THE VALUE OF THE NUMBER $g(k)$ IN WARING'S PROBLEM*

BY

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1. **Introduction.** The number $g(k)$ is defined to be such that (a) every integer is a sum of $g(k)$ kth powers $\geqq 0$; (b) there is at least one integer which is not a sum of $g(k)-1$ kth powers $\geqq 0$. It is well known that $g(2)=4$, $g(3)=9$, but the exact value of $g(k)$ is not known when $k \geqq 4$.

The number $G(k)$ is defined to be such that every integer $>C=C(k)$ is a sum of $G(k)$ kth powers $\geqq 0$. Hardy and Littlewood‡ have proved that

$$G(k) \leqq (k-2)2^{k-2} + k + 5 + \zeta_k,$$

where

$$\zeta_k = \left[\frac{(k-2)\log 2 - \log k + \log (k-2)}{\log k - \log (k-1)}\right].$$

In this paper we obtain a similar bound for $g(k)$ when $k \geqq 6$. We shall prove the

THEOREM. *Let L be a number $>k^k$ such that every integer $\leqq L$ is a sum of s_3 kth powers $\geqq 0$. Let*

$$D = (d+2)(k-1) - 2^{d+1} + 1/10, \quad d = [\log (k-1)/\log 2];$$

$$E = s_3 + \frac{3\log k + \log 20 - \log (\log L - k\log k)}{\log k - \log (k-1)};$$

$$F = \log 2(\log k - \log (k-1))^{-1}; \quad H = (k-2)2^{k-2} + k;$$

$$Q = 2 + s_2 = 6 + \zeta_k; \quad R = (1 + (1-a)^{s_2-2})k2^{k-2} - DQ.$$

Then

$$g(k) \leqq [\tfrac{1}{2}(H + FD + Q + E + ((H + FD + Q - E)^2 + 4F(ED + R))^{1/2})] + 1. \tag{1}$$

The method of proof is as follows: We determine the constants as they occur at each step of the Hardy-Littlewood analysis as functions of k, s, and

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‡ G. H. Hardy and J. E. Littlewood, *Some problems of "partitio numerorum"* (VI): *Further researches in Waring's problem*, Mathematische Zeitschrift, vol. 23 (1925), pp. 1–37. See also E. Landau, *Vorlesungen über Zahlentheorie*, vol. 1, part 6 (referred to as L), and M. Gelbcke, *Zum Waringschen Problem*, Mathematische Annalen, vol. 105 (1931), pp. 637–652 (referred to as G).

ϵ, where ϵ is a small positive number. In this way we conclude that every integer $>C(k, s, \epsilon)$ is a sum of s kth powers $\geqq 0$ when $s \geqq g_1(k, \epsilon)$ (Theorem 46). Then, using a theorem proved by L. E. Dickson,* we show that every integer $\leqq C(k, s, \epsilon)$ is a sum of s kth powers $\geqq 0$ when $s \geqq g_2(k, \epsilon)$ (Theorem 50). We choose ϵ as a function of k so that $g_1(k, \epsilon(k)) = g_2(k, \epsilon(k))$ and then $g(k) \leqq g_1(k, \epsilon(k)) = g_2(k, \epsilon(k))$.

In Theorem 48 we give a general method for the determination of L and s_3 and prove that

$$s_3 < 2^k + \left(\frac{3}{2}\right)^k + 2\left(\frac{4}{3}\right)^k + 2\left(\frac{2}{3}\right)^k + 2\left(\frac{1}{2}\right)^k + \frac{k(2k+7)}{9} - 9$$

when $L = (k+1)^k - k^k > k^k$. It then follows from (1) that

$$\overline{\lim_{k=\infty}} \frac{g(k)}{k2^{k-1}} \leqq \tfrac{1}{2}.$$

For particular values of k we may obtain better values of L and s_3. For example, since $25 \cdot 2^8 = 6400$, $3^8 = 6561$, $26 \cdot 2^8 = 6656$, every integer from 6400 to 6656 is a sum of 185 8th powers $\geqq 0$. Repeated application of Theorem 47 yields the result that every integer from 1 to $10^{713 \cdot 7}$ is a sum of 279 8th powers $\geqq 0$. With $L = 10^{713 \cdot 7}$, $s_3 = 279$, we get $g(8) \leqq 622$. Again, since $25 \cdot 2^8 + 9 \cdot 3^8 = 65449$, $4^8 = 65536$, $10 \cdot 3^8 = 65610$, $26 \cdot 2^8 + 9 \cdot 3^8 = 65705$, every integer from 65449 to 65705 is a sum of 120 8th powers $\geqq 0$ and this gives $L = 10^{3,920,000}$, $s_3 = 279$, $g(8) \leqq 595$. It is obvious that the larger we can make L for a given s_3 the better will be the resulting bound for $g(k)$. In the table below we summarize the known results for $g(k)$ and $G(k)$ when $6 \leqq k \leqq 10$. The first line gives the bounds for $g(k)$ obtained by algebraic methods separately for each k; the second gives the bounds obtained by the methods of this paper; the third gives the bounds for $G(k)$ obtained by the Hardy-Littlewood method; and the fourth gives the lower bounds for $g(k)$.

k	6	7	8	8	10
$g(k) \leqq$	478	3806	31353	—	140004
$g(k) \leqq$	183	322	595	1177	2421
$G(k) \leqq$	87	193	425	949	2113
$g(k) \geqq$	73	143	279	548	1079

* L. E. Dickson, *Proof of a Waring theorem on fifth powers*, Bulletin of the American Mathematical Society, vol. 37 (1931), pp. 549–553.

The numbers in the last line are probably the exact values of $g(k)$. In order to prove $g(10)=1079$, for example, it would be necessary to prove some inequality like $G(10)\leqq 700$. On the basis of an unproved hypothesis, Hardy and Littlewood (loc. cit.) have shown that $G(10)$ would be $\leqq 21$. It seems likely, then, that a far less drastic assumption would be sufficient to prove $g(10)=1079$ and this assumption may be capable of proof.

The possibility of evaluating the constants of the Hardy-Littlewood analysis was suggested by Professor L. E. Dickson. The case of fifth powers was considered in the author's doctor's dissertation written under Professor Dickson's direction at the University of Chicago.

2. Notation. We shall use the following notation throughout the paper. Let

$$T(m)=\text{the number of divisors of } m;$$
$$\pi(x)=\text{the number of primes}\leqq x;$$
$$\vartheta(x)=\text{the sum of the logarithms of all primes}\leqq x;$$
$$[x]=\text{the greatest integer}\leqq x;$$
$$\{x\}=\min\,(x-[x],\ [x]+1-x);$$
$$M(p^t,\ n)=\text{the number of solutions of the congruence}\textstyle\sum_{i=1}^{s}h_i^k\equiv n \pmod{p^t};$$
$$N(p^t,\ n)=\text{the number of solutions of the same congruence in which not every } h_i \text{ is divisible by } p \text{ (primitive solutions)};$$
$$k=\text{an integer}\geqq 6;\ a=1/k;\ K=2^{k-1};\ A=1/K;$$
$$\epsilon_i=\text{a small positive number},\ i=1,2,3;\ \eta_i=1/\epsilon_i,\ i=1,2,3;$$
$$s=[(H+(1+(1-a)^{s_2-2})k2^{k-2}\epsilon_1)\,(1-D\epsilon_1)^{-1}]+3;$$
$$s_2=[((k-2)\log 2-\log k+\log(k-2))\,(\log k-\log(k-1))^{-1}]+4; \tag{2}$$
$$\lambda=2A+(1-a)^{s_2-2}\epsilon_1;$$
$$\Theta=\text{the highest power of a prime } p \text{ which divides } k;$$
$$\gamma=\begin{cases}\Theta+2 \text{ if } p=2,\\ \Theta+1 \text{ if } p>2;\end{cases}\qquad p^\gamma=P;$$
$$(a,\ b)=\text{the greatest common divisor of } a \text{ and } b;$$
$$r=((P-1)/(p-1))\,(k,\ p-1);$$
$$r(n)=r_{k,s}(n)=\text{the number of solutions of}\textstyle\sum_{i=1}^{s}h_i^k=n,\ h_i\geqq 0;$$
$$\rho=e^{2\pi ib/q},\ (b,q)=1;$$
$$S_\rho=\textstyle\sum_{z=1}^{q}\rho^{z^k};$$
$$A(q)=A_{k,s}(q,j)=q^{-s}\textstyle\sum_\rho S_\rho^s\rho^{-j},\ \text{where } \rho \text{ ranges over all primitive } q\text{th roots of unity};$$
$$\chi_p=\textstyle\sum_{t=0}^{\infty}A(p^t);$$
$$\mathfrak{S}(j,k,s,w)=\textstyle\sum_{q=1}^{w}A(q);$$
$$f(x)=\textstyle\sum_{h=0}^{na}x^{h^k};$$
$$\psi_\rho(x)=\Gamma(1+a)q^{-1}S_\rho(1+\textstyle\sum_{j=1}^{n}(a(a+1)\cdots(a+j-1)/j!)\,(x/\rho)^j);$$
$$\phi_\rho(x)=\Gamma(1+a)q^{-1}S_\rho\textstyle\sum_{j=n+1}^{\infty}(a(a+1)\cdots(a+j-1)/j!)\,(x/\rho)^j;$$

$$(2)\qquad \begin{aligned}\Psi_\rho(x)&=\psi_\rho(x)+\phi_\rho(x)=\Gamma(1+a)q^{-1}S_\rho(1-x/\rho)^{-a};\\ \sigma(j)&=r(j)-(\Gamma^s(1+a)/\Gamma(sa))\,(\Gamma(sa+j)/j!)\mathfrak{S}(j,k,s,n^a).\end{aligned}$$

The letters A, α, b, B, c, C are numbered in the same way as the corresponding letters in paper L, while the letters G correspond to the C of paper G.

