ $(s = 1, 2, \cdots)$, $|c_s| \leq 1$, and*

(21.1) $|\gamma_{st}| < \eta_s/2^{t+1}$ $(s, t = 1, 2, \cdots).$

Then there are numbers λ_s $(s=1, 2, \cdots)$ such that

$$(21.2) \qquad \sum_{t=1}^{\infty}(\gamma_{st}+\delta_{st})\lambda_t = \sum_{t=1}^{\infty}\gamma_{st}\lambda_t + \lambda_s = c_s \qquad (s=1, 2, \cdots),$$

and

$$(21.3) \qquad |\lambda_s - c_s| \leq \eta_s \qquad (s=1, 2, \cdots).$$

Using the method of successive approximations, put

$$(21.4) \qquad \lambda_{1s} = c_s, \quad \lambda_{ps} = -\sum_{t=1}^{\infty}\gamma_{st}\lambda_{p-1,t} \qquad (p=2, 3, \cdots).$$

It is readily proved by induction that

$$(21.5) \qquad |\lambda_{ps}| < \eta_s/2^{p-1} \qquad (p=2, 3, \cdots).$$

Hence the series $\lambda_{1s}+\lambda_{2s}+\cdots$ converges to a limit λ_s $(s=1, 2, \cdots)$, and

$$\sum_{t=1}^{\infty}(\gamma_{st}+\delta_{st})\lambda_t = \sum_{p=1}^{\infty}\sum_{t=1}^{\infty}(\gamma_{st}+\delta_{st})\lambda_{pt} = \sum_{p=1}^{\infty}(\lambda_{ps}-\lambda_{p+1,s}) = c_s,$$

$$|\lambda_s - c_s| = |\sum_{p=2}^{\infty}\lambda_{ps}| \leq \sum_{p=2}^{\infty}\eta_s/2^{p-1} = \eta_s.$$

22. We are now ready to prove

LEMMA 9. *Let R, m, a_ν, $m_\nu(\nu=1, 2, \cdots)$ be defined as in §19. Then there are numbers $\beta_\nu > 0$ $(\nu=1, 2, \cdots)$ with the following property. Given any set of numbers $f_{\nu k}$ defined for $\sigma_k \leq m_\nu$, $\nu=1, 2, \cdots$, such that (19.4) holds, there exists a function $G(x)$ analytic in R, such that*

$$(22.1) \qquad D_k G(a_\nu) = f_{\nu k} \qquad (\sigma_k \leq m_\nu, \quad \nu=1, 2, \cdots),$$

and such that if we set $G(x) \equiv 0$ in $E-R$, then $G(x)$ is of class C^m in E, and

$$(22.2) \qquad D_k G(x) = 0 \text{ in } E - R \qquad (\sigma_k \leq m).$$

We define the $\omega_{\nu k}(x)$ and the β_ν as in §19. Now take any $f_{\nu k}$ satisfying (19.4), and define the $c_{\rho(\nu k)}$ by (19.5). Define the ϵ_s and the transformation L as in §20. Set

$$(22.3) \qquad \eta_{\rho(\nu k)} = \beta_\nu \qquad (\sigma_k \leq m_\nu, \nu=1, 2, \cdots),$$

and

$$(22.4) \quad \gamma_{st} = \gamma_{\rho(\nu k)\rho(\mu l)} = D_k L\omega_{\mu l}(a_\nu) - \delta_{st} = D_k L\omega_{\mu l}(a_\nu) - D_k\omega_{\mu l}(a_\nu).$$

Let $u = \rho(\theta j)$ be the larger of the two numbers $s = \rho(\nu k)$, $t = \rho(\mu l)$. Then using (20.6) or (20.7) according as $u = s$ or $u = t$, we find (as $\beta_{\rho(\theta j)} \leq \beta_{\rho(\nu k)} \leq \beta_\nu$)

$$\left|\gamma_{st}\right| < \epsilon_{\rho(\theta j)} \leqq \beta_\nu/2^{u+1} \leqq \eta_s/2^{t+1}.$$

