

THE BERTINI TRANSFORMATION IN SPACE*

BY

F. R. SHARPE AND L. A. DYE

1. Introduction. Examples are known of involutorial transformations I having an invariant pencil of planes through a line l , such that in each plane of the pencil there is a transformation of the Geiser or the de Jonquières type. We have shown in this paper that involutorial transformations exist which have in each plane through l a Bertini transformation with 6 of the fundamental points lying on a C_6 , $p=3$, the other two being on l , and either fixed or variable. There is another type in which 6 of the 8 fundamental points lie on a C_6 , $p=4$, and in every plane through l there is a degenerate Bertini transformation. A third type is discussed in which there is a net of invariant quartic surfaces through a C_{11} , $p=14$. The method of obtaining this last transformation leads also to an involutorial transformation with a net of invariant surfaces of order $n+1$ through a C_{5n-3} of genus $12n-19$. This type has on each plane through l a Geiser transformation having the 7 fundamental points on C_{5n-3} .

2. The involutorial Bertini transformation I_B on a cubic surface F_3 . The conics tangent to a cubic surface F_3 at two fixed points O_1, O_2 meet F_3 in two residual points P, P' which are conjugate points of an involutorial Bertini transformation I_B on F_3 . The web of quadrics tangent to F_3 at O_1, O_2 meet F_3 in a web of sextic curves of genus 2 which is invariant under I_B as is also the pencil of plane sections through the line $l:O_1+O_2$. If the space (y) of F_3 is transformed into a space (z) by means of the web of cubic surfaces through a fixed C_6 , $p=3$, on F_3 , then F_3 is transformed into a plane meeting the fundamental sextic of the transformation in 6 points Q_3, \dots, Q_8 . If Q_1, Q_2 are the transforms of O_1, O_2 , then I_B becomes a plane transformation of order 17, a line going into a $C_{17}:8Q^6$. The image of each six-fold point Q_i is a $C_6:Q_i^3+7Q_j^2$ ($j \neq i$). The line Q_1Q_2 is the transform of a cubic curve on F_3 through O_1, O_2 .

Analytically, if $y_2=0, y_1=0$ are the planes tangent to F_3 at $O_1 \equiv (1, 0, 0, 0)$, $O_2 \equiv (0, 1, 0, 0)$, the equation of F_3 may be written

$$(1) \quad Ay_1 + By_2 + C \equiv y_1(a^2y_1y_2 + \alpha) + y_2(b^2y_1y_2 + \beta) + \gamma y_1y_2 + \delta = 0,$$

where $\alpha, \beta, \gamma, \delta$ are binary forms in y_3, y_4 . The transformation I_B is defined by

$$(2) \quad Ay_1' = By_2, \quad By_2' = Ay_1, \quad y_3' = y_3, \quad y_4' = y_4.$$

* Presented to the Society, December 27, 1933; received by the editors November 20, 1933.

The images of O_1, O_2 are the sextics in which the quadrics $A = 0, B = 0$ meet F_3 . Since the second polars of O_1, O_2 differ from A, B by terms containing y_2, y_1 respectively, the quadrics have second-order contact with F_3 at O_1, O_2 respectively and meet F_3 in sextics having triple points at O_1, O_2 respectively.

3. The transformation I_B for a pencil of cubic surfaces. Since a point P determines an F_3 of the pencil (parameter λ) on which P' can be found by the method of §2, we can define involutorial space transformations by either taking O_1, O_2 as fixed points lying on each F_3 of the pencil, or by taking one or both of them variable (the coordinates being functions of λ), on a rational curve lying on each F_3 .

4. Case I, O_1, O_2 are fixed. For this case we take a pencil of cubic surfaces F_3 having in common a $C_9, p = 10$, through $O_1 \equiv (1, 0, 0, 0), O_2 \equiv (0, 1, 0, 0)$, and write the equation of the F_3 in the form

$$(3) \quad \begin{aligned} F_3 &\equiv F_3' - \lambda F_3'' \\ &\equiv ax_1^2 x_2 + bx_1 x_2^2 + cx_1^2 + dx_1 x_2 + ex_2^2 + fx_1 + gx_2 + h = 0, \end{aligned}$$

where $a \equiv a' - \lambda a''$, etc., and c', c'' , etc., are binary forms in x_3, x_4 . A change of coordinate system given by

$$(4) \quad y_1 = bx_1 + e, \quad y_2 = ax_2 + c, \quad y_3 = x_3, \quad y_4 = x_4$$

will express F_3 in the form (1). The transformation (2) in terms of x_i is

$$(5) \quad \begin{aligned} x_1' &= (ax_2 B + cB - eA)Ba, \\ x_2' &= (bx_1 A - cB + eA)Ab, \\ x_3' &= x_3 A B a b, \quad x_4' = x_4 A B a b. \end{aligned}$$

The surface of invariant points $K \equiv y_1 A - y_2 B = 0$ contains ab as a factor as does the transformation (5). The forms A, B are of degree 4 in λ and of degree 2 in x_i , so that the x_i' are of degree 5 in x_i and of degree 8 in λ . If λ is replaced by F_3'/F_3'' we have an I_{29} in which the image of O_1 is $A = 0$, the image of O_2 is $B = 0$, and the image of C_9 can be obtained by applying the transformation to an S_{29} . The table of characteristics of the I_{29} is

$$\begin{aligned} O_1 &\sim F_{14}: O_1^7 + O_2^6 + C_9^4, \\ O_2 &\sim F_{14}: O_1^6 + O_2^7 + C_9^4, \\ C_9 &\sim F_{56}: O_1^{23} + O_2^{28} + C_9^{15}, \\ S_1 &\sim S_{29}: O_1^{14} + O_2^{14} + C_9^8, \\ K_{12} &: O_1^6 + O_2^6 + C_9^3. \end{aligned}$$

