IDEAL THEORY AND ALGEBRAIC DIFFERENTIAL EQUATIONS*

BY

H. W. RAUDENBUSH, JR.

INTRODUCTION

J. F. Ritt† introduced the idea of irreducible system of algebraic differential equations and showed that every system of such equations is equivalent to a finite set of irreducible systems.

One of the objects of this paper is to develop a special type of abstract ideal theory which has Ritt's theorem as a consequence. The elements of our ideals are polynomials in unknowns $y_1, \cdots, y_n$ and a certain number of their derivatives. Following Ritt, we call these polynomials *forms*. The coefficients in these forms are assumed to be elements of a *differential field* $\mathfrak{F}$ of characteristic zero.‡ A *differential field* is a commutative field (as in abstract algebra) whose elements $a, b, \cdots$ have unique derivatives $a_1, b_1, \cdots$ which are elements of the field. These derivatives must satisfy the rules $(a+b)_1 = a_1 + b_1$ and $(ab)_1 = a_1 b + a b_1$.§ The totality of these forms with coefficients in $\mathfrak{F}$ is a differential ring $\mathfrak{R}$.|| We consider *differential ideals*, which are ideals containing together with any element its derivative.¶ An example given by Ritt shows that there exists a differential ideal of $\mathfrak{R}$ having no finite subset, such that every element of the ideal is a linear combination of elements of the subset and their derivatives with forms of $\mathfrak{R}$ as coefficients.**

Certain results of Ritt suggested that we consider, as our purpose permits, only differential ideals which have the property that if they contain an ele-

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† J. F. Ritt, *Differential Equations from the Algebraic Standpoint*, Colloquium Publications of this Society, vol. 14. Cf. p. 14.

‡ For definitions of terms of abstract algebra see B. L. van der Waerden, *Moderne Algebra*.

§ Abstract differential fields have been treated by R. Baer, *Algebraische Theorie der differentierbaren Funktionenkörper*, I, Heidelberger Akademie der Wissenschaften, Sitzungsberichte, Mathematische-Naturwissenschaftliche Klasse, 1927–1928, and by the author, *Differential fields and ideals of differential forms*, Annals of Mathematics, vol. 34 (1933), pp. 509–517. They have been used by O. Ore, *Formale Theorie der linearen Differentialgleichungen*, Journal für Mathematik, vol. 167 (1932), pp. 221–234, and vol. 168 (1932), pp. 233–252.

|| Raudenbush, loc. cit., p. 514. In the definition of differential field, substitute ring for field to obtain the definition of differential ring.

¶ Raudenbush, loc. cit., p. 516.

** Ritt, loc. cit., p. 12.

ment a of $\mathfrak{R}$, they contain any element b of $\mathfrak{R}$ such that a positive power of b is a. We call these differential ideals *perfect differential ideals*. We show that *every perfect differential ideal of $\mathfrak{R}$ is the intersection of a finite number of prime perfect differential ideals.*

The use of perfect differential ideals was suggested by the following two results of Ritt:

(a) *Every infinite system of forms has a finite subsystem whose manifold of solutions is identical with that of the infinite system.**

(b) *Let $F_1, \cdots, F_r$; G be forms such that G has every solution of the system $F_1, \cdots, F_r$. Then some power of G is a linear combination of the F_i and a certain number of their derivatives with forms for coefficients.*†

We obtain abstract theorems that specialize to a combination of these results of Ritt. For instance, we show that *every perfect differential ideal of $\mathfrak{R}$ has a finite subset such that every form of the ideal has a power which is a linear combination of the forms of the subset and their derivatives with forms of $\mathfrak{R}$ for coefficients.* The proof of this basis theorem is like the proof of Ritt's result (a) in fundamental respects, but there are essential differences. We also obtain an abstract generalization of Ritt's result (b). The conciseness of the proof of this theorem is an indication of the simplicity of our theory.

Having established the basis theorem, the development of our ideal theory follows approximately the well known methods of E. Noether.‡

Perfect differential ideals

1. We consider a fixed differential ring $\mathfrak{R}$ of characteristic zero.

The intersection of any arbitrary set of differential ideals is a differential ideal. For let a be any element of the intersection. Then a is an element of every ideal of the set; hence the derivative a_1 is in the intersection. The intersection, which is known to be an ideal, is then a differential ideal. The intersection of any arbitrary set of perfect differential ideals is a perfect differential ideal. Let a and b be elements of $\mathfrak{R}$ such that a is in the intersection and some power of b is a. Then a is in every ideal of the set, hence also b. Therefore the intersection is a perfect differential ideal.