Also $\left|c_s\right| = \left|f_{\nu k}\right| \leqq \beta_\nu < 1$. Therefore the equations (21.2) have a solution $\lambda_1, \lambda_2, \cdots$, and

$$(22.5) \qquad \sum_{t=1}^{\infty}(\gamma_{st} + \delta_{st})\lambda_t = \sum_{\mu,l}\lambda_{\rho(\mu l)}D_kL\omega_{\mu l}(a_\nu) = c_s = c_{\rho(\nu k)} = f_{\nu k}.$$

By (22.8) below, the λ's are admissible (§20), and we can define the analytic function $G(x)$ in R by the equation

$$(22.6) \qquad\qquad G(x) = L\sum_{t=1}^{\infty}\lambda_t\omega_t'(x).$$

(20.5) and (22.5) give

$$(22.7) \qquad D_kG(a_\nu) = D_k\sum_{t=1}^{\infty}\lambda_tL\omega_t'(a_\nu) = \sum_{\mu,l}\lambda_{\rho(\mu l)}D_kL\omega_{\mu l}(a_\nu) = f_{\nu k}.$$

(19.6) and (20.3) show that the last sum above is uniformly convergent in any $\bar{R}_p$; hence the termwise differentiation is permissible.

Set $G(x) \equiv 0$ in $E - R$; we must show that $G(x)$ is of class C^m in E. (This is trivial if $m = -1$.) First note that, by (19.4) and (21.3),

$$(22.8) \qquad\qquad \left|\lambda_{\rho(\nu k)}\right| \leqq \left|c_{\rho(\nu k)}\right| + \eta_{\rho(\nu k)} = \left|f_{\nu k}\right| + \beta_\nu \leqq 2\beta_\nu;$$

this with (19.3) gives (replacing ν, k and l by μ, l and k)

$$(22.9) \qquad \left|\lambda_{\rho(\mu l)}D_k\omega_{\mu l}(x)\right| < 2/[\mu(m_\mu + 1)^n] \text{ in } E$$
$$(\sigma_k, \sigma_l \leqq m_\mu, \mu = 1, 2, \cdots).$$

Now take any boundary point x^0 of R, any integer $m' \leqq m$, and any $\epsilon > 0$. Take $q \geqq m'$ so that $\epsilon_q < \epsilon/2$. Take $\delta > 0$ so that R_q has no points within δ of x^0, and so that if ν is any number such that U_ν has points within δ of x^0, then $m_\nu \geqq m'$ and $2/\nu < \epsilon/2$ (see (19.1)). Consider any point y of R within δ of x^0, and take any k, $\sigma_k \leqq m'$. Either $D_k\omega_{\mu l}(y) = 0$ for all μ, l, or else for some μ, y lies in U_μ, in which case there are at most $(m_\mu + 1)^n$ such numbers $\neq 0$, and $2/\mu < \epsilon/2$, and $m_\mu \geqq m'$. Thus if we replace x by y in (22.9) and sum over μ and l, we find

$$\left|D_k\sum_{t=1}^{\infty}\lambda_t\omega_t'(y)\right| = \left|\sum_{\mu,l}\lambda_{\rho(\mu l)}D_k\omega_{\mu l}(y)\right| < \epsilon/2 \qquad (\sigma_k \leqq m').$$

As y is in $R - R_q$, replacing α_q by $q \geqq m'$ in (16.1) gives

$$\left|D_kL\sum_{t=1}^{\infty}\lambda_t\omega_t'(y) - D_k\sum_{t=1}^{\infty}\lambda_t\omega_t'(y)\right| < \epsilon_q < \epsilon/2 \qquad (\sigma_k \leqq m').$$

This with the last inequality gives

$$\left| D_k G(y) \right| < \epsilon \text{ in } R \text{ if } r_{z_y} < \delta \qquad (\sigma_k \leqq m');$$

the proof is completed with the aid of Lemma 1.

23. **Functions analytic at the isolated points of A.** Lemmas 4, 6 and 9 lead directly to the following theorem.

THEOREM II. *Let A be a closed set in E, and let $a_1, a_2, \cdots$ be isolated points of A. Set $A' = A - (a_1 + a_2 + \cdots)$. Let m be an integer $\geqq -1$ or ∞, and let the integers $m_\nu \geqq 0$, $\nu = 1, 2, \cdots$, satisfy (19.1). Let $f_k(x)$ be defined for x in $A'(\sigma_k \leqq m)$, and for $x = a_\nu$ $(\sigma_k \leqq m_\nu)$, so that $f(x)$ is of class C^m in A. Then there is a function $F(x)$ of class C^m in E such that*

(1) $F(x) = f(x)$ in A,
(2) $D_k F(x) = f_k(x)$ in A' for $\sigma_k \leqq m$ and at each a_ν for $\sigma_k \leqq m_\nu$,
(3) $F(x)$ is analytic in $E - A'$.