Every plane through the line $l: O_1 + O_2$ cuts from the F_{56} a composite curve of order 56, the 7 components of which are the images of the 7 residual intersections of C_9 with the plane. If O_i ($i = 3, \dots, 9$) is any one of these 7 points and $6O_j$ ($j = 3, \dots, 9$) are the others, then

$$O_i \sim C_8: O_1^4 + O_2^4 + O_i^3 + 6O_j^2 \quad (i, j = 3, \dots, 9; i \neq j).$$

In each of these planes there is a transformation of the Bertini type of order 29. If it is transformed by a quadratic transformation having O_1, O_2, O_i for fundamental points it becomes the usual Bertini transformation of order 17 with 8 six-fold points at $O_1, O_2, 6O_j$.

Since C_9 is of genus 10 there are 11 trisecants of the C_9 which pass through O_1 or O_2 . Any one of these 22 lines meets an S_{29} in $14 + 2 \cdot 8 = 30$ points and therefore lies on the S_{29} . These lines are the fundamental lines of the second species in the I_{29} . The surface R_{42} of trisecants of C_9 contains C_9 as an 11-fold curve. The line l meets R_{42} in 20 points not on C_9 from which trisecants of C_9 may be drawn. In any one of the 20 planes determined by one of these trisecants and l , the 6 residual intersections of C_9 lie on a conic. Each of these 20 conics meets an S_{29} in $2 \cdot 14 + 4 \cdot 8 = 60$ points and therefore lies doubly on the S_{29} . They are the fundamental conics of the second species in the I_{29} . The tangent planes to the pencil of cubic surfaces at O_1 form a pencil of planes through the tangent line to C_9 at O_1 . The plane of the pencil which passes through O_2 cuts from the corresponding F_3 a cubic curve with a double point at O_1 and through O_2 and the 6 residual intersections of C_9 with the plane. There is another such cubic curve with the roles of O_1, O_2 interchanged. Each of these cubics meets an S_{29} in $28 + 14 + 6 \cdot 8 = 90$ points and hence lies triply on the S_{29} . They are the fundamental cubics of the second species in the I_{29} . There exist then 22 lines, 20 conics, and 2 cubics which are parasitic curves in the involutorial transformation I_{29} .

If the C_9 is composed of a space cubic C_3 through O_1, O_2 and a C_6 , $p = 3$, $[C_3, C_6] = 8$, the surface F_{56} breaks up into an $F_8: O_1^4 + O_2^4 + C_3^3 + C_6^2$, the image of C_3 , and an $F_{48}: O_1^{24} + O_2^{24} + C_3^{12} + C_6^{13}$, the image of C_6 . If we transform the space (x) into a space (z) by means of the cubic transformation $T_{3,3}: C_6$ the pencil of F_3 's becomes a pencil of planes through the line l' which is the transform of C_3 . In each plane through l' there is a Bertini transformation of order 17. The transform of the surface F_8 of trisecants of C_6 is C_6' , and to O_1, O_2 correspond the points Q_1, Q_2 . The characteristics of the I_{29} in the (z) space are

$$\begin{aligned} Q_1 &\sim F_{10}: Q_1^7 + Q_2^6 + l'^4 + C_6'^2, \\ Q_2 &\sim F_{10}: Q_1^6 + Q_2^7 + l'^4 + C_6'^2, \end{aligned}$$

$$\begin{aligned}
 l' &\sim F_8 : Q_1^4 + Q_2^4 + l'^3 + C_6'^2, \\
 C_6' &\sim F_{64} : Q_1^{40} + Q_2^{40} + l'^{28} + C_6'^{13}, \\
 S_1 &\sim S_{29} : Q_1^{18} + Q_2^{18} + l'^{12} + C_6'^6, \\
 K_{12} &: Q_1^6 + Q_2^6 + l'^3 + C_6'^3.
 \end{aligned}$$

In the (x) space in place of the surface $R_{42}:C_9^{11}$ of trisecants of C_9 , we have the surface $R_8:C_6^3$ of trisecants of C_6 , the surface $R_8':C_3^4+C_6$ of bisecants of C_3 which meet C_6 , and the surface $R_{26}:C_3^7+C_6^7$ of bisecants of C_6 which meet C_3 . The 22 parasitic lines in the (x) space are (a) the 4 bisecants of C_3 through O_1 which meet C_6 , (b) the 4 bisecants of C_3 through O_2 which meet C_6 , (c) the 7 bisecants of C_6 through O_1 , (d) the 7 bisecants of C_6 through O_2 . The lines of types (a), (b) correspond to parasitic conics in the (z) space through Q_1 or Q_2 and meeting C_6' in 5 points. The lines of types (c), (d) correspond to parasitic lines which are bisecants of C_6' from Q_1 or Q_2 . The 8 trisecants of C_6' meeting l' are parasitic and correspond in the (x) space to the 8 points $[C_3, C_6]$.