Let σ be an arbitrary set of elements of $\mathfrak{R}$. We notice that $\mathfrak{R}$ is a perfect differential ideal. The intersection of the differential ideals containing σ will be called the *differential ideal* $[\sigma]$ *determined by* σ. $[\sigma]$ is uniquely defined. The intersection of all perfect differential ideals containing σ we call *the perfect differential ideal* $\{\sigma\}$ *determined by* σ. $\{\sigma\}$ is uniquely defined.

* Ritt, loc. cit., p. 10.

† Ritt, loc. cit., p. 108.

‡ E. Noether, *Idealtheorie in Ringbereichen*, Mathematische Annalen, vol. 86 (1921), pp. 24–66.

Let α be any set of elements of $\mathfrak{R}$. We shall denote by α' the set consisting of all elements of $\mathfrak{R}$ which have a positive integral power in α. Using the set σ of the preceding paragraph, we define σ_n recursively as follows:

$$\sigma_1 = [\sigma],$$
$$\sigma_n = [\sigma_{n-1}'] \qquad (n = 2, 3, 4, \cdots).$$

Let β denote the totality of elements of the sets σ_n. Then β is a perfect differential ideal and is contained in $\{\sigma\}$, hence is $\{\sigma\}$. This means that any element t of $\{\sigma\}$ is in some σ_n with a sufficiently large subscript.

LEMMA. *If a differential ideal δ contains a positive integral power a^p of an element a it contains the positive integral power a_1^{2p-1} of the derivative a_1 of a.*

δ contains $(a^p)_1 = pa^{p-1}a_1$ hence δ contains $a^{p-1}a_1$.* Assume that δ contains $a^{p-r}a_1^s$, where $r<p$. Then δ contains

$$a_1(a^{p-r}a_1^s)_1 - sa_2(a^{p-r}a_1^s) = (p-r)a^{p-r-1}a_1^{s+2},$$

where $a_2 = (a_1)_1$; hence δ contains $a^{p-r-1}a_1^{s+2}$. Applying this result $p-1$ times to $a^{p-1}a_1$ we find that δ contains a_1^{2p-1}.

Let t be any element of $\{\sigma\}$ not in σ_1. There is a least positive integer $n>1$ such that σ_n contains t. As an element of σ_n, t is equal to a linear homogeneous expression in a finite number of elements of σ_{n-1}' and a finite number of derivatives of elements of σ_{n-1}' with elements of the ring or integers for coefficients.† But, by the lemma, each of these elements has a power in σ_{n-1}. Let r be their number and s the maximum of the powers to which each must be raised to give an element of σ_{n-1}. Then t^{rs-r+1} is in σ_{n-1}, for each term of the same power of the linear expression contains an sth power of some one of the elements and hence each term is in σ_{n-1}.

This power of t by the same reasoning has a power in σ_{n-2}. Hence t has a power in σ_{n-2}. Continuing this process a finite number of steps gives the

THEOREM 1. *If t is any element of a perfect differential ideal $\{\sigma\}$ of a differential ring $\mathfrak{R}$ of characteristic zero determined by a set σ, then some positive integral power of t is in the differential ideal $[\sigma]$ determined by σ.*‡

2.§ LEMMA. *If a perfect differential ideal π contains the product ab of any two elements a and b then it contains the product a_pb_q of any derivatives of a and b.*‖

* If $\mathfrak{R}$ were of characteristic p we could not draw this conclusion.

† If n is an integer and a an element of the ring, $1a=a$, $-1a=-a$, $na=(n-1)a+a$, $na=an$.

‡ The theorem is not true for rings of non-zero characteristic.

§ The results of this and the next article are independent of Theorem 1 and true for non-zero characteristic.

‖ p may be zero; $a_0=a$ and we shall speak of the zero derivative of a. $a_p=(a_{p-1})_1$.

Assume that $a_m b_n$ is in π. Then π contains

$$a_{m+1}b_n(a_m b_n)_1 - a_{m+1}b_{n+1}(a_m b_n) = a_{m+1}^2 b_n^2.$$

Hence, by the definition of a perfect differential ideal, π contains $a_{m+1}b_n$. Similarly, π contains $a_m b_{n+1}$. Since, by hypothesis, π contains $a_0 b_0$, the lemma is obtained by induction.

THEOREM 2. *The intersection* $\{\sigma, a\} \wedge \{\sigma, b\}$ *of the perfect differential ideals determined by the sets obtained by adjoining elements* a *and* b, *respectively, to the set* σ *of elements is the perfect differential ideal* $\{\sigma, ab\}$ *determined by the set obtained by adjoining the product* ab *to* σ.*

Every element of $\{\sigma, ab\}$ is in the intersection. We have only to show that any element t of the intersection is in $\{\sigma, ab\}$.