We asked that $f(x)$ be of class C^m in A, while $f_k(a_\nu)$ may not be defined for certain values of ν and k $(\sigma_k \leqq m)$. We require merely that after setting $f_k(a_\nu) = 0$ $(\sigma_k > m_\nu)$, $f(x)$ shall be of class C^m in A.

A special case of interest is $m = -1$. The m_ν and the $f_k(a_\nu)$ are then unrestricted. A' may be void, in which case $f(x)$ is analytic throughout E. A' may of course contain isolated points.

To prove the theorem, set $R = E - A'$ and determine the open sets R_ν and the numbers β_ν $(\nu = 1, 2, \cdots)$, as in §19. Let $g'(x)$ be the extension of $f(x)$ of class C^m in E and of class C^∞ in $E - A'$ given by Lemma 4 (setting $f_k(a_\nu) = 0$ for $\sigma_k > m_\nu$). Let $G'(x)$ be the analytic function in R given by Lemma 6 (with α_p replaced by p) such that

$$(23.1) \qquad \left| D_k G'(x) - D_k g'(x) \right| < \beta_p \text{ in } R - R_p \qquad (\sigma_k \leqq p),$$

and set $G'(x) = f(x)$ in A'. $G'(x)$ is of class C^m in E, and

$$(23.2) \qquad D_k G'(x) = f_k(x) \text{ in } A' \qquad (\sigma_k \leqq m)$$

(see §18). Set

$$(23.3) \qquad f_{\nu k} = D_k g'(a_\nu) - D_k G'(a_\nu) \qquad (\sigma_k \leqq m_\nu, \nu = 1, 2, \cdots).$$

As a_ν lies in $R - R_{\rho'(\nu k)}$ and $\rho'(\nu k) > m_\nu$, (23.1) gives $|f_{\nu k}| < \beta_{\rho'(\nu k)} < \beta_\nu (\sigma_k \leqq m_\nu)$. Thus the conditions of Lemma 9 are satisfied, and there is a function $G(x)$ analytic in R, $\equiv 0$ in A', of class C^m in E, and such that (22.1) and (22.2) hold. Set

$$(23.4) \qquad F(x) = G'(x) + G(x);$$

then $F(x)$ is our required function. It is of class C^m in E as the same is true of $G'(x)$ and $G(x)$; it is analytic in $R = E - A'$ as the same is true of $G'(x)$ and $G(x)$; it equals $f(x)$ in A' as $G'(x) = f(x)$ and $G(x) = 0$ there. (22.1), (23.3) and (14.2) show that $D_k F(a_\nu) = D_k g'(a_\nu) = f_k(a_\nu)$ $(\sigma_k \leq m_\nu)$; (22.2) and (23.2) show that $D_k F(x) = f_k(x)$ in A', completing the proof.

24. **An extension-approximation theorem.** We prove here

THEOREM III.† *Let A be closed, and let $A_{-1}, A_0, A_1, \cdots$ be closed subsets of A such that each A_s lies in A_{s+1}. Let $a_{s1}, a_{s2}, \cdots$ be points of $A_s - A_{s-1}$ which are isolated points of A, and set $A' = A - \sum a_{st}$. Let B_{-1} be void, and let B_0, $B_1, \cdots$ be sets whose sum B lies in $E - A$, such that each B_s lies in B_{s+1}, such that each set $B_s - B_{s-1}$ has limit points in $B - B_{s-1} + A_s$ only, and such that each set $A + B - B_s$ is closed. Let $f_k(x)$ be defined in each set $\Gamma_s = A + B - (A_{s-1} + B_{s-1})$ for $\sigma_k \leq s$ $(s = 0, 1, \cdots)$ so that $f(x) = f_0(x)$ is of class C^s in Γ_s in terms of the $f_k(x)$ for each s. Let $\epsilon(x)$ be a continuous function, positive in $E - A'$ and zero in A'. Then there is a function $F(x)$ defined in $E - A_{-1}$ such that*