3. **Preliminary theorems.** We shall not repeat the proof of a known theorem if the constants involved are explicitly given in the original proof.

THEOREM 1.* *For every* $\epsilon_1>0$

$$T(m)\leqq A_1 m^{\epsilon_1},$$

where

$$A_1=\frac{2^{\pi(2^{\eta_1})}\cdot(3/2)^{\pi((3/2)^{\eta_1})}\cdot(4/3)^{\pi((4/3)^{\eta_1})}\cdots}{\exp\left(\epsilon_1(\vartheta(2^{\eta_1})+\vartheta((3/2)^{\eta_1})+\vartheta((4/3)^{\eta_1})+\cdots)\right)}.$$

THEOREM 2. (L, Theorem 112.) *For* $\xi\geqq 2$,

$$\alpha_1\xi/\log\xi<\pi(\xi)<\alpha_2\xi/\log\xi,$$

where $8\alpha_1\geqq\log 2$ *and* $\alpha_2\leqq 7\log 2$.

Since

$$[\eta]-2[\eta/2]<\eta-2(\eta/2-1)=2$$

and the left side is an integer, it follows that

$$(3)\qquad [\eta]-2[\eta/2]\leqq 1.$$

Let $n\geqq 2$. For every prime $p\leqq 2n$ let f denote the greatest integer such that $p^f\leqq 2n$ (i.e., $f=[\log(2n)/\log p]$). We show first that

$$(4)\qquad \prod_{n<p\leqq 2n} p\;\Big|\;\frac{(2n)!}{n!n!}\;\Big|\;\prod_{p\leqq 2n} p^f.$$

The first part of (4) follows at once since every p for which $n<p\leqq 2n$ divides $(2n)!$ but not $n!n!$. Also, since the highest power of a prime p which divides $x!$ is

$$\sum_{1\leqq m\leqq\log x/\log p}[x/p^m]$$

(see, for example, L, Theorem 27), the highest power of p which divides $(2n)!/(n!\,n!)$ is

$$\sum_{m=1}^{f}([2n/p^m]-2[n/p^m])\leqq\sum_{m=1}^{f}1=f$$

* S. Ramanujan, *Highly composite numbers*, Proceedings of the London Mathematical Society, (2), vol. 14 (1915), p. 392.

by (3). This proves the second part of (4). Next, the left side of (4) has $\pi(2n)-\pi(n)$ factors each $>n$, and the right side has $\pi(2n)$ factors each $\leqq 2n$. Hence

$$n^{\pi(2n)-\pi(n)} < \prod_{n<p\leqq 2n} p \leqq \frac{(2n)!}{n!n!} \leqq \prod_{p\leqq 2n} p^{\nu} \leqq (2n)^{\pi(2n)},$$

$$(\pi(2n) - \pi(n))\log n < \log ((2n)!/(n!n!)) \leqq \pi(2n) \log (2n).$$

Therefore

$$(\pi(2n) - \pi(n)) \log n < \log\left(\binom{2n}{n}\right) \leqq \log\left(\sum_{j=1}^{2n}\binom{2n}{j}\right) = \log 2^{2n} = 2n \log 2,$$

$$\pi(2n) - \pi(n) < 2(\log 2)n/\log n = \alpha_3 n/\log n; \tag{5}$$

and

$$\begin{aligned}\pi(2n) \log (2n) &\geqq \log\left(\binom{2n}{n}\right) = \log\left(\prod_{j=1}^{n} \frac{n+j}{j}\right)\\ &\geqq \log\left(\prod_{j=1}^{n} 2\right) = \log 2^n = n \log 2,\end{aligned}$$

$$\begin{aligned}\pi(2n) &\geqq n \log 2/\log (2n) = n \log 2/(\log n + \log 2)\\ &\geqq n \log 2/(2 \log n) = \alpha_4\, n/\log n.\end{aligned} \tag{6}$$

From (6) when $\xi \geqq 4$

$$\pi(\xi) \geqq \pi(2[\xi/2]) \geqq \alpha_4[\xi/2]/\log [\xi/2] \geqq \alpha_4\xi/(4 \log \xi) = \alpha_5\xi/\log \xi.$$

When $2 \leqq \xi \leqq 4$ we have

$$\pi(\xi) \geqq 1 = ((\log 2)/4)(4/\log 2) \geqq ((\log 2)/4)(\xi/\log \xi).$$

Hence for all $\xi \geqq 2$,

$$\pi(\xi) > \alpha_1\xi/\log \xi,$$

where $8\alpha_1 \geqq 8 \min (\alpha_5, (\log 2)/4) = 2\alpha_4 = \log 2$. This proves the first inequality of the theorem.

Now, since $\eta = 2+2(\eta/2-1) < 2+2[\eta/2]$ it follows from (5) when $\eta \geqq 8$ that

$$\begin{aligned}\pi(\eta) - \pi(\eta/2) &= \pi(\eta) - \pi([\eta/2]) \leqq 2 + \pi(2[\eta/2]) - \pi([\eta/2])\\ &< 2 + \alpha_3[\eta/2]/\log [\eta/2] < 2 + \alpha_3\eta/(2 \log (\eta/2 - 1))\\ &= 2 + \alpha_3\eta/(2(\log (\eta - 2) - \log 2))\\ &= 2 + (\alpha_3 \log \eta/(2(\log (\eta - 2) - \log 2)))(\eta/\log \eta)\\ &\leqq (\log 8/4)(8/\log 8) + (\alpha_3 \log 8/(2(\log 6 - \log 2)))(\eta/\log \eta)\\ &\leqq ((\log 8/4) + (\alpha_3 \log 8/(2 \log 3)))(\eta/\log \eta).\end{aligned}$$

When $2 \leqq \eta \leqq 8$ we have

$$\pi(\eta) - \pi(\eta/2) \leqq 2 = (\log 8)2/\log 8 \leqq (\log 8)\eta/\log \eta.$$

Therefore for all $\eta \geqq 2$

$$\pi(\eta) - \pi(\eta/2) \leqq \alpha_7 \eta/\log \eta$$

where $\alpha_7 = \max((\log 8/4) + (\alpha_3 \log 8/(2 \log 3)), \log 8) = \log 8$. Then

$$\begin{aligned}\pi(\eta) \log \eta - \pi(\eta/2) \log (\eta/2) &= (\pi(\eta) - \pi(\eta/2)) \log \eta + \pi(\eta/2) \log 2 \\ &< \alpha_7(\log \eta)\eta/\log \eta + \eta(\log 2)/2 = (\alpha_7 + (\log 2)/2)\eta = \alpha_8\eta.\end{aligned}$$

For $\xi \geqq 2$ we have

$$\pi(\xi/2^m) \log (\xi/2^m) - \pi(\xi/2^{m+1}) \log (\xi/2^{m+1}) < \alpha_8\xi/2^m,$$

$$\begin{aligned}\pi(\xi) \log \xi &= \sum_{m=0}^{\infty} (\pi(\xi/2^m) \log (\xi/2^m) - \pi(\xi/2^{m+1}) \log (\xi/2^{m+1})) \\ &< \alpha_8 \sum_{m=0}^{\infty} (\xi/2^m) = 2\alpha_8\xi;\end{aligned}$$

that is,

$$\pi(\xi) < \alpha_2\xi/\log \xi,$$

where $\alpha_2 \leqq 2\alpha_8 = 2(\alpha_7 + (\log 2)/2) = 2(\log 8 + (\log 2)/2) = 7 \log 2$.

THEOREM 3.* *We have*

$$\vartheta(x) < 6cx/5 + 3 \log^2 x + 8 \log x + 5,$$
$$\vartheta(x) > cx - 12cx^{1/2}/5 - 3 \log^2 x/2 - 13 \log x - 15$$

where $c = 2^{1/2} \cdot 3^{1/3} \cdot 5^{1/5}/30^{1/30} = 0.92129 \cdots$.