The line l meets the surface $R_8:C_6^3$ in 8 points, hence 8 trisecants of C_6 meet l . Each of the planes determined by l and one of these trisecants meets the F_3 containing the trisecant in a residual conic which is parasitic. The 8 conics go into parasitic cubics in the (z) space which have double points on C_6' and pass through Q_1, Q_2 , and 5 points on C_6' . The surface $R_{26}:C_3^7+C_6^7$ is met by l in 12 points, hence 12 bisecants of C_6 meet C_3 and l . In each of the 12 planes determined by these lines and l there is a parasitic conic which corresponds to a parasitic conic in the (z) space through Q_1, Q_2 , and 4 points of C_6' . The two parasitic cubics with double points at O_1 or O_2 and through O_2 or O_1 and 6 points of C_6 correspond to similar cubics in the (z) space. The I_{29} in the (z) space has 22 lines, 20 conics, and 10 cubics which are fundamental curves of the second species.

5. Case II, O_1 is variable on a space cubic curve C_3 . We take a pencil of cubic surfaces (parameter λ) through a space cubic curve C_3 containing the points $O_1 \equiv (1, \lambda^3, \lambda^2, \lambda), O_2 \equiv (0, 1, 0, 0)$, and having the equation

$$(6) \quad F_3 \equiv \begin{vmatrix} (px) & x_1 & x_4 \\ (qx) & x_4 & x_3 \\ (rx) & x_3 & x_2 \end{vmatrix} \equiv F_3' - \lambda F_3'' \equiv (px)H_p + (qx)H_q + (rx)H_r = 0,$$

where $(px) = p_1x_1 + p_2x_2 + p_3x_3 + p_4x_4, p_i = p_i' - \lambda p_i''$, etc., and H_p, H_q, H_r are quadrics through C_3 . A point $P(x)$ determines an F_3 of the pencil and a definite point O_1 so that by the construction of §2 we can determine a point $P'(x')$. A change of variables is made by

$$\begin{aligned}y_1 &= p_2x_4 - q_2x_1, \\y_2 &= p(x_2 - 2\lambda x_3 + \lambda^2x_4) - q(x_3 - 2\lambda x_4 + \lambda^2x_1), \\y_3 &= x_3 - \lambda x_4, \\y_4 &= x_4 - \lambda x_1,\end{aligned}$$

where $p \equiv \lambda P - Q$, $q \equiv \lambda Q - R$, $P \equiv p_1 + p_2\lambda^3 + p_3\lambda^2 + p_4\lambda$, etc. The planes $y_1=0$, $y_2=0$ are the tangent planes at O_2 , O_1 respectively, and the planes $y_3=0$, $y_4=0$ are a pair of planes through the line $l:O_1+O_2$. The pencil of cubic surfaces is now of the form (1) and the transformation (2) gives I_B .

The equations of the surfaces $A=0$, $B=0$, $K=0$ may be written in terms of x_i and y_i as follows:

$$\begin{aligned}A &\equiv m p \{ p(PH_p + QH_q + RH_r) + (y_3 - \lambda y_4) \{ \lambda q(p x) - M(q x) + p(r x) \} \} \\&\quad - y_2 [m(y_3 - \lambda y_4)(\lambda q p_2 - M q_2 + p r_2) + p y_4(Q p_2 - P q_2)] = 0, \\B &\equiv m p \{ m(p_2 H_p + q_2 H_q + r_2 H_r) - y_4 \{ q_2(p x) - p_2(q x) \} \} \\&\quad - y_1 [m(y_3 - \lambda y_4)(\lambda q p_2 - M q_2 + p r_2) + p y_4(Q p_2 - P q_2)] = 0, \\K &\equiv y_1 \{ p(PH_p + QH_q + RH_r) + (y_3 - \lambda y_4) \{ \lambda q(p x) - M(q x) + p(r x) \} \} \\&\quad - y_2 [m(p_2 H_p + q_2 H_q + r_2 H_r) - y_4 \{ q_2(p x) - p_2(r x) \}] = 0,\end{aligned}$$

where $m \equiv \lambda p_2 - q_2$, $M \equiv \lambda p + q$. The surfaces $A=0$, $B=0$ are of the second degree in x_i and of degrees 16, 10 in λ respectively. The surface $K=0$ is of the third degree in x_i and of degree 9 in λ .

We transform the space (x) into a space (z) as was done in the latter part of §4. A surface of the web in the (x) space goes into a surface of the web in the (z) space such that in any plane through l' there is a Bertini transformation of order 17. By this I_{17} the image of l' is a $C_5:Q_1+Q_2+6Q^2$ which with l' makes up a $C_6:8Q^2$ that is the plane section of a sextic surface having Q_1 , Q_2 , C'_6 as double elements. This sextic surface is the transform of a quadric surface through C_3 and tangent to F_3 at O_1 , O_2 . This quadric which is the image of C_3 and is of the sixth degree in λ has the equation

$$p p_2 H_p + p q_2 H_q + (\lambda p q_2 - \lambda q p_2 + q q_2) H_r = 0.$$

Any plane through l' is invariant under I_B , hence a pencil of surfaces of the web in the (z) space is made up of the pencil of planes through l' together with the image surfaces of l' , Q_1 , Q_2 . Since the image of Q_1 by the I_{17} in a plane through l' is a $C_6:Q_1^3+Q_2^2+6Q^2$ and since $A=0$ is of order 16 in λ , the image of Q_1 by the I_B in the (z) space is a surface of order $6+16=22$ on which l' is a 16-fold line with 3 sheets of the surface having contact along l' . The point Q_1 is $16+3=19$ -fold; Q_2 is a $16+2=18$ -fold point; C'_6 is a double curve. In the same way we obtain the surfaces corresponding to Q_2 and l' , and

the invariant surface K . The table of characteristics of the I_{51} is

$$\begin{aligned}
 l' &\sim F_{12} : l'^{7+t} + C_6'^2 + Q_1^8 + Q_2^8, \\
 Q_1 &\sim F_{22} : l'^{16+3t} + C_6'^2 + Q_1^{19} + Q_2^{18}, \\
 Q_2 &\sim F_{16} : l'^{10+2t} + C_6'^2 + Q_1^{12} + Q_2^{13}, \\
 C_6' &\sim F_{112} : l'^{76+12t} + C_6'^{13} + Q_1^{88} + Q_2^{88}, \\
 S_1 &\sim S_{51} : l'^{34+6t} + C_6'^6 + Q_1^{40} + Q_2^{40}, \\
 K_{18} &: l'^{9+3t} + C_6'^3 + Q_1^{12} + Q_2^{12},
 \end{aligned}$$

where the coefficient of t indicates the number of fixed tangent planes at a point of l' .