(1) $F(x)$ *is of class C^s in $E - A_{s-1}$ $(s = 0, 1, \cdots)$*,
(2) $D_k F(x) = f_k(x)$ *in $A - A_{s-1}$ $(\sigma_k \leq s, s = 0, 1, \cdots)$*,
(3) $|D_k F(x) - f_k(x)| < \epsilon(x)$ *in $B - B_{s-1}$ $(\sigma_k \leq s, s = 0, 1, \cdots)$*,
(4) $F(x)$ *is analytic in $E - A'$.*

Any number of sets A_s, B_s may be void; any of the points a_{st} may not exist. Note that if $A_\infty = A - (A_{-1} + A_0 + \cdots)$, then $F(x)$ is of class C^∞ at all points of A_∞. Theorem I for m finite is obtained by letting B and $A_{-1}, \cdots$, A_{m-1} be void, and setting $A = A_m$; and for m infinite, by letting B and every A_s be void. Theorem II is obtained similarly; we arrange the a_{st} in a sequence $a_1, a_2, \cdots$, and set $m_\nu = s$ if a_ν is in $A_s - A_{s-1}$. Lemma 6 is obtained by setting $A = A_{-1} = E - R$, $B_s = R_{s+1}$ $(s = 0, 1, \cdots)$, and taking $\epsilon(x)$ so that $\epsilon(x) \leq \epsilon_s$ in $R - R_s$.

We turn now to the proof. Take a subdivision of the open set $E - (A + B)$ as in §8, let $y^{0\nu}$ $(\nu = 1, 2, \cdots)$ be the vertices of the cubes, and let $x^{0\nu}$ be a point of Γ_0 whose distance from $y^{0\nu}$ is not more than twice the distance from $y^{0\nu}$ to Γ_0. Define the functions $\phi_{0\nu}(x)$ in $E - (A + B)$ as in §9, and define $g_0'(x)$ by (11.1), using the functions $\phi_{0\nu}(x)$ and $\psi_{0;0}(x; x^{0\nu}) = f(x^{0\nu})$, and replacing $E - A$ and A by $E - (A + B)$ and Γ_0 respectively. Then $g_0'(x)$ is defined throughout $E - A_{-1}$, and is easily seen to be a continuous extension of $f(x)$. Let $g_0(x)$ be a function $= g_0'(x)$ in $A - A_{-1} + \Gamma_1$ and analytic in the open set $E - (A + \Gamma_1)$ so that

† A special case of this theorem has been proved by A. Besikowitsch, *Über analytische Funktionen mit vorgeschriebenen Werten ihrer Ableitungen*, Mathematische Zeitschrift, vol. 21 (1924), pp. 111–118.

(24.1) $$| g_0(x) - g_0'(x) | < \theta_1(x)/4 \text{ in } E - (A + \Gamma_1),$$

where $\theta_p(x) = \min \left[\epsilon(x), \text{ distance from } x \text{ to } A + \Gamma_p \right]$ $(p = 1, 2, \cdots)$. Then $g_0(x)$ is continuous in $E - A_{-1}$.

We shall now define in succession functions $g_1(x)$, $g_2(x)$, $\cdots$, with the following properties:

(a) $g_p(x)$ is defined in $E - A_{-1}$, is of class C^s in $E - A_{s-1}$ $(s = 0, \cdots, p)$, and is analytic in $E - (A + \Gamma_{p+1})$.

(b) $D_k g_p(x) = f_k(x)$ in $A - A_{s-1} + \Gamma_{p+1}$ $(\sigma_k \leq s, s = 0, \cdots, p)$.

(c) $| D_k g_p(x) - D_k g_{p-1}(x) | < \epsilon(x)/2^{p+2}$ in $B_{p-1} - B_{s-1}$ $(\sigma_k \leq s, s = 0, \cdots, p-1)$.

(d) $| D_k g_p(x) - f_k(x) | < \epsilon(x)/2^{p+2}$ in $B_p - B_{p-1}$ $(\sigma_k \leq p)$.