THEOREM 4. (L, Theorem 264.) *Let t be an integer, $m > 0$, $z \gtrless 0$, and*

$$S = \sum_{h=t+1}^{t+m} e^{2\pi i z h^k}.$$

Then

$$|S|^K < 4^K\left(m^{K-1} + m^{K-k} \sum_{h_1, \cdots, h_{k-1}=1}^{m} \min (m, 1/\{zk!h_1 \cdots h_{k-1}\})\right).$$

THEOREM 5. *The number of solutions of the equation*

$$h_1h_2 \cdots h_{k-1} = v, \qquad 0 \leqq h_i \leqq m, \tag{7}$$

is at most $A_2m^{\epsilon_2}$ where $A_2 = A_1^{k-2}$ and ϵ_2 is given by

$$\begin{aligned}&\epsilon_2(2) = 0; \quad \epsilon_2(k) = (k-1)\epsilon_1 + \epsilon_2(k/2) + \epsilon_2(k/2+1), \quad k \text{ even} \geqq 4; \\ &\epsilon_2(3) = 2\epsilon_1; \quad \epsilon_2(k) = (k-1)\epsilon_1 + 2\epsilon_2((k+1)/2), \quad k \text{ odd} \geqq 5.\end{aligned} \tag{8}$$

* E. Landau, *Handbuch der Lehre von der Verteilung der Primzahlen*, §22.

(i) Let $k=2$. Then $h_1=v$ has at most $1=A_1^{2-2}m^0=A_2m^{\epsilon_2(2)}$ solutions. (ii) Let $k=3$. Then $h_1h_2=v$ has at most $T(v)$ solutions since h_1 must divide v. By Theorem 1,

$$T(v) \leqq A_1v^{\epsilon_1} \leqq A_1^{3-2}m^{2\epsilon_1} = A_2m^{\epsilon_2(3)}$$

since $v=h_1h_2\leqq m^2$. (iii) Let k be even $\geqq 4$ and assume that the theorem is true for all integers $<k$. In equation (7) write

$$h_1h_2 \cdots h_{(k-2)/2} = v_1, \tag{9}$$

$$h_{k/2} \cdots h_{k-1} = v_2. \tag{10}$$

There are at most $A_1^{k/2-2}m^{\epsilon_2(k/2)}$ solutions of (9) and at most $A_1^{k/2+1-2}m^{\epsilon_2(k/2+1)}$ solutions of (10). The equation $v=v_1v_2$ has at most $T(v)$ solutions. Hence the number of solutions of (7) is

$$\begin{aligned}
&\leqq T(v)\cdot A_1^{k/2-2}m^{\epsilon_2(k/2)}\cdot A_1^{k/2-1}m^{\epsilon_2(k/2+1)}\\
&\leqq A_1v^{\epsilon_1}A_1^{k-3}m^{\epsilon_2(k/2)+\epsilon_2(k/2+1)}\\
&\leqq A_1m^{(k-1)\epsilon_1}A_1^{k-3}m^{\epsilon_2(k/2)+\epsilon_2(k/2+1)} = A_2m^{\epsilon_2(k)}.
\end{aligned}$$

(iv) Let k be odd $\geqq 5$ and assume that the theorem is true for all integers $<k$. As in the proof of (iii) with

$$h_1h_2 \cdots h_{(k-1)/2} = v_1,$$
$$h_{(k+1)/2} \cdots h_{k-1} = v_2,$$

the number of solutions of (7) is

$$\begin{aligned}
&\leqq T(v)\cdot A_1^{(k+1)/2-2}m^{\epsilon_2((k+1)/2)}\cdot A_1^{(k+1)/2-2}m^{\epsilon_2((k+1)/2)}\\
&\leqq A_1m^{(k-1)\epsilon_1}\cdot A_1^{k-3}m^{2\epsilon_2((k+1)/2)} = A_2m^{\epsilon_2(k)}.
\end{aligned}$$

COROLLARY. *Let* $d=[\log (k-1)/\log 2]$. *Then*

$$\epsilon_2(k) = ((d+2)(k-1) - 2^{d+1})\epsilon_1. \tag{11}$$

(i) Let $k=2$ and hence $d=0$. Then by Theorem 5

$$\epsilon_2(2) = 0 = ((d+2)(k-1) - 2^{d+1})\epsilon_1.$$

(ii) Let $k>2$ and hence $d>0$. We have $2^d+1\leqq k\leqq 2^{d+1}$. Assume that (11) is true for all integers $<2^d+1$. If k is even and $<2^{d+1}$ we have

$$2^{d-1}+1 \leqq k/2 \leqq 2^d-1,$$
$$2^{d-1}+2 \leqq (k+2)/2 \leqq 2^d.$$

By (8)

$$\begin{aligned}\epsilon_2(k) &= (k-1)\epsilon_1 + ((d+1)(k/2-1) - 2^d)\epsilon_1 + ((d+1)(k/2) - 2^d)\epsilon_1 \\ &= ((d+2)(k-1) - 2^{d+1})\epsilon_1.\end{aligned}$$

If $k=2^{d+1}$ we have $k/2=2^d$, $(k+4)/4=2^{d-1}+1$ and

$$\begin{aligned}\epsilon_2(k) &= (k-1)\epsilon_1 + \epsilon_2(k/2) + \epsilon_2(k/2+1) \\ &= (k-1)\epsilon_1 + \epsilon_2(k/2) + (k/2)\epsilon_1 + 2\epsilon_2((k+4)/4) \\ &= (k-1)\epsilon_1 + ((d+1)(k/2-1) - 2^d)\epsilon_1 + 2^d\epsilon_1 + 2((d+1)(k/4) - 2^d)\epsilon_1 \\ &= ((d+2)(k-1) - 2^{d+1})\epsilon_1.\end{aligned}$$

If k is odd then $2^{d-1}+1 \leqq (k+1)/2 \leqq 2^d$ and we have

$$\begin{aligned}\epsilon_2(k) &= (k-1)\epsilon_1 + 2\epsilon_2((k+1)/2) \\ &= (k-1)\epsilon_1 + 2((d+1)((k+1)/2-1) - 2^d)\epsilon_1 \\ &= ((d+2)(k-1) - 2^{d+1})\epsilon_1.\end{aligned}$$

THEOREM 6. (L, Theorem 266.) *Under the hypotheses of Theorem* 4,

$$|S|^K < C_{15} m^{\epsilon_2}\left(m^{K-1} + m^{K-k} \sum_{v=1}^{k!m^{k-1}} \min(m, 1/\{zv\})\right),$$

where $C_{15} = 4^K A_2$.

In the summation in Theorem 4 write $k!h_1h_2 \cdots h_{k-1} = v$. By Theorem 5 each v appears at most $A_2 m^{\epsilon_2}$ times. Hence

$$\sum_{h_1,\cdots,h_{k-1}=1}^{m} \min(m, 1/\{zk!h_1 \cdots h_{k-1}\}) \leqq A_2 m^{\epsilon_2} \sum_{v=1}^{k!m^{k-1}} \min(m, 1/\{zv\}),$$

$$\begin{aligned}|S|^K &< 4^K\left(m^{K-1} + m^{K-k} A_2 m^{\epsilon_2} \sum_{v=1}^{k!m^{k-1}} \min(m, 1/\{zv\})\right) \\ &\leqq 4^K A_2 m^{\epsilon_2}\left(m^{K-1} + m^{K-k} \sum_{v=1}^{k!m^{k-1}} \min(m, 1/\{zv\})\right).\end{aligned}$$

THEOREM 7. *Let* $x \geqq 1$, $b > 0$. *Then for every* $\epsilon_3 > 0$,

$$b + \log x \leqq A_3 x^{\epsilon_3},$$

where $A_3 = 1/(\epsilon_3 e^{1-b\epsilon_3})$.

Consider the function

$$y = (b + \log x)x^{-\epsilon_3}, \qquad y' = (1 - \epsilon_3(b + \log x))x^{-1-\epsilon_3},$$
$$y'' = -(\epsilon_3 + \epsilon_3(1+\epsilon_3)(b + \log x))x^{-2-\epsilon_3}.$$

We have $y'=0$ when $x=\infty$ or $b+\log x=1/\epsilon_3$. The second value gives a maximum. Hence

$$\max y = 1/(\epsilon_3 e^{1-b\epsilon_3}),$$

$$b + \log x \leqq x^{\epsilon_3}/(\epsilon_3 e^{1-b\epsilon_3}) = A_3 x^{\epsilon_3}.$$

4. **The singular series.** The series

$$\mathfrak{S} = \mathfrak{S}(j, k, s, \infty) = \sum_{q=1}^{\infty} A(q)$$

is called the singular series. In this section we show that $\mathfrak{S}=\prod_p \chi_p$ and from this that $\mathfrak{S}>b_4$. We shall follow closely part 6, chapter 2, of paper L.