By Theorem 1 some power of t, say t^r, is in $[\sigma, a]$, and some power, say t^s, is in $[\sigma, b]$. Hence t^{r+s} is in $\{\sigma, ab\}$ since each term of the product of the linear expression for t^r, in terms of the elements of σ and a and their derivatives, and for t^s, in terms of the elements of σ and b, contains either elements of σ or a product $a_p b_q$ of derivatives of a and b. By the definition of perfect differential ideal, $\{\sigma, ab\}$ contains t.

DECOMPOSITION OF PERFECT DIFFERENTIAL IDEALS

3. We shall say that a perfect differential ideal π which is determined by a set σ has σ as a *basis*. If every perfect differential ideal of a differential ring $\mathfrak{R}$ has a finite basis, we say that $\mathfrak{R}$ is *a differential ring with a basis theorem*.

THEOREM 3. *Let*

$$\pi_1 \leqq \pi_2 \leqq \pi_3 \leqq \cdots$$

be an infinite sequence of perfect differential ideals of a differential ring with a basis theorem such that each ideal contains its predecessor in the sequence. There exists an integer n *such that*

$$\pi_n = \pi_{n+1} = \cdots .\dagger$$

Let π be the totality of elements in the ideals of the sequence. Let a be any element of π. Then a is contained in some ideal of the sequence with a sufficiently high subscript. Therefore π contains a_1 or any element b having a power equal to a and hence is a perfect differential ideal. π has a finite basis

* A more general theorem could be proved but this is sufficient to our purpose.

† Cf. van der Waerden, loc. cit., vol. II, p. 25.

which must be contained in some ideal of the sequence with a sufficiently large subscript n. But $\pi_n = \pi$, hence $\pi_n = \pi_{n+1} = \cdots$.

A perfect differential ideal π will be called *reducible* if there exist perfect differential ideals α and β such that π is a proper subset of α and of β and is their intersection $\alpha \wedge \beta$. If a perfect differential ideal is not reducible, it is said to be *irreducible*.*

THEOREM 4. *A perfect differential ideal which is irreducible is prime.*†

We show that a perfect differential ideal which is not prime is reducible. Let π be a perfect differential ideal which is not prime. There exist two elements a and b such that π contains ab but neither a nor b. Form the perfect differential ideals $\{\pi, a\}$ and $\{\pi, b\}$. Each contains π as a proper subset. Their intersection $\{\pi, a\} \wedge \{\pi, b\}$ by §2 is $\{\pi, ab\}$ but since ab is in π, the intersection is π. Hence π is reducible.

THEOREM 5. *In a differential ring with a basis theorem, any perfect differential ideal is the intersection of a finite set of irreducible or prime perfect differential ideals.*

We suppose that the theorem is not true. Then there exists a perfect differential ideal π which is not the intersection of a finite number of irreducible perfect differential ideals. π must be reducible. Hence π is the intersection of two perfect differential ideals α and β each containing π as a proper subset. At least one of the perfect differential ideals α and β is not the intersection of a finite number of irreducible perfect differential ideals. Let π_1 denote this perfect differential ideal. By the same reasoning π_1 is a proper subset of a perfect differential ideal π_2 which is not the intersection of a finite number of irreducible perfect differential ideals. Continuing in this manner we obtain an infinite sequence of perfect differential ideals, each containing its predecessor as a proper subset. This contradiction of Theorem 3 proves the theorem.

In such a finite set of irreducible or prime perfect differential ideals, we may delete in turn all ideals which contain other ideals of the set. The remaining set will be called an *essential set*.

THEOREM 6. *If a perfect differential ideal π is the intersection of each of two essential sets $\alpha_1, \cdots, \alpha_r$ and $\beta_1, \cdots, \beta_s$, then $r = s$ and the α's coincide with the β's after a suitable rearrangement.*

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* This use of the word "irreducible" is analogous to its use in algebra. Cf. van der Waerden, loc. cit., vol. **II**, p. 36.

† An ideal is prime if it contains together with the product of any two elements at least one of the elements.

α_1 is contained in some β_i. If not, each β_i would contain an element b_i not in α_1. By a repeated application of Theorem 2, π contains the product $b_1 \cdots b_s$. Hence α_1 contains this product, which contradicts the fact that α_1 is prime.