Assuming $g_0(x)$, $\cdots$, $g_{p-1}(x)$ are defined, we shall define $g_p(x)$. Consider any point of Γ_p; it is at a positive distance from the closed set A_{p-1}, and hence we can enclose it in an open set lying at a positive distance from A_{p-1}. We thus enclose Γ_p in an open set Γ_p' containing no points of A_{p-1}, and having no limit points in A_{p-1} other than limit points of Γ_p. Take a subdivision of the open set $E - (A + \Gamma_p)$, let $y^{p\nu}$ $(\nu = 1, 2, \cdots)$ be the vertices of the cubes, and let $x^{p\nu}$ be a point of Γ_p whose distance from $y^{p\nu}$ is not more than twice the distance from $y^{p\nu}$ to Γ_p $(\nu = 1, 2, \cdots)$. Define the functions $\phi_{p\nu}(x)$ in $E - (A + \Gamma_p)$ as in §9 and define $\psi_{p;k}(x'; x)$ by (6.1) $(\sigma_k \leq p)$, replacing m by p. Remembering that $f(x)$ is of class C^p in Γ_p, set

(24.2) $$g_p'(x) = \sum_\nu \phi_{p\nu}(x) \psi_{p;0}(x; x^{p\nu}) \text{ in } \Gamma_p' - \Gamma_p,$$

and set $g_p'(x) = g_{p-1}(x)$ in Γ_p. From the proof in §11 it is seen that $g_p'(x)$ is an extension of class C^p of the values of $f(x)$ in Γ_p.

Set $\zeta_p(x) = g_p'(x) - g_{p-1}(x)$ in Γ_p'; then $\zeta_p(x)$ is of class C^{p-1} in Γ_p', and

(24.3) $$D_k \zeta_p(x) = 0 \text{ in } \Gamma_p \qquad\qquad (\sigma_k \leq p - 1).$$

Set $\eta_p = 1 / \{ 2^{p+4} c [(p+1)!]^n (36 n^{1/2})^p N^{(p)} \}$ $(p = 0, 1, \cdots)$, where $N^{(p)} = \max N_k$ for $\sigma_k \leq p$. Let K_{p-1} be the set of points of B_{p-1} for which

$$| D_k \zeta_p(x) | \geq \eta_{p-1} \epsilon(x) \delta_x^{p-1-\sigma_k} \text{ for some } k \qquad (\sigma_k \leq p - 1),$$

where δ_x is the distance from x to Γ_p, or 1 if that is smaller. Each point of $A - A_{p-1}$ is at a positive distance from K_{p-1}, as B_{p-1} has no limit points in $A - A_{p-1}$, and each point of $B - B_{p-1}$ is at a positive distance from K_{p-1} on account of (24.3), as $\epsilon(x) > 0$ in B; hence each point of Γ_p is at a positive distance from K_{p-1}, and we can enclose Γ_p in an open set Γ_p'' which lies in Γ_p' and contains no points of K_{p-1}. Now

(24.4) $\left| D_k \zeta_p(x) \right| < \eta_{p-1} \epsilon(x) \delta_x{}^{p-1-\sigma_k}$ in $\Gamma_p'' \cdot B_{p-1}$ ($\sigma_k \leqq p - 1$).

We can also take Γ_p'' so that if $\rho_p(x)$ is the distance from x to A_{p-1}, then

(24.5) $\left| D_k \zeta_p(x) \right| < \eta_{p-1} \rho_p(x) \delta_x{}^{p-1-\sigma_k}$ in Γ_p'' ($\sigma_k \leqq p - 1$).

Let I_{ν_t} be those sets I_ν of the subdivision of $E-(A+\Gamma_p)$ lying wholly in Γ_p'' ($t=1, 2, \cdots$), and set

(24.6) $g_p''(x) = g_{p-1}(x) + \zeta_p(x) \sum_{t=1}^{\infty} \phi_{\nu_t}(x)$ in $E - (A + \Gamma_p)$,

and $g_p''(x) = g_{p-1}(x) = f(x)$ in $A - A_{-1} + \Gamma_p$. Then $g_p''(x)$ is of class C^p in $E - A_{p-1}$, and with the help of (24.4) we find

(24.7) $\left| D_k g_p''(x) - D_k g_{p-1}(x) \right| < \epsilon(x)/2^{p+3}$ in B_{p-1} ($\sigma_k \leqq p - 1$)

(see §13). Also (24.5) gives

(24.8) $\left| D_k g_p''(x) - D_k g_{p-1}(x) \right| < \rho_p(x)/2^{p+3}$ in $E - A$ ($\sigma_k \leqq p - 1$),

and hence $g_p''(x)$ is of class C^s in $E-A_{s-1}$ ($s=0, \cdots, p$), as the same is true of $g_{p-1}(x)$ ($s=0, \cdots, p-1$).