THEOREM 8. (L, Theorem 293.) *Let* $n=n_0 p^{\beta k+\sigma}\neq 0$, *where* $\beta\geqq 0$, $0\leqq\sigma<k$, $(n_0, p)=1$. *Let* $t_0=\max(\beta k+\sigma+1, \beta k+\gamma)$. *Then*

$$A(p^t) = 0 \text{ when } t > t_0, \tag{12}$$

$$\chi_p = P^{1-s}N(P, 0)\sum_{\alpha=0}^{\beta-1} p^{\alpha(k-s)} + p^{\beta(k-s)}P^{1-s}N(P, n/p^{\beta k}), \tag{13}$$

where the summation is omitted if $\beta=0$.

Remark. This theorem shows that the terms of the series $\chi_p=\sum_{t=0}^{\infty}A(p^t)$ are all zero after a certain one. Also, since everything on the right side of (13) is positive or zero it follows that $\chi_p\geqq 0$. If $p\nmid 2kn$ we have $\gamma=1$, $\beta=\sigma=0$, $t_0=1$ and thus $\chi_p=1+A(p)$.

THEOREM 9. (L, Theorem 301.) *For* $s\geqq r$ *and every* $n\not\equiv 0 \pmod p$,

$$N(P, n) > 0;$$

for $s\geqq r+1$ *and every* n,

$$N(P, n) > 0.$$

COROLLARY. *For* $s\geqq r+1$ *and every* n,

$$N(P, n) \geqq P^{s-r-1}.$$

In the congruence

$$h_1^k + h_2^k + \cdots + h_s^k \equiv n \pmod P \tag{14}$$

write $n_1=n-h_{r+2}^k-\cdots-h_s$. By Theorem 9 the congruence

$$h_1^k + h_2^k + \cdots + h_{r+1}^k \equiv n \pmod P$$

has at least one primitive solution. Since $h_{r+2}, \cdots, h_s$ may be chosen arbitrarily mod P it follows that (14) has at least P^{s-r-1} primitive solutions.

THEOREM 10. (L, Theorem 302.) *If* $k \geqq 5$, $s \geqq 4k$, *then*

$$\chi_p \geqq P^{-r} = b(p).$$

For $k \geqq 5$,

$$r = \frac{p^\gamma - 1}{p-1}\left(\frac{k}{p^\Theta},\ p-1\right) \leqq \begin{cases} (2^{\Theta+2}-1)\dfrac{k}{2^\Theta} < 4k,\ p = 2, \\ \dfrac{p^{\Theta+1}-1}{p-1}\cdot\dfrac{k}{p^\Theta} < \dfrac{p}{p-1}k < 2k,\ p > 2. \end{cases}$$

Hence $s \geqq 4k$ implies $s \geqq r+1$ and from the Corollary to Theorem 9 we get $N(P, n) \geqq P^{s-r-1}$. By Theorem 8 either $\chi_p = P^{1-s}N(P, n)$ $(\beta = 0)$ or $\chi_p \geqq P^{1-s}N(P, 0)$ $(\beta > 0)$. In both cases it follows that

$$\chi_p \geqq P^{1-s}\cdot P^{s-r-1} = P^{-r} = b(p).$$

THEOREM 11. (L, Theorem 307.) *If* $q = p^t$, $p \nmid k$, $2 \leqq t \leqq k$, *then*

$$S_\rho = p^{t-1}.$$

THEOREM 12. (L, Theorem 311.) *If* $q = p$, *then*

$$|S_\rho| \leqq (k-1)p^{1/2}.$$

THEOREM 13. (L, Theorem 313.) *Let* $T_\rho = q^{a-1}S_\rho$. *Then if* $q = p^t$, $t > k$,

$$T_\rho = T_{\rho p^k}.$$

THEOREM 14. (L, Theorem 314.) *If* $q = p^t$, $t \geqq 1$, *then*

$$|T_\rho| \leqq \begin{cases} 1 \text{ if } p > c_{38}, \\ c_{39} \text{ always}, \end{cases}$$

where $c_{38} = k^{2k/(k-2)}$, $c_{39} = k$.

If $\rho = e^{2\pi ib/p^t}$, $(b, p) = 1$ is a primitive p^tth root of unity, then $\rho^{p^k} = e^{2\pi ib/p^{t-k}}$ is a primitive p^{t-k}th root of unity. Hence in view of Theorem 13 we may assume $1 \leqq t \leqq k$.

(i) If $p \nmid k$, $2 \leqq t \leqq k$, then by Theorem 11

$$|T_\rho| = p^{at-t}|S_\rho| = p^{at-t}p^{t-1} = p^{at-1} \leqq 1.$$

(ii) If $p \nmid k$, $t = 1$, Theorem 12 gives

$$|T_\rho| = p^{a-1}|S_\rho| \leqq p^{a-1}(k-1)p^{1/2} < kp^{a-1/2}.$$

(iii) If $p \mid k$, then

$$|T_\rho| = p^{at-t}|S_\rho| \leqq p^{at-t}\cdot p^t \leqq p^{ak} = p \leqq k.$$

It follows from (i), (ii), and (iii) that $|T_\rho| \leqq 1$ when $p > \max(k^{2k/(k-2)}, k) = c_{38}$ and for all p we have $|T_\rho| \leqq \max(1, k, k) = k = c_{39}$.

THEOREM 15. (L, Theorem 315.) *We have*

$$|T_\rho| < c_{15} \text{ and hence } |S_\rho| < c_{15} q^{1-a},$$

where $\log c_{15} = (k-1)\log k + \alpha_2(k-2)k^{2k/(k-2)}/(2k)$.

When $(q_1, q_2) = 1$ we have $S_{\rho_1\rho_2} = S_{\rho_1}S_{\rho_2}$ (L, Theorem 281). Therefore

$$T_{\rho_1\rho_2} = (q_1q_2)^{a-1}S_{\rho_1\rho_2} = q_1^{a-1}S_{\rho_1}q_2^{a-1}S_{\rho_2} = T_{\rho_1}T_{\rho_2}.$$

For $q > 1$ write $q = p_1^{t_1}p_2^{t_2}\cdots p_m^{t_m} = \prod p_i^{t_i}$. Then

$$|T_\rho| = \prod |T_{\rho_i}| = \prod_1 |T_{\rho_i}| \cdot \prod_2 |T_{\rho_i}| \cdot \prod_3 |T_{\rho_i}|,$$

where Π_1 contains those ρ_i for which $p_i | k$ and $p_i \leqq c_{38}$, Π_2 contains those for which $p_i \nmid k$ and $p_i \leqq c_{38}$, and Π_3 contains those for which $p_i > c_{38}$. By the proof of Theorem 14 we have

$$\prod_1 |T_{\rho_i}| \leqq \prod_{p \leqq k} k \leqq k^{k-1};$$

$$\prod_3 |T_{\rho_i}| \leqq \prod_{p > c_{38}} 1 = 1;$$

$$\prod_2 |T_{\rho_i}| \leqq \prod_{p \leqq c_{38}} kp^{a-1/2},$$

$$\log \prod_2 < \sum_{p \leqq c_{38}} \log k = \pi(c_{38}) \log k < \alpha_2 c_{38} \log k / \log c_{38} \quad \text{(Theorem 2)}$$

$$= \alpha_2(k-2)k^{2k/(k-2)}/(2k).$$

Hence when $q > 1$

$$\log |T_\rho| = \log \prod_1 + \log \prod_2 + \log \prod_3$$
$$< (k-1)\log k + \alpha_2(k-2)k^{2k/(k-2)}/(2k) = \log c_{15}.$$

If $q = 1$ then $|T_\rho| = 1 < c_{15}$.

THEOREM 16. (L, Theorem 316.) *We have*

$$|A(q)| < b_{19} q^{1-sa},$$

where $b_{19} = c_{15}^s$.

We use Theorem 15:

$$|A(q)| \leqq \sum_\rho |q^{-1}S_\rho|^s < q(c_{15}q^{-a})^s = b_{19}q^{1-sa}.$$

THEOREM 17. (L, Theorem 317.) *If* $p \nmid n$ *then*

$$|A(p)| < b_{20} p^{1/2 - s/2},$$

where $b_{20} = k^s$.

THEOREM 18. (L, Theorem 318.) *We have*

$$|A(p)| < b_{21}p^{1-s/2},$$

where $b_{21} = \max(b_{20}, b_{22}) = k^s$.