We may suppose that α_1 is contained in β_1 after a suitable rearrangement of the β's. β_1 is contained in some α which must be α_1. For if β_1 were contained in α_k, $k \neq 1$, then α_1 would be contained in α_k contradicting the assumption that the α's form an essential set. Hence α_1 is β_1.

α_2 is contained in some β which cannot be β_1. Suppose that it is β_2. Then β_2 is in α_2 and is α_2. Continuing in this manner the theorem is proved.

The basis theorem

4. In what follows we will need the following

Lemma. *If the perfect differential ideal $\{\sigma\}$ has a finite basis, it has a finite basis consisting of elements of σ.*

Let $s_1, \cdots, s_n$ be the elements of a finite basis of $\{\sigma\}$. Each s_i as an element of $\{\sigma\}$ is an element of $\{\sigma_i\}$ where σ_i is a suitably chosen finite subset of σ. Every s_i is in $\{\sigma_1, \cdots, \sigma_n\}$, hence $\{\sigma\}$ is $\{\sigma_1, \cdots, \sigma_n\}$.

5. We prove the following theorem:

Theorem 7. *The differential ring $\mathfrak{R}$ of forms in a finite set of indeterminates $y_1, \cdots, y_n$ with coefficients in a differential field $\mathfrak{F}$ of characteristic zero is a differential ring with a basis theorem.*

We suppose that the theorem is not true and force a contradiction.

Lemma. *Let Σ be a perfect differential ideal without a finite basis. Let $F_1, \cdots, F_s$ be forms such that by multiplying each form of Σ by some product of non-negative powers of $F_1, \cdots, F_s$ a system Λ is obtained such that $\{\Lambda\}$ has a finite basis. Then $\{\Sigma, F_1 \cdots F_s\}$ has no finite basis.*

Suppose, as we may by the lemma of the preceding article, that $\{H_1, \cdots, H_t; F_1 \cdots F_s\}$ is $\{\Sigma, F_1 \cdots F_s\}$, where $H_1, \cdots, H_t$ are forms of Σ. Let a finite basis of $\{\Lambda\}$ be chosen from the forms of Λ and let $K_1, \cdots, K_v$ be forms of Σ such that the forms which they yield, after the above described multiplications, form this basis of $\{\Lambda\}$. Let Π be the totality of H's and K's. Then $\{\Pi, F_1 \cdots F_s\}$ is $\{\Sigma, F_1 \cdots F_s\}$ and $\{\Pi\}$ contains $\{\Lambda\}$.

Since Σ has no finite basis, there exists a form L of Σ not in $\{\Pi\}$. Some $F_1^{g_1} \cdots F_s^{g_s} L$ is in $\{\Lambda\}$ and hence in $\{\Pi\}$. Consequently, if g is the maximum of the g_i's, $F_1^g \cdots F_s^g L^g$ and hence $F_1 \cdots F_s L$ are in $\{\Pi\}$. L is in $\{\Pi, F_1 \cdots F_s\}$ by our assumption and obviously in $\{\Pi, L\}$; hence by §2 it is in $\{\Pi, F_1 \cdots F_s L\}$ which is $\{\Pi\}$. This contradiction proves the lemma.

LEMMA. *Let Σ and $\{\Sigma, F_1 \cdots F_s\}$ be perfect differential ideals having no finite basis sets. Then at least one of the perfect differential ideals $\{\Sigma, F_1\}, \cdots, \{\Sigma, F_s\}$ has no finite basis.*

We may limit ourselves to the case of $s=2$. Let $\{\Sigma, F_1\}$ and $\{\Sigma, F_2\}$ be $\{\Phi_1, F_1\}$ and $\{\Phi_2, F_2\}$ respectively, where Φ_1 and Φ_2 are finite sets taken, according to the lemma of the preceding article, as subsets of Σ. $\{\Sigma, F_1\}$ and $\{\Sigma, F_2\}$ are also $\{\Phi_1, \Phi_2, F_1\}$ and $\{\Phi_1, \Phi_2, F_2\}$ respectively. $\{\Sigma, F_1F_2\}$ is the intersection $\{\Sigma, F_1\} \wedge \{\Sigma, F_2\}$ by §2 and hence also $\{\Phi_1, \Phi_2, F_1F_2\}$ which contradiction proves the lemma.

We consider the totality of perfect differential ideals of $\mathfrak{R}$ without finite basis sets.* We form a *basic set*† for each. By a lemma of Ritt's†, we know that there is a perfect differential ideal Σ without a finite basis whose basic sets are not of higher rank‡ than the basic sets of any other perfect differential ideal without a finite basis. Let

$$A_1, \cdots, A_r \tag{1}$$

be a basic set of Σ. Then A_1 is not an element of $\mathfrak{F}$, as otherwise Σ would have unity as a finite basis.