In determining the number of parasitic lines, conics, and cubics the methods in the previous section have to be changed when the variable point Q_1 is involved. There are 7 bisecants of C_6' through Q_2 and 8 trisecants of C_6' meeting l' which are parasitic. In any plane λ through l' the 15 bisecants of C_6' meet l' in 15 points μ ; through any point μ on l' the 7 bisecants of C_6' determine 7 planes λ through l' . The number of coincidences in the (λ, μ) correspondence is $15+7=22$, and hence in 22 positions of Q_1 a bisecant of C_6' can be drawn from Q_1 in the plane through l' associated with Q_1 . These bisecants are parasitic lines.

There are 4 conics through Q_2 and 5 points of C_6' in planes through l' which are parasitic. In any plane λ through l' the 6 conics through 5 points of C_6' meet l' in 12 points μ ; through any point μ on l' the 4 conics through 5 points of C_6' lie in 4 planes λ through l' . There are $12+4=16$ coincidences in this (λ, μ) correspondence and therefore 16 positions of Q_1 such that Q_1 and 5 points of C_6' lie on a conic in the plane through l' associated with Q_1 . In any plane λ through l' the 15 conics through Q_2 and 4 points of C_6' meet l' in 15 points μ ; through any point μ on l' the 12 conics through Q_2 and 4 points of C_6' lie in 12 planes λ through l' . The $15+12=27$ coincidences of this (λ, μ) correspondence determine 27 positions of Q_1 such that Q_1, Q_2 , and 4 points of C_6' lie on a conic in the plane through l' associated with Q_1 . These 47 conics are all parasitic.

There are 2 values of λ given by $m=0$ for which the tangent plane to F_3 at O_2 contains O_1 , and there are 5 values of λ given by $p=0$ for which the tangent plane to F_3 at O_1 contains O_2 . In each of these 7 planes there is a cubic with a double point at O_2 or O_1 and passing through O_1 or O_2 and 6 points of C_6 . These 7 cubics correspond to 7 similar cubics in the (z) space. In any plane λ through l' there are 6 cubics with a double point on C_6' and through Q_2 and the 5 remaining points of C_6' . These 6 cubics meet l' in 12 points μ .

Through any point μ on l' there are 8 cubics with a double point on C'_6 and through Q_2 and the other 5 points of C'_6 . These cubics lie in 8 planes λ through l' so that there are $12+8=20$ coincidences in the (λ, μ) correspondence and 20 positions of Q_1 such that Q_1, Q_2 , and the 6 points of C'_6 lie on a cubic with a double point at one of these latter points. These cubics lie in planes through l' associated with the positions of Q_1 . There are then 37 lines, 47 conics, and 27 cubics which are fundamental curves of the second species in the I_{61} .

6. Case III, O_1, O_2 are both variable on a space cubic C_3 . To illustrate the case where the points O_1, O_2 are variable on a rational curve which is part of the basis curve of a pencil of cubic surfaces we again utilize a rational space cubic. Other rational curves might be considered and other arrangements of the points O_1, O_2 might be used, but the transformations obtained resemble the I_{61} in Case II and the I_{61} derived in the following case.

The points $O_1 \equiv (1, \mu^3, \mu^2, \mu), O_2 \equiv (1, -\mu^3, \mu^2, -\mu)$, where $\lambda = \mu^2$, lie on the C_3 which with C_6 makes up the basis of the pencil of cubic surfaces F_3 given by (6). A change of coordinate system is made by

$$\begin{aligned} y_1 &= \bar{p}(x_2 + 2\mu x_3 + \mu^2 x_4) - \bar{q}(x_3 + 2\mu x_4 + \mu^2 x_1), \\ y_2 &= p(x_2 - 2\mu x_3 + \mu^2 x_4) - q(x_3 - 2\mu x_4 + \mu^2 x_1), \\ y_3 &= x_2 - \mu^2 x_4, \\ y_4 &= x_3 - \mu^2 x_1, \end{aligned}$$

where $\bar{p} = \mu P - Q, \bar{q} = \mu Q - R, P = p_1 + p_2 \mu^3 + p_3 \mu^2 + p_4 \mu$, etc., and the dashed letters indicate a change of sign in μ . The surface F_3 is now in the form (1) and the involutorial transformation (2) is determined. We have the following expressions for A, B, K written in terms of x_i and y_i for the sake of conciseness:

$$\begin{aligned} A &\equiv 4\mu^2 \bar{M} M [M(PH_p + QH_q + RH_r) - (y_3 - \mu y_4) \{ -\mu q(p x) + M(q x) - \bar{p}(r x) \}] \\ &\quad + y_2 [y_3 \{ \bar{M}(\mu \bar{P} q - \bar{Q} M + \bar{R} p) + M(-\mu P \bar{q} - Q \bar{M} + R \bar{p}) \} \\ &\quad + \mu y_4 \{ -\bar{M}(\mu \bar{P} q - \bar{Q} M + \bar{R} p) + M(-\mu P \bar{q} - Q \bar{M} + R \bar{p}) \}], \\ B &\equiv 4\mu^2 M \bar{M} [\bar{M}(\bar{P} H_p + \bar{Q} H_q + \bar{R} H_r) - (y_3 + \mu y_4) \{ \mu \bar{q}(p x) + \bar{M}(q x) - \bar{p}(r x) \}] \\ &\quad + y_1 [y_3 \{ \bar{M}(\mu \bar{P} q - \bar{Q} M + \bar{R} p) + M(-\mu P \bar{q} - Q \bar{M} + R \bar{p}) \} \\ &\quad + \mu y_4 \{ -\bar{M}(\mu \bar{P} q - \bar{Q} M + \bar{R} p) + M(-\mu P \bar{q} - Q \bar{M} + R \bar{p}) \}], \\ K &\equiv y_1 [M(PH_p + QH_q + RH_r) - (y_3 - \mu y_4) \{ -\mu q(p x) + M(q x) - \bar{p}(r x) \}] \\ &\quad - y_2 [\bar{M}(\bar{P} H_p + \bar{Q} H_q + \bar{R} H_r) - (y_3 + \mu y_4) \{ \mu \bar{q}(p x) + \bar{M}(q x) - \bar{p}(r x) \}], \end{aligned}$$

where $\bar{M} = \mu p + q$.

The surfaces $A=0, B=0$ are of order 2 in x_i and of order 13 in μ^2 after the removal of a factor μ^2 . The invariant surface $K=0$ is of order 9 in μ^2 and of order 3 in x_i . The image of C_3 which is the quadric which contains C_3 and is tangent to F_3 at O_1, O_2 is of order 6 in μ^2 and has the equation

$$(\bar{p}M - p\bar{M})H_p + \mu(p\bar{q} + q\bar{p})H_q + \mu(\bar{q}M + q\bar{M})H_r = 0.$$

The table of characteristics of the I_B in the (z) space may be obtained as in Case II and with the same results except that the images of the points Q_1, Q_2 combine and the joint image is

$$(Q_1, Q_2) \sim F_{38}: \nu'^{26+54} + C_6'^4 + (Q_1, Q_2)^{31}.$$

In any plane λ through l' the 15 bisecants of C_6' meet l' in 15 points μ ; through any point μ on l' the 7 bisecants of C_6' determine 7 planes λ through l' . In the correspondence (λ, μ) there are $15+7+7=29$ coincidences since $\lambda=\mu^2$, and hence in 29 positions of the pair of points Q_1, Q_2 a bisecant of C_6' can be drawn from one of them in the plane through l' associated with the pair. There are 8 trisecants of C_6' which meet l' . These 37 lines are parasitic in I_{51} .

In any plane λ through l' the 6 conics through 5 points of C_6' meet l' in 12 points μ ; through any point μ on l' the 4 conics through 5 points of C_6' lie in 4 planes λ through l' . The number of coincidences in the (λ, μ) correspondence is $12+4+4=20$ and hence in 20 positions of the pair of points Q_1, Q_2 , one of the pair and 5 points of C_6' lie on a conic in the plane through l' associated with the pair. In any plane λ there is a pencil of conics through each of the 15 sets of 4 of the 6 points of C_6' . Each pencil determines an involution on l' which has one pair in common with the involution of points μ , hence 15 pairs of points μ^2 are determined. Given any pair of points μ^2 on l' there are 12 planes λ through l' in which there are conics through the pair μ^2 and 4 points of C_6' . In the correspondence (λ, μ^2) the $15+12=27$ coincidences fix 27 positions of the pair Q_1, Q_2 such that conics in the associated planes pass through them and 4 of the points of C_6' .

The 7 values of λ given by $M\bar{M}=0$ determine 7 planes tangent to F_3 at O_1 or O_2 which pass through O_2 or O_1 . From the associated F_3 each of these planes cuts a cubic with a double point at O_1 or O_2 and passing through O_2 or O_1 and 6 points of C_6 . These 7 cubics correspond to similar cubics in the (z) space which are parasitic in the I_{51} . In any plane λ through l' there are 6 pencils of cubics through the 6 points of C_6' and with a double point at one of them. Each pencil determines an involution of the third order on l' which has 2 pairs in common with the involution of points μ , hence to a λ correspond 12 pairs of points μ^2 . Given any pair of points μ^2 on l' there are 8 planes λ

through l' in which there are cubics through the pair μ^2 and 6 points of C'_6 and which have a double point at one of the points of C'_6 . The correspondence (λ, μ^2) has $12+8=20$ coincidences which determine 20 positions of the pair Q_1, Q_2 such that in the associated plane there will be a cubic through Q_1, Q_2 and the 6 points of C'_6 and having a double point at one of the points of C'_6 . Hence as in Case II we have 37 lines, 47 conics, and 27 cubics which are fundamental curves of the second species in the I_{61} .