If $p \nmid n$ Theorem 17 gives

$$|A(p)| < b_{20}p^{1/2-s/2} < b_{20}p^{1-s/2}.$$

If $p \mid n$ it follows from Theorem 12 that

$$|A(p)| < p\,|p^{-1}S_p|^s \leqq p(p^{-1}(k-1)p^{1/2})^s$$
$$< k^s p^{1-s/2} = b_{22}p^{1-s/2}.$$

THEOREM 19. (L, Theorem 319.) *For* $s \geqq 4$,

$$\mathfrak{S} = \sum_{q=1}^{\infty} A(q)$$

converges absolutely and

$$\mathfrak{S} = \prod_p \chi_p.$$

THEOREM 20. (L, Theorem 320.) *If* $p \nmid n$ *then*

$$|\chi_p - 1| < b_{23}p^{1/2-s/2},$$

where $b_{23} = \max(b_{20}, b_{24}) = b_{20} = k^s$.

(i) Let $p \nmid (2k)$. By the remark after Theorem 8 and by Theorem 17,

$$|\chi_p - 1| = |A(p)| < b_{20}p^{1/2-s/2}.$$

(ii) Let $p \mid (2k)$. Then since $p \nmid n$ we have $\beta = 0$ and $t_0 = \max(1, \gamma) \leqq k$. By (12)

$$|\chi_p - 1| = \left|\sum_{t=1}^{t_0} A(p^t)\right| \leqq \sum_{t=1}^{k} p^t \leqq \sum_{t=1}^{k} (2k)^t$$
$$= 2k\,\frac{(2k)^k - 1}{2k - 1} < 2^{k+1}k^k$$
$$= \frac{2^{k+1}k^k}{(2k)^{1/2-s/2}}(2k)^{1/2-s/2} \leqq \frac{2^{k+1}k^k}{(2k)^{1/2-s/2}}p^{1/2-s/2}$$
$$= b_{24}p^{1/2-s/2}.$$

THEOREM 21. (L, Theorem 220.) *Let $D_q(m)$ denote the sum of the mth powers of all primitive qth roots of unity. Then*

$$D_q(m) = \sum_{d|(q,m)} d\cdot\mu(q/d),$$

where

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1, \\ (-1)^j & \text{if } n \text{ is the product of } j \text{ distinct primes}, \\ 0 & \text{otherwise}. \end{cases}$$

COROLLARY. *If p is a prime then*

$$D_{p^t}(m) = \begin{cases} p^t - p^{t-1} & \text{if } p^t|m, \\ -p^{t-1} & \text{if } p^{t-1}|m,\ p^t\nmid m, \\ 0 & \text{if } p^{t-1}\nmid m. \end{cases}$$

Proof: $D_{p^t}(m)=\sum_{d|(p^t,m)} d\cdot\mu(p^t/d)$ and $\mu(p^t/d)=1$ if $d=p^t$, $\mu(p^t/d)=-1$ if $d=p^{t-1}$, and $\mu(p^t/d)=0$ in all other cases.

THEOREM 22. (L, Theorem 321.) *If $p^k\nmid n$ then*

$$|\chi_p - 1| < b_{25}p^{1-s/2},$$

where $b_{25}=\max(b_{23}, b_{26}, b_{27})=1+k^s$.

If $p\nmid n$ Theorem 20 gives

$$|\chi_p - 1| < b_{23}p^{1/2-s/2} < b_{23}p^{1-s/2}.$$

Hence we may assume $p\,|\,n$ and since $p^k\nmid n$ we have $\beta=0$, $1\leqq\sigma\leqq k-1$. (i) Let $p\nmid(2k)$. Then $\gamma=1$, $t_0=\max(\sigma+1, 1)=\sigma+1$ and

$$\chi_p - 1 = A(p) + \sum_{t=2}^{\sigma+1} A(p^t). \tag{15}$$

Also, by Theorem 11, $S_\rho=p^{t-1}$ since $2\leqq t\leqq\sigma+1\leqq k$ and then from the Corollary to Theorem 21 we obtain

$$\begin{aligned} A(p^t) &= \sum_\rho (p^{-t}S_\rho)^s\rho^{-n} = \sum(p^{-1})^s\rho^{-n} \\ &= p^{-s}D_{p^t}(-n) \\ &= p^{-s}\begin{cases} p^t - p^{t-1} & \text{when } p^t|\,n, \text{ that is, when } 2\leqq t\leqq\sigma, \\ -p^{t-1} & \text{when } p^{t-1}|\,n,\ p^t\nmid n, \text{ that is, when } t=\sigma+1, \\ 0 & \text{when } p^{t-1}\nmid n, \text{ which does not occur}. \end{cases} \end{aligned} \tag{16}$$

Therefore from (15), (16), and Theorem 18 we get

$$\begin{aligned}|\chi_p - 1| &= \left| A(p) + \sum_{t=2}^{\sigma+1} A(p^t) \right| \\ &= \left| A(p) + p^{-s}\left(\sum_{t=2}^{\sigma} (-p^{t-1} + p^t) - p^\sigma \right) \right| \\ &= |A(p) - p^{1-s}| < b_{21}p^{1-s/2} + p^{1-s} \\ &< (b_{21} + 1)p^{1-s/2} = b_{26}p^{1-s/2}.\end{aligned}$$

(ii) Let $p \mid (2k)$. Then $t_0 = \max(\sigma+1, \gamma) \leqq k$ and

$$|\chi_p - 1| \leqq \sum_{t=1}^{k} (2k)^t < b_{24}p^{1/2-s/2} < b_{24}p^{1-s/2}.$$

THEOREM 23. (L, Theorem 322.) *We have*

$$\chi_p > 1 - b_{28}p^{1-s/2},$$

where $b_{28} = \max(b_{25}, b_{29}) = 1 + k^s$.

If $p^k \nmid n$, Theorem 22 gives $\chi_p > 1 - b_{25}p^{1-s/2}$. Hence let $p^k \mid n$, that is, $\beta > 0$. (i) Let $p > k$ so that $\gamma = 1, P = p$. Applying Theorem 8 twice we get

$$\chi_p \geqq p^{1-s}N(p, 0) = p^{1-s}N(p, p) = \chi_p(p).$$

Since $p^k \nmid p$ we have

$$\chi_p \geqq \chi_p(p) > 1 - b_{25}p^{1-s/2}$$

by Theorem 22. (ii) Let $p \leqq k$. Then

$$\chi_p \geqq 0 = 1 - k^{s/2-1}k^{1-s/2} \geqq 1 - k^{s/2-1}p^{1-s/2} = 1 - b_{29}p^{1-s/2}.$$

THEOREM 24. (L, Theorem 324.) *If* $p > (1+k^s)^{2/(s-5)} = b_{18}$ *then*

$$\chi_p > 1 - p^{-3/2}.$$

By Theorem 23

$$\chi_p > 1 - b_{28}p^{1-s/2} = 1 - (1 + k^s)p^{5/2-s/2}p^{-3/2} \geqq 1 - p^{-3/2}$$

when $p > (1+k^s)^{2/(s-5)} = b_{18}$.

THEOREM 25. (L, Theorems 325–326.) *We have*

$$\mathfrak{S} > b_4$$

where

$$b_4 = \prod_{p \leqq b_{18}} b(p) \cdot \prod_{p > b_{18}} (1 - p^{-3/2}).$$

We use Theorems 19, 10, and 24:

$$\mathfrak{S} = \prod_p \chi_p = \prod_{p \leqq b_{18}} \chi_p \cdot \prod_{p > b_{18}} \chi_p > \prod_{p \leqq b_{18}} b(p) \cdot \prod_{p > b_{18}} (1 - p^{-3/2}) = b_4.$$

5. The main lemma for the third Hardy-Littlewood theorem. In this section we follow the methods of paper G. We shall assume that

$$\eta_1 \geqq 17 \text{ when } k = 6, \qquad \eta_1 \geqq D + 2^{k-3} \text{ when } k \geqq 7, \tag{17}$$

so that from (2)

$$(k-2)2^{k-2} + k + 2 \leqq s < 4(k-2)2^{k-2} + 4k. \tag{18}$$

As we shall see later our final choice of η_1 satisfies the conditions (17). It follows from the second part of (18) that

$$\frac{K-1}{K} - \frac{2s-2K}{2s-K} = \frac{K^2+K-2s}{K(2s-K)} \geqq \frac{1}{K^2},$$

$$\frac{2s-2k}{2s-k} - \frac{2s-2K}{2s-K} = \frac{2s(K-k)}{(2s-k)(2s-K)} \geqq \frac{K-k}{2s-K} \geqq \frac{1}{K^2}.$$

We may then choose $\theta = \theta(k, s, \epsilon)$ so that

$$\frac{2s-2k}{2s-k} - \theta \geqq \frac{1}{2K^2}, \tag{19}$$

$$\frac{K-1}{K} - \theta \geqq \frac{1}{2K^2}, \tag{20}$$

$$\theta - \frac{2s-2K}{2s-K} \geqq \frac{1}{2K^2}. \tag{21}$$

The purpose of this section is to find an approximation for

$$\int_C \left| f^s(x) - \sum_{q,\rho} \psi_\rho^s(x) \right|^2 |dx|$$

taken around the unit circle $|x| = 1$. We divide the circumference into subarcs in the following manner. On the circle we take the points $\rho = e^{2\pi i b/q}$ which correspond to the Farey fractions * with denominators $q \leqq n^{1-\sigma}$. The mediants between two neighboring Farey fractions form the end points of our subarcs. It is known that if x is any point of an arc which contains the point ρ then

$$x = \rho e^{2\pi i \nu} = e^{2\pi i(b/q+\nu)},$$

* For the definition and properties of Farey fractions see L, pp. 98–100.

where

$$-y_1 \leqq y \leqq y_2,$$

$$1/(2qn^{1-a}) \leqq y_1 < 1/(qn^{1-a}), \quad 1/(2qn^{1-a}) \leqq y_2 < 1/(qn^{1-a}).$$

The arcs for which $n^a < q \leqq n^{1-a}$ are called *minor arcs* and are denoted by m; those for which $1 \leqq q \leqq n^a$ are called *major arcs* and denoted by M. Each major arc is further divided into two sub-arcs denoted by M_1 when $|y| \leqq 1/(2q^\theta n^{1-\theta a})$, and by M_2 when $|y| > 1/(2q^\theta n^{1-\theta a})$.