For every form of Σ not in (1), let a remainder§ with respect to (1) be found. Let Λ be the system composed of the forms of (1) and the products of the forms of Σ not in (1) by the products $S_1^{s_1} \cdots S_r^{s_r} I_1^{t_1} \cdots I_r^{t_r}$ of the separants‖ S_i and the initials‖ I_i of the forms of (1) used in their reduction.‖ Let Ω be the system composed of (1) and the remainders of the forms of Σ not in (1).

$\{\Omega\}$ has a finite basis. If not, Ω would have non-zero forms not in (1). Such forms would be reduced† with respect to (1) and $\{\Omega\}$ would have lower basic sets than Σ, contradicting our assumption. Consequently $\{\Lambda\}$ has a finite basis, for $\{\Lambda\}$ is $\{\Omega\}$.

The lemmas show that some $\{\Sigma, S_i\}$ or some $\{\Sigma, I_i\}$ has no finite basis. But for every i, S_i and I_i are distinct from zero, and reduced with respect to (1). Hence the basic sets of $\{\Sigma, S_i\}$ and $\{\Sigma, I_i\}$ are lower than (1). This contradiction proves the theorem.

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* Cf. Ritt, loc. cit. In what follows we use the concepts and theorems of §§2 to 5 in this book. The reader will have no difficulty seeing that these articles with slight changes in language apply to our abstract forms.

† Ritt, loc. cit., p. 6.

‡ Ritt, loc. cit., p. 4.

§ Ritt, loc. cit., p. 9 and p. 7 for the existence. Notice that in case of non-zero characteristic a separant may vanish.

‖ Ritt, loc. cit., p. 7.

THEOREM 8. *Any infinite system of forms contains a finite subset such that every form of the system has a power which is a linear combination of the forms of the subset and their derivatives with forms for coefficients.*

This follows at once from Theorems 1 and 7, and the lemma of the preceding article.

ANALOGUE OF THE HILBERT-NETTO THEOREM

We prove the following

THEOREM 9. *Let Σ and G be a system of forms and a form respectively of the differential $\mathfrak{R}$ of forms in a finite set of indeterminates $y_1, \cdots, y_n$ and with coefficients in a differential field $\mathfrak{F}$ of characteristic zero. If G is not in $\{\Sigma\}$ then there exists a set of elements $a_1, \cdots, a_n$ of an extension of $\mathfrak{F}$ such that every form of Σ vanishes when the a's are substituted for the indeterminates and such that G does not vanish for the same substitution.*

Let $\Pi_1, \cdots, \Pi_n$ be prime perfect differential ideals whose intersection is $\{\Sigma\}$. Some Π_i, say Π', does not contain G. By a theorem of the author's dissertation,* there exists a set of elements $a_1, \cdots, a_n$ of an extension of $\mathfrak{F}$ such that Π' is the set of forms of $\mathfrak{R}$ that vanish when the a's are substituted for the indeterminates. The a's are then solutions of Σ but not of G.

In what follows, we suppose that $\mathfrak{F}$ is a differential field of functions of a complex variable x meromorphic on an open region $\mathfrak{A}$. Let Π' have $A_1, \cdots, A_p$ as its basic set. A_1 is of class greater than zero.† Let S_i and I_i be the separant and initial, respectively, of A_i. We show that the basic set has analytic solutions, when regarded as polynomials in the y_{ij} that they contain‡, for which G and no separant or initial vanishes. We suppose that every analytic solution of the basic set is a solution of $T = S_1 \cdots S_p I_1 \cdots I_p G$. Then by the Hilbert-Netto theorem for polynomials, T is in Π'. This contradicts the fact that Π' is prime, for Π' can contain no separant or initial of the forms of its basic set, and was chosen so as not to contain G. For a suitable value of x the values of the analytic functions in such a solution provide initial conditions for a regular§ analytic solution of the basic set which is not a solution of G. By a theorem of Ritt's‖, such a solution is a solution of Π' and hence of Σ. This together with Theorem 1 gives Ritt's result (b).

* Raudenbush, loc. cit., p. 517, Theorem V.

† Ritt, loc. cit., p. 3.

‡ Certain of the y_{ij} may be indeterminate.

§ Ritt, loc. cit., p. 20.

‖ Ritt, loc. cit., p. 25. The theorem is true if the system is not closed, provided it is an ideal.

BARNARD COLLEGE, COLUMBIA UNIVERSITY,
NEW YORK, N. Y.