7. A Bertini transformation on a cubic variety in S_4 . In a space of four dimensions we take a cubic variety V_3 with a double point at $O_5 \equiv (0, 0, 0, 0, 1)$ and through the points $O_1 \equiv (1, 0, 0, 0, 0)$, $O_2 \equiv (0, 1, 0, 0, 0)$. The equation of the variety is

$$V_3 \equiv \phi_2 x_5 + \phi_3 = 0,$$

where ϕ_2, ϕ_3 are quaternary forms in x_1, x_2, x_3, x_4 with the x_1^3, x_2^3 terms missing in ϕ_3 . The conics tangent to V_3 at the points O_1, O_2 meet V_3 in two residual points P, P' which are conjugate points in a Bertini involution J_B on V_3 . This involution can be mapped on the 3-space $x_5=0$, and a Bertini involution I_B in 3-space is thus determined. The hyperplane $x_5=0$ meets V_3 in the cubic surface $\phi_3=0$, and meets the tangent hypercone to V_3 at O_5 in the quadric $\phi_2=0$. The surfaces $\phi_2=0, \phi_3=0$ meet in a sextic curve C_6 of genus 4. Any plane π through the line O_1O_2 meets C_6 in 6 points R which lie on a conic. The hyperplanes through O_1, O_2 are invariant under J_B , and the planes π are invariant under I_B . Since the 6 points R lie on a conic in each plane π , the Bertini involution in such a plane is degenerate and of the form $I_{13}: O_1^6 + O_2^6 + 6R^4$, with an invariant curve $k_7: O_1^3 + O_2^3 + 6R^2$. The I_B in the space $x_5=0$ has the characteristics

$$\begin{aligned} O_1 &\sim F_6 : O_1^3 + O_2^2 + C_6^2, \\ O_2 &\sim F_6 : O_1^2 + O_2^3 + C_6^2, \\ C_6 &\sim F_{24} : O_1^{12} + O_2^{12} + C_6^7, \\ S_1 &\sim S_{13} : O_1^6 + O_2^6 + C_6^4, \\ K_7 &: O_1^3 + O_2^3 + C_6^2. \end{aligned}$$

The 6 bisecants of C_6 from O_1 and the 6 from O_2 are parasitic lines in I_B and correspond to lines on the V_3 through O_1 or O_2 . To determine the number of parasitic conics we must find the number of conics which lie on V_3 and pass through O_1 and O_2 , since in any such conic the construction used to determine J_B will fail in the sense that to a point on the conic corresponds the whole conic. By a proper choice of coordinate system we can write the equation of any cubic variety in the form

$$(7) \quad x_1^2x_2 + x_1x_2^2 + ax_1 + bx_2 + cx_1x_2 + d = 0$$

where a, b, c, d are ternary forms in x_3, x_4, x_5 . The left hand member of (7) can be factored as follows:

$$(x_1x_2 + b)(x_1 + x_2 + c),$$

if

$$(8) \quad a - b = 0 \text{ and } ac - d = 0.$$

Equations (8) represent two hypercones of the second and third orders respectively whose rulings are planes. The 6 planes common to the two hypercones cut conics from the cubic variety through the points O_1, O_2 . Hence there are 6 fundamental conics of the second species in the I_{13} besides the 12 lines of the second species.

8. A family of space Bertini transformations. A net of planes $\pi \equiv \lambda_1x_1 + \lambda_2x_2 + \lambda_3x_3 = 0$ through the point $(0, 0, 0, 1)$ and a net of cubic surfaces

$$(9) \quad F_3 \equiv x_4(ax)^2 + x_1x_2(bx) \equiv \lambda_1F'_3 + \lambda_2F''_3 + \lambda_3F'''_3 = 0,$$

where $(ax)^2$ and (bx) are quaternary forms in x_1, x_2, x_3, x_4 , and

$$a_{ij} \equiv \lambda_1a'_{ij} + \lambda_2a''_{ij} + \lambda_3a'''_{ij}, \text{ and } b_i \equiv \lambda_1b'_i + \lambda_2b''_i + \lambda_3b'''_i,$$

through the lines $l_1 \equiv x_2 = x_4 = 0$ and $l_2 \equiv x_1 = x_4 = 0$ may be used to determine a transformation of the Bertini type. A point $P(x)$ determines a set of λ_i and hence a plane π and a surface F_3 . The plane π cuts the lines l_1, l_2 in a pair of points $O_1(\lambda_3, 0, -\lambda_1, 0), O_2(0, \lambda_3, -\lambda_2, 0)$. The conic through $P(x)$ and tangent to F_3 at O_1 and O_2 will meet F_3 in a residual point $P'(x')$ which is the conjugate of $P(x)$ in an involutorial transformation I .

If we make the linear transformation

$$\begin{aligned} y_1 &= \lambda_3B_2x_1 + A_2x_4, \\ y_2 &= \lambda_3B_1x_2 + A_1x_4, \\ y_3 &= \lambda_1x_1 + \lambda_2x_2 + \lambda_3x_3, \\ y_4 &= x_4, \end{aligned}$$

where

$$\begin{aligned} B_1 &\equiv b_1\lambda_3 - b_3\lambda_1, \\ B_2 &\equiv b_2\lambda_3 - b_3\lambda_2, \\ A_1 &\equiv a_{11}\lambda_3^2 - 2a_{13}\lambda_1\lambda_3 + a_{33}\lambda_1^2, \\ A_2 &\equiv a_{22}\lambda_3^2 - 2a_{23}\lambda_2\lambda_3 + a_{33}\lambda_2^2, \end{aligned}$$

then equation (9) is in the form of (1), and the transformation (2) may be used to obtain

$$\begin{aligned} x_1' &= B_1B(By_2 - A_2Ay_4), \\ x_2' &= B_2A(Ay_1 - A_1By_4), \\ x_3' &= -B_1B(By_2 - A_2Ay_4) - B_2A(Ay_1 - A_1By_4), \\ x_4' &= \lambda_3B_1B_2AB_4y_4, \end{aligned}$$