THEOREM 26. (G, Theorem 1; L, Theorem 140.) *Let* $|y| \leqq \frac{1}{2}$, $a_0 \geqq a_1 \geqq \cdots \geqq 0$. *Then*

$$\left| \sum_{j=0}^{N} a_j e^{2\pi i y j} \right| \leqq a_0/\sin \pi |y|.$$

THEOREM 27. (G, Theorem 2; L, Theorem 223.)

$$\int_{-1/2}^{1/2} \left| \sum_{j=0}^{N} a_j e^{2\pi i y j} \right|^2 dy = \sum_{j=0}^{N} |a_j|^2.$$

THEOREM 28. (G, Theorem 3; L, Theorem 262.)

$$\sum_{j=0}^{N} r_{k,2}^2(j) < G_3 N^{2a+\epsilon_1},$$

where $G_3 = 4(k-1)A_1$.

If k is even $r_{k,2}(j)$ is at most equal to the number of solutions of $h_1^2 + h_2^2 = j$ and this is

$$4 \sum_{\substack{u \mid j \\ u \text{ odd}}} (-1)^{(u-1)/2} \leqq 4T(j) \leqq 4A_1 j^{\epsilon_1}$$

by Theorem 1. If k is odd,

$$j = h_1^k + h_2^k = (h_1 + h_2)(h_1^{k-1} + \cdots + h_2^{k-1})$$

implies that $h_1 + h_2$ divides j. For each positive divisor d of j the two equations

$$h_1 + h_2 = d, \quad h_1^k + h_2^k = j$$

have at most $k-1$ solutions in common since the elimination of h_2 between them yields an equation of degree $k-1$. Therefore when k is odd

$$r_{k,2}(j) \leqq (k-1)T(j) \leqq (k-1)A_1 j^{\epsilon_1}.$$

Thus for all $k \geqq 6$

$$r_{k,2}(j) \leqq \max(4, k-1)A_1 j^{\epsilon_1} = (k-1)A_1 j^{\epsilon_1}.$$

Next, $\sum_{j=0}^{N} r_{k,2}(j)$ is the number of solutions of $h_1^k+h_2^k \leqq N$, $h_1 \geqq 0$, $h_2 \geqq 0$. Since $h_1 \leqq N^a$, $h_2 \leqq N^a$, this is at most $(1+N^a)(1+N^a) \leqq 4N^{2a}$. Finally

$$\sum_{j=0}^{N} r_{k,2}^2(j) \leqq \max_{0 \leqq j \leqq N} r_{k,2}(j) \cdot \sum_{j=0}^{N} r_{k,2}(j) \leqq (k-1)A_1 N^{\epsilon_1} \cdot 4N^{2a} = G_2 N^{2a+\epsilon_1}.$$

THEOREM 29. (G, Theorem 4; L, Theorem 277.) *For $\beta > 0$ and j a positive integer,*

$$\left| \frac{\Gamma(\beta+1+j)}{j!} - j^\beta \right| < \gamma(\beta) j^{\beta-1},$$

where $\gamma(\beta) = 4\beta(2^\beta+1)e^\beta\Gamma(\beta+1)$ is independent of j.

It is known that*

$$\lim_{j=\infty} \Gamma(\beta+1+j)/(j!j^\beta) = 1.$$

Let $\Phi(j) = \Gamma(\beta+1+j)/(j!j^\beta)$. Then

$$\Phi(j) = 1 + \sum_{v=j+1}^{\infty} (\Phi(v-1) - \Phi(v)),$$

$$\Phi(v-1) - \Phi(v) = \Phi(v-1)(1 - (1+\beta/v)(1-1/v)^\beta). \tag{22}$$

Also,

$$\begin{aligned}
&\left| 1 - \left(1+\frac{\beta}{v}\right)\left(1-\frac{1}{v}\right)^\beta \right| \\
&= \left| 1 - \left(1+\frac{\beta}{v}\right)\left(1 - \binom{\beta}{1}\frac{1}{v} + \binom{\beta}{2}\frac{1}{v^2} - \cdots\right) \right| \\
&= \left| 1 - \left(1 - \left(\beta\binom{\beta}{1} - \binom{\beta}{2}\right)\frac{1}{v^2} \right.\right. \\
&\qquad \left.\left. + \left(\beta\binom{\beta}{2} - \binom{\beta}{3}\right)\frac{1}{v^3} - \cdots\right) \right| \\
&= \left| \binom{\beta+1}{2}\frac{1}{v^2} - \binom{\beta+1}{3}\frac{2}{v^3} + \cdots \right|.
\end{aligned}$$

If β is an integer this expression is

$$\leqq \sum_{t=1}^{\beta} \binom{\beta+1}{t+1}\frac{t}{v^{t+1}} < \frac{\beta}{v^2} \sum_{t=1}^{\beta} \binom{\beta+1}{t+1} = \frac{\beta \cdot 2^{\beta+1}}{v^2}.$$

Next, suppose that β is not an integer. Since

* See, for example, Whittaker and Watson, *Modern Analysis*, Chapter XII.

$$\left|\binom{\beta+1}{t+2}(t+1)\right| \leqq \left|\binom{\beta+1}{t+1}t\right|$$

when $t \geqq [\beta]+1$, and since

$$1 \geqq \binom{\beta+1}{[\beta]+2} = \frac{(\beta+1)\beta\cdots(\beta-[\beta])}{([\beta]+2)!} > 0,$$

we have

$$\begin{aligned}
&\left|\binom{\beta+1}{2}\frac{1}{v^2} - \binom{\beta+1}{3}\frac{2}{v^3} + \cdots\right| \\
&\qquad \leqq \sum_{t=1}^{[\beta]}\binom{\beta+1}{t+1}\frac{t}{v^{t+1}} + \sum_{t=[\beta]+1}^{\infty}\binom{\beta+1}{[\beta]+2}\frac{[\beta]+1}{v^{t+1}} \\
&\qquad \leqq \frac{2\beta}{v^2}\sum_{t=1}^{[\beta]}\binom{[\beta]+1}{t+1} + \frac{[\beta]+1}{v^2}\sum_{t=[\beta]}^{\infty}\frac{1}{v^t} \\
&\qquad < \left(2\beta\cdot 2^{\beta+1} + 2\beta\frac{v}{v-1}\right)\frac{1}{v^2} \leqq (2\beta\cdot 2^{\beta+1} + 4\beta)\frac{1}{v^2}.
\end{aligned}$$

Hence

$$(23) \qquad |1-(1+\beta/v)(1-1/v)^\beta| < \max(\beta\cdot 2^{\beta+1}, 4\beta(2^\beta+1))v^{-2} = \gamma_1(\beta)v^{-2}.$$

Also,

$$\begin{aligned}
\log\Phi(v-1) &= \log\Gamma(\beta+v) - \log((v-1)!) - \beta\log(v-1) \\
&= \log\Big((\beta+v-1)\cdots(\beta+1)\Gamma(\beta+1)\Big) - \sum_{n=1}^{v-1}\log n - \beta\log(v-1) \\
&= \sum_{n=1}^{v-1}(\log(\beta+n) - \log n) + \log\Gamma(\beta+1) - \beta\log(v-1) \\
&\leqq \sum_{n=1}^{v-1}(\beta n^{-1}) + \log\Gamma(\beta+1) - \beta\log(v-1) \\
&\leqq \beta + \beta\int_1^{v-1} u^{-1}du + \log\Gamma(\beta+1) - \beta\log(v-1) \\
&= \beta + \beta\log(v-1) + \log\Gamma(\beta+1) - \beta\log(v-1) \\
&= \beta + \log\Gamma(\beta+1).
\end{aligned}$$

Therefore

$$(24) \qquad \Phi(v-1) \leqq e^\beta\Gamma(\beta+1) = \gamma_2(\beta).$$

From (22), (23), and (24) we get

$$|\Phi(v-1)-\Phi(v)| \leqq \gamma_1(\beta)\cdot\gamma_2(\beta)v^{-2} = \gamma(\beta)v^{-2};$$

$$|\Gamma(\beta+1+j)/(j!j^\beta)-1| = |\Phi(j)-1| = \left|\sum_{v=j+1}^{\infty}(\Phi(v-1)-\Phi(v))\right|$$

$$< \gamma(\beta)\sum_{v=j+1}^{\infty} v^{-2} \leqq \gamma(\beta)\left((j+1)^{-2}+\int_{j+1}^{\infty} u^{-2}du\right) < \gamma(\beta)j^{-1}.$$

COROLLARY. *For $\beta>0$ and j an integer $\geqq 1$,*

$$\Gamma(\beta+1+j)/j! > \gamma_3 j^\beta,$$

where $\gamma_3=(1+\gamma(1))^{-1}=(1+12e)^{-1}$.