Finally let $g_p(x)$ be an analytic function such that

(24.9) $\left| D_k g_p(x) - D_k g_p''(x) \right| < \theta_{p+1}(x)/2^{p+3}$ in $E - (A + \Gamma_{p+1})$ ($\sigma_k \leqq p$);

set $g_p(x) = g_{p-1}(x) = f(x)$ in $A - A_{-1} + \Gamma_{p+1}$. Then $g_p(x)$ has all the required properties. (c) is a direct consequence of the above inequality and (24.7); (d) follows from (24.9) and the fact that $D_k g_p''(x) = f_k(x)$ in $\Gamma_p(\sigma_k \leqq p)$; (a) and (b) follow with the aid of Lemma 1.

Set

(24.10) $g(x) = \lim_{p \to \infty} g_p(x)$ in $E - A_{-1}$.

By (24.8) and (24.9), $g(x)$ exists and is of class C^∞ in $E-A$. Let x be any point of any $A_s - A_{s-1}$; by (a), $g_p(x)$ is of class C^s in a neighborhood of x for $p \geqq s$, and by (24.8) and (24.9), the same is true of $g(x)$. The same argument, using (b), shows that

(24.11) $D_k g(x) = f_k(x)$ in $A - A_{s-1}$ ($\sigma_k \leqq s, s = 0, 1, \cdots$).

Finally (c), (d), (24.1) and the definition of $g_0'(x)$ show that

(24.12) $\left| D_k g(x) - f_k(x) \right| < \epsilon(x)/2$ in $B - B_{s-1}$ ($\sigma_k \leqq s, s = 0, 1, \cdots$).

We have now found an extension $g(x)$ with all the properties but (4). It is replaced by an analytic extension $F(x)$ just as in §23; we must be careful merely to make

(24.13) $\left| D_k F(x) - D_k g(x) \right| < \epsilon(x)/2$ in $B - B_{s-1}$ $(\sigma_k \leqq s, s = 0, 1, \cdots)$.

Let $a_1, a_2, \cdots$ be the a_{st} arranged in a sequence, and set $m_\nu = s$ if a_ν is in $A_s - A_{s-1}$. Set $R = E - A'$. Let R_p consist of those points of the R_p of §19 whose distances from the closed set $A + B - B_{p-1}$ are $> 1/p$ $(p = 1, 2, \cdots)$. Every point of $E - A'$ lies in some R_p, as $B - (B_0 + B_1 + \cdots)$ is void. Take the β_ν (§19) small enough so that if $\left| \lambda_{\rho(\nu k)} \right| \leqq 2\beta_\nu$ (see (22.8)) and $g^*(x) = \sum \lambda_t \omega_t'(x)$, then

(24.14) $\left| D_k g^*(x) \right| < \epsilon(x)/8$ in $R_{p+1} - R_p$ $(\sigma_k \leqq p)$.

Let ϵ_s' be one eighth the lower bound of $\epsilon(x)$ for x in R_{s+1} $(s = 1, 2, \cdots)$, or 1 if that is smaller. Replace the ϵ_s of (20.2) by min $(\epsilon_s, \epsilon_s')$. Then for any such $g^*(x)$, (16.1) gives

(24.15) $\left| D_k L g^*(x) \right| < \epsilon_p + \left| D_k g^*(x) \right| < \epsilon(x)/4$ in $R_{p+1} - R_p$ $(\sigma_k \leqq p)$.

Replace the $g'(x)$ of §23 by the present $g(x)$, and determine $G'(x)$ so that

(24.16) $\left| D_k G'(x) - D_k g(x) \right| < \min \left[\beta_p, \epsilon(x)/4 \right]$ in $R - R_p$ $(\sigma_k \leqq p)$,

and, in particular, in $B - B_{p-1}$. Now if $G(x)$ and $F(x)$ are determined as in §23, then (24.13) and hence property (3) hold; the other properties are easily verified, and the proof of the theorem is complete.

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