where

$$\begin{aligned} A &\equiv B_1^2y_1y_2 + B_2y_4^2[A_1^2B_2 + 2\lambda_3^2B_1^2(a_{14}\lambda_3 - a_{34}\lambda_1) \\ &\quad + A_1B_1(2a_{13}\lambda_2\lambda_3 + 2a_{23}\lambda_1\lambda_3 - 2a_{12}\lambda_3^2 - 2a_{33}\lambda_1\lambda_2 - b_4\lambda_3^2)], \\ B &\equiv B_2^2y_1y_2 + B_1y_4^2[A_2^2B_1 + 2\lambda_3^2B_2^2(a_{24}\lambda_3 - a_{34}\lambda_2) \\ &\quad + A_2B_2(2a_{13}\lambda_2\lambda_3 + 2a_{23}\lambda_1\lambda_3 - 2a_{12}\lambda_3^2 - 2a_{33}\lambda_1\lambda_2 - b_4\lambda_3^2)]. \end{aligned}$$

The λ_i are now replaced by

$$\begin{aligned} \lambda_1 &\equiv \phi_1 \equiv x_2F_3''' - x_3F_3'', \\ \lambda_2 &\equiv \phi_2 \equiv x_3F_3' - x_1F_3''', \\ \lambda_3 &\equiv \phi_3 \equiv x_1F_3'' - x_2F_3'. \end{aligned}$$

The quartic surfaces $\phi_i = 0$ have in common the lines l_1, l_2 , and a residual curve C_{11} of order 11 and genus 14. The surfaces $A = 0, B = 0$ which are the images of the lines l_1, l_2 are of order 8 in ϕ_i and 2 in x_i after a factor ϕ_3^2 is removed. The factor $\phi_3^5 B_1 B_2$ can be removed from the transformation, and the invariant surface $K \equiv y_1 A - y_2 B = 0$ has the factor $\phi_3^3 B_1 B_2$. The characteristics of the transformation are

$$\begin{aligned} l_1 &\sim F_{34} : l_1^{11} + l_2^{10} + C_{11}^8, \\ l_2 &\sim F_{34} : l_1^{10} + l_2^{11} + C_{11}^8, \\ C_{11} &\sim F_{204} : l_1^{66} + l_2^{66} + C_{11}^{47}, \\ S_1 &\sim S_{69} : l_1^{22} + l_2^{22} + C_{11}^{16}, \\ K_{27} &: l_1^9 + l_2^9 + C_{11}^6. \end{aligned}$$

The x parasitic lines of I are trisecants of C_{11} which meet either l_1 or l_2 . Since C_{11} meets l_1 in 4 points there are 7 residual intersections R_i in a plane through l_1 . In any such plane a line $R_i R_j$ meets l_1 in a point P , and through each of R_i and R_j pass 5 other bisecants of C_{11} meeting l_1 in 10 points Q . If h' is the number of bisecants of C_{11} through any point of l_1 , the points P, Q are in $(10h', 10h')$ correspondence. The $20h'$ coincidences are determined by the x trisecants of C_{11} meeting l_1 , the r' tangents of C_{11} meeting l_1 , and the 4 tangents to C_{11} where it meets l_1 . Hence

$$20h' = 6x + 5r' + 30 \cdot 4.$$

Since the C_{11} is of class $r = 48$ and has $h = 31$ apparent double points, then

$h' = h - 4 \cdot 3/2 = 25$, and $r' = r - 2 \cdot 4 = 40$. These values make $x = 30$, but among the 30 trisecants the line l_2 , which is a quadrisecant, is counted 4 times. Hence there are 26 trisecants of C_{11} meeting l_1 and 26 more which meet l_2 . These 52 lines are the parasitic lines of the transformation I .

Let y be the number of parasitic conics and z be the number of parasitic cubics of I . The complete intersection of two surfaces of the web of S_{69} is made up of

$$69^2 = 69 + 22^2 + 22^2 + 11 \cdot 16^2 + 52 + 8y + 27z,$$

and the complete intersection of an S_{69} and the K_{27} is made up of

$$69 \cdot 27 = 27 + 9 \cdot 22 + 9 \cdot 22 + 6 \cdot 16 \cdot 11 + 52 + 4y + 9z.$$

The solution of these equations is $y = 45$, $z = 18$, whence we can conclude that the fundamental curves of the second species in I consist of 52 lines, 45 conics, and 18 cubics.

9. A family of space Geiser transformations. If $F_n = 0$ is a surface of order n with an $(n-2)$ -fold line $l \equiv x_3 = x_4 = 0$, the equations

$$(10) \quad \pi \equiv \lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3 = 0,$$

$$(11) \quad F_n \equiv a x_1^2 + b x_2^2 + 2h x_1 x_2 + 2f x_2 + 2g x_1 + c \equiv \lambda_1 F_n' + \lambda_2 F_n'' + \lambda_3 F_n''' = 0,$$

where $a \equiv \lambda_1 a' + \lambda_2 a'' + \lambda_3 a'''$, etc., and a' , a'' , a''' , etc., are binary forms in x_3 , x_4 , define a net of plane curves C_n of order n with an $(n-2)$ -fold point $Q \equiv (\lambda_2, -\lambda_1, 0, 0)$. A line through Q and a point $P(x)$ on C_n meets it in a residual point $P'(x')$, thus defining an involutorial transformation I having the invariant net of surfaces

$$\begin{aligned} k_1 \phi_1 + k_2 \phi_2 + k_3 \phi_3 \\ \equiv k_1(x_2 F_n''' - x_3 F_n'') + k_2(x_3 F_n' - x_1 F_n''') + k_3(x_1 F_n'' - x_2 F_n') = 0. \end{aligned}$$