Since $\Gamma(\beta+2+j)=(\beta+1+j)\ \Gamma(\beta+1+j)$ it follows that $\Gamma(\beta+2+j)/j! >\gamma_3 j^{\beta+1}$ if $\Gamma(\beta+1+j)/j!>\gamma_3 j^\beta$. Hence we may assume $0<\beta\leqq 1$.

(i) Let $j\geqq 1+\gamma(1)$. Then

$$\begin{aligned}\Gamma(\beta+1+j)/j! &> j^\beta-\gamma(\beta)j^{\beta-1} > j^\beta-\gamma(1)j^{\beta-1}\\ &= j^\beta(1+\gamma(1))^{-1}+\gamma(1)j^\beta(1+\gamma(1))^{-1}-\gamma(1)j^{\beta-1}\\ &\geqq j^\beta(1+\gamma(1))^{-1} = \gamma_3 j^\beta.\end{aligned}$$

(ii) Let $1\leqq j<1+\gamma(1)$. Then

$$\begin{aligned}\Gamma(\beta+1+j)/j! &= (\beta+j)\cdots(\beta+2)\Gamma(\beta+2)/j! > j!\,\Gamma(2)/j! = 1\\ &= (1+\gamma(1))(1+\gamma(1))^{-1} > j(1+\gamma(1))^{-1}\\ &\geqq j^\beta(1+\gamma(1))^{-1} = \gamma_3 j^\beta.\end{aligned}$$

THEOREM 30. (G, Theorem 5; L, Theorem 267.) *Let t be an integer, $m>0$, and*

$$S = \sum_{h=t+1}^{t+m}\rho^{h^k}.$$

Then for every $\epsilon_2>0$, $\epsilon_3>0$,

$$|S|^K < G_4 m^{\epsilon_2}q^{\epsilon_3}(m^{K-1}+m^K q^{-1}+m^{K-k}q),$$

where $G_4=C_{18}=8k!\ C_{15}A_3$.

By Theorem 6 with $z=b/q$ we have

$$|S|^K < C_{15}m^{\epsilon_2}\left(m^{K-1}+m^{K-k}\sum_{v=1}^{k!m^{k-1}}\min(m,1/\{bv/q\})\right). \tag{25}$$

Divide the summation into partial sums according to the j in

$$bv \equiv j \pmod q, \qquad 0 \leqq j \leqq q-1.$$

Since $\{b_1/q\} = \{b_2/q\}$ when $b_1 \equiv b_2 \pmod q$ we have

$$\min(m, 1/\{bv/q\}) = \min(m, 1/\{j/q\}).$$

Each partial sum has at most $k!\, m^{k-1}q^{-1}+1$ terms and thus

$$\sum_{v=1}^{k!m^{k-1}} \min(m, 1/\{bv/q\}) \leqq (k!\, m^{k-1}q^{-1}+1)\sum_{j=0}^{q-1}\min(m, 1/\{j/q\}).$$

Also

$$\sum_{j=0}^{q-1}\min(m, 1/\{j/q\}) \leqq m + \sum_{j=1}^{q-1}(1/\{j/q\}) \leqq m + 2\sum_{1\leqq j\leqq q/2}(1/\{j/q\})$$

$$= m + 2\sum_{1\leqq j\leqq q/2}(1/(j/q)) < m + 2\sum_{j=1}^{q} qj^{-1} \leqq m + 2q(1+\log q)$$

$$\leqq m + 2qA_3q^{\epsilon_3} \qquad \text{(Theorem 7)}.$$

Therefore

$$\sum_{v=1}^{k!m^{k-1}} \min(m, 1/\{bv/q\}) < k!(m^{k-1}q^{-1}+1)(m+2A_3q^{1+\epsilon_3})$$

$$\leqq 2k!A_3q^{\epsilon_3}(m^{k-1}q^{-1}+1)(m+q)$$

$$< 4k!A_3q^{\epsilon_3}(m^{k-1}q^{-1}+m^{k-1}+q).$$

Combining this result with (25) we obtain

$$|S|^K < C_{15}m^{\epsilon_2}(m^{K-1}+m^{K-k}\cdot 4k!A_3q^{\epsilon_3}(m^{k-1}q^{-1}+m^{k-1}+q))$$

$$\leqq 4k!C_{15}A_3m^{\epsilon_2}q^{\epsilon_3}(m^{K-1}+m^Kq^{-1}+m^{K-1}+m^{K-k}q)$$

$$< 8k!C_{15}A_3m^{\epsilon_2}q^{\epsilon_3}(m^{K-1}+m^Kq^{-1}+m^{K-k}q).$$

We choose $\epsilon_3 = \epsilon_1/(10k)$ so that $\epsilon_2 + k\epsilon_3 = D\epsilon_1$.

THEOREM 31. (G, Theorem 6.) *On the entire circle* $|x|=1$

$$|\psi_\rho(x)| < G_5n^aq^{-a},$$

where $G_5 = c_{15}(\Gamma(1+a)+2a\gamma(a)\ (1-a^2)^{-1}+(2a+1)\ (a+1)^{-1})$.

We use Theorems 15 and 29:

$$|\psi_\rho(x)| = |q^{-1}S_\rho|\cdot\left|\Gamma(1+a)+a\sum_{j=1}^{n}(\Gamma(1+a+j)/j!)(a+j)^{-1}(x/\rho)^j\right|$$

$$< c_{15}q^{-a}\left(\Gamma(1+a)+a\sum_{j=1}^{n}(j^a+\gamma(a)j^{a-1})(a+j)^{-1}\right)$$

$$\leqq c_{15}q^{-a}\left(\Gamma(1+a)+a\left((1+\gamma(a))(a+1)^{-1}\right.\right.$$
$$\left.\left.+\int_1^n (j^{a-1}+\gamma(a)j^{a-2})dj\right)\right)$$
$$< c_{15}q^{-a}\left(\Gamma(1+a)+a((1+\gamma(a))(a+1)^{-1}+n^a a^{-1}+\gamma(a)(1-a)^{-1})\right)$$
$$\leqq c_{15}q^{-a}\left(\Gamma(1+a)+a((1+\gamma(a))(a+1)^{-1}+k+\gamma(a)(1-a^{-1}))\right)n^a$$
$$= G_5 n^a q^{-a}.$$

THEOREM 32. (G, Theorem 7.) *We have*

$$|\phi_\rho(x)| < G_7 n^{a-1}q^{-a}|y|^{-1}; \qquad |\Psi_\rho(x)| < G_8 q^{-a}|y|^{-a};$$

where $G_7=\frac{1}{2}ac_{15}(1+\gamma(a))$, $G_8=\Gamma(1+a)c_{15}$.

(i) By Theorems 15, 26, and 29,

$$|\phi_\rho(x)| < c_{15}q^{-a}\Gamma(1+a)\left|\sum_{j=n+1}^{\infty}(a(a+1)\cdots(a+j-1)/j!)(x/\rho)^j\right|$$
$$\leqq c_{15}q^{-a}\Gamma(1+a)\cdot a(a+1)\cdots(a+n)/((n+1)!\sin\pi|y|)$$
$$= c_{15}q^{-a}a(n+1)^{-1}(\Gamma(1+a+n)/n!)(\sin\pi|y|)^{-1}$$
$$< c_{15}q^{-a}n^{-1}a(n^a+\gamma(a)n^{a-1})(2|y|)^{-1}$$
$$\leqq \tfrac{1}{2}ac_{15}(1+\gamma(a))n^{a-1}q^{-a}|y|^{-1}.$$

(ii) From Theorem 15 it follows that

$$|\Psi_\rho(x)| = |\Gamma(1+a)q^{-1}S_\rho(1-x/\rho)^{-a}|$$
$$= |\Gamma(1+a)|\cdot|q^{-1}S_\rho|\cdot|1-e^{2\pi iy}|^{-a}$$
$$= |\Gamma(1+a)|\cdot|q^{-1}S_\rho|\cdot|e^{\pi iy}-e^{-\pi iy}|^{-a}$$
$$< \Gamma(1+a)c_{15}q^{-a}|2\sin\pi y|^{-a}$$
$$\leqq \Gamma(1+a)c_{15}q^{-a}(4|y|)^{-a} < G_8 q^{-a}|y|^{-a}.$$

THEOREM 33. (G, Theorem 8.) *We have*

$$|\psi_\rho(x)| < G_9 q^{-a}\min(n^a, |y|^{-a}),$$

where $G_9=\max(G_5, G_7+G_8)$.