The pencil of planes $p \equiv x_4 - \mu x_3 = 0$ through l are invariant under I and in any such plane F_n takes the form

$$(12) \quad a x_1^2 + b x_2^2 + c x_3^2 + 2h x_1 x_2 + 2f x_2 x_3 + 2g x_1 x_3 = 0,$$

where the coefficients are polynomials in μ . This net of conics enables us to map I on a double space $S(\lambda_1 : \lambda_2 : \lambda_3, \mu)$. A plane

$$(13) \quad m_1 x_1 + m_2 x_2 + m_3 x_3 + m_4 x_4 = 0$$

is mapped on S by eliminating x_i between (10) and (13) and using $x_4 = \mu x_3$. The values of x_i thus obtained are substituted in (12) giving

$$\begin{aligned}
 &a(m_2\lambda_3 - \bar{m}_3\lambda_2)^2 + b(\bar{m}_3\lambda_1 - m_1\lambda_3)^2 + c(m_1\lambda_2 - m_2\lambda_1)^2 \\
 &+ 2h(m_2\lambda_3 - \bar{m}_3\lambda_2)(\bar{m}_3\lambda_1 - m_1\lambda_3) + 2f(\bar{m}_3\lambda_1 - m_1\lambda_3)(m_1\lambda_2 - m_2\lambda_1) \\
 &+ 2g(m_2\lambda_3 - \bar{m}_3\lambda_2)(m_1\lambda_2 - m_2\lambda_1) = 0, \text{ where } \bar{m}_3 \equiv m_3 + \mu m_4,
 \end{aligned}$$

which must be identical with

$$(m_1x_1 + m_2x_2 + m_3x_3 + m_4x_4)(m_1x'_1 + m_2x'_2 + m_3x'_3 + m_4x'_4) = 0.$$

From this identity we have

$$\begin{aligned}
 x_1x'_1 &= b\lambda_3^2 - 2f\lambda_2\lambda_3 + c\lambda_2^2, \\
 x_2x'_2 &= c\lambda_1^2 - 2g\lambda_1\lambda_3 + a\lambda_3^2, \\
 x_3x'_3 &= a\lambda_2^2 - 2h\lambda_1\lambda_2 + b\lambda_2^2, \\
 x'_4 &= \mu x'_3.
 \end{aligned}$$

If we replace μ by x_4/x_3 and λ_i by ϕ_i ; we have the transformation I in the form

$$\begin{aligned}
 x_1x'_1 &= b\phi_3^2 - 2f\phi_2\phi_3 + c\phi_2^2, \\
 x_2x'_2 &= c\phi_1^2 - 2g\phi_1\phi_3 + a\phi_3^2, \\
 x'_3 &= x_3(a\phi_2^2 - 2h\phi_1\phi_2 + b\phi_2^2), \\
 x'_4 &= x_4(a\phi_2^2 - 2h\phi_1\phi_2 + b\phi_2^2),
 \end{aligned}$$

where x_1, x_2 are factors of the first two equations respectively. The surfaces $\phi_i=0$ are of order $n+1$ and have l as an $(n-2)$ -fold line. The residual basis curve of the net of ϕ_i is a C_{5n-3} of order $5n-3$ and genus $12n-19$ through the point $(0, 0, 0, 1)$. The image of l in I is the surface $L \equiv a\phi_2^2 - 2h\phi_1\phi_2 + b\phi_2^2 = 0$, which is of order 3 in ϕ_i and of order $n-2$ in x_3, x_4 . The image in S of the invariant surface K has the equation

$$\begin{vmatrix}
 a & h & g & \lambda_1 \\
 h & b & f & \lambda_2 \\
 g & f & c & \lambda_3 \\
 \lambda_1 & \lambda_2 & \lambda_3 & 0
 \end{vmatrix} = 0,$$

which corresponds to K^2 in the space (x) . Hence K is of order 2 in ϕ_i and of order $n-1$ in x_3, x_4 . The table of characteristics of I is

$$\begin{aligned}
 l &\sim L_{4n+1} : l^{4n-7} + C_{5n-3}^3, \\
 C_{5n-3} &\sim F_{12n+3} : l^{12n-18} + C_{5n-3}^8, \\
 S_1 &\sim S_{4n+2} : l^{4n-6} + C_{5n-3}^3, \\
 K_{3n+1} &: l^{3n-5} + C_{5n-3}^2.
 \end{aligned}$$

In any plane p through l there is an ordinary Geiser transformation, therefore the C_{5n-3} meets such a plane in the 7 fundamental points R_i of the Geiser transformation and in $5n-10$ points on l . The section of C_{5n-3} by the plane $x_3=0$ is the point $(0, 0, 0, 1)$ and 6 points lying on the conic $x_3=0, F_n'''=0$. Hence on this plane the Geiser transformation degenerates and the conic is parasitic for I .

The x parasitic lines of I are trisecants of C_{5n-3} meeting l . Since C_{5n-3} meets any plane p in 7 points not on l the method of §8 may be used in determining the number of trisecants of C_{5n-3} which meet l . The number $x=15n-15$ is obtained from the equation

$$20h' = 6x + 5r' + 30(5n - 10),$$

where $r'=24n-26$, and $h'=18n-26$. Therefore the fundamental curves of the second species for I consist of $15n-15$ lines and one conic.

CORNELL UNIVERSITY,
ITHACA, N. Y.