(i) Let $|y|\leqq 1/n$. By Theorem 31

$$|\psi_\rho(x)| < G_5 n^a q^{-a} = G_5 q^{-a}\min(n^a, |y|^{-a}).$$

(ii) Let $|y| \geqq 1/n$. From Theorem 32 we get

$$\begin{aligned}|\psi_\rho(x)| &\leqq |\phi_\rho(x)| + |\Psi_\rho(x)| < G_7 n^{a-1} q^{-a} |y|^{-1} + G_8 q^{-a} |y|^{-a} \\ &= G_7 n^{a-1} |y|^{a-1} q^{-a} |y|^{-a} + G_8 q^{-a} |y|^{-a} \\ &\leqq G_7 q^{-a} |y|^{-a} + G_8 q^{-a} |y|^{-a} \\ &= (G_7 + G_8) q^{-a} \min (n^a, |y|^{-a}).\end{aligned}$$

THEOREM 34. (G, Theorem 9.) *We have*

$$\sum_{M_1} \int_{C-M_1} |\psi_\rho^s(x)|^2 |dx| < G_{10} n^{2sa-1-\lambda-2a},$$

where

$$G_{10} = \frac{2^{2sa}}{2sa-1} \frac{2sa-1-(2sa-1)\theta}{2sa-2-(2sa-1)\theta} G_9^{2s}.$$

The integral is taken around the entire circle with the exception of the arc M_1 itself and the summation extends over all M_1-arcs.

From Theorem 33 we get

$$\begin{aligned}\int_{C-M_1} |\psi_\rho^s(x)|^2 |dx| &< 2G_9^{2s} q^{-2sa} \int_{1/(2q^\theta n^{1-\theta a})}^{\infty} y^{-2sa} dy \\ &= 2^{2sa} G_9^{2s} (2sa-1)^{-1} n^{2sa-1-(2sa-1)\theta a} q^{-2sa+(2sa-1)\theta}.\end{aligned}$$

The exponent of q is < -2 since $\theta < (2sa-2)/(2sa-1)$ by (19). Also, for each q there are at most q arcs. Hence

$$\begin{aligned}&\sum_{M_1} \int_{C-M_1} |\psi_\rho^s(x)|^2 |dx| \\ &< 2^{2sa} G_9^{2s} (2sa-1)^{-1} \cdot n^{2sa-1-(2sa-1)\theta a} \sum_{1 \leqq q \leqq n^a} q^{1-2sa+(2sa-1)\theta} \\ &\leqq 2^{2sa} G_9^{2s} (2sa-1)^{-1} \cdot n^{2sa-1-(2sa-1)\theta a} \left(1 + \int_1^{n^a} q^{1-2sa+(2sa-1)\theta} dq\right) \\ &< 2^{2sa} G_9^{2s} (2sa-1)^{-1} \cdot n^{2sa-1-(2sa-1)\theta a} (1 + (2sa-2-(2sa-1)\theta)^{-1}) \\ &= G_{10} n^{2sa-1-(2sa-1)\theta a}.\end{aligned}$$

In the exponent of n we have

$$\begin{aligned}(2sa-1)\theta a &> ((2s-k)/k^2)((2s-2K)/(2s-K)) && \text{(by (21))} \\ &\geqq ((2s-k)/k^2)(((k-2)K-2K)/((k-2)K-K)) && \text{(by (2))} \\ &= ((2s-k)/k^2)((k-4)/(k-3))\end{aligned}$$

$$> \frac{((k-2)K + 2k + (1+(1-a)^{\epsilon_2-2})kK\epsilon_1)(k-4)}{k^2(k-3)}$$

$$> 2a + 2A + (1-a)^{\epsilon_2-2}\epsilon_1 = 2a + \lambda.$$

Therefore $G_{10}n^{2sa-1-(2sa-1)\theta a} < G_{10}n^{2sa-1-\lambda-2a}$.

THEOREM 35. (G, Theorem 10.) *On m we have*

$$|f(x)| < G_{13}n^{a-aA+aAD\epsilon_1},$$

where $G_{13} = (2\pi+1)2^{1+A}G_4^A$.

Let $\tau(j) = \sum_{h=0}^{j^a}\rho^{h^k}$, $j \geqq 0$, $\tau(-1) = 0$. Then

$$(26) \qquad f(x) = \sum_{h=0}^{n^a}\rho^{h^k}(x/\rho)^{h^k} = \sum_{j=0}^{n}(\tau(j) - \tau(j-1))(x/\rho)^j$$

$$= \sum_{j=0}^{n-1}\tau(j)((x/\rho)^j - (x/\rho)^{j+1}) + \tau(n)(x/\rho)^n$$

$$= (1 - x/\rho)\sum_{j=0}^{n-1}\tau(j)(x/\rho)^j + \tau(n)(x/\rho)^n.$$

By Theorem 30 with $m = [j^a]+1$, $t = -1$,

$$|\tau(j)|^K = \left|\sum_{h=0}^{j^a}\rho^{h^k}\right|^K < G_4([j^a]+1)^{\epsilon_2}q^{\epsilon_3}\Big(([j^a]+1)^{K-1}$$

$$+ ([j^a]+1)^K q^{-1} + ([j^a]+1)^{K-k}q\Big)$$

$$\leqq G_4(2j^a)^{\epsilon_2}q^{\epsilon_3}((2j^a)^{K-1} + (2j^a)^K q^{-1} + (2j^a)^{K-k}q)$$

$$\leqq G_4(2n^a)^{\epsilon_2}q^{\epsilon_3}((2n^a)^{K-1} + (2n^a)^K q^{-1} + (2n^a)^{K-k}q)$$

$$\leqq G_4(2n^a)^{\epsilon_2}n^{(1-a)\epsilon_3}(2^{K-1} + 2^K + 2^{K-k})n^{aK-a} \qquad (n^a < q \leqq n^{1-a} \text{ on } m)$$

$$< 2^{K+1}G_4(n^a)^{K-1+D\epsilon_1} \qquad (\epsilon_2 + (k-1)\epsilon_3 < \epsilon_2 + k\epsilon_3 = D\epsilon_1);$$

$$(27) \qquad |\tau(j)| < 2^{1+A}G_4^A n^{a-aA+aAD\epsilon_1}.$$

Also,

$$|1 - x/\rho| = |1 - e^{2\pi i y}| = |e^{\pi i y} - e^{-\pi i y}| = 2|\sin \pi y| \leqq 2\pi|y| < 2\pi/(qn^{1-a}).$$

Therefore from (26) and (27)

$$|f(x)| \leqq |1 - x/\rho|\sum_{j=0}^{n-1}|\tau(j)| + |\tau(n)|$$

$$< (2\pi n/(qn^{1-a}) + 1)(2^{1+A}G_4^A)n^{a-aA+aAD\epsilon_1}$$

$$\leqq (2\pi + 1)2^{1+A}G_4^A n^{a-aA+aAD\epsilon_1}.$$

THEOREM 36. (G, Theorem 11.) *We have*

$$\sum_m \int_m |f^s(x)|^2 |dx| < G_{15} n^{2sa-1-\lambda},$$

where $G_{15} = 2^{2a+\epsilon_1} G_3 G_{13}^{2s-4}$.

By Theorem 35,

$$\sum_m \int_m |f^s(x)|^2 |dx| \leqq \max |f(x)|^{2s-4} \cdot \sum_m \int_m |f^2(x)|^2 |dx|$$

$$< G_{13}^{2s-4} n^{(2s-4)a-(2s-4)(1-D\epsilon_1)aA} \int_{-1/2}^{1/2} |f^2(e^{2\pi i y})|^2 dy. \tag{28}$$

But $f^2(x) = \sum_{j=0}^{2n} R(j) x^j$, where

$$R(j) = r_{k,2}(j), \qquad 0 \leqq j \leqq n;$$

$$0 \leqq R(j) \leqq r_{k,2}(j), \qquad n < j \leqq 2n;$$

and by Theorems 27 and 28,

$$\int_{-1/2}^{1/2} |f^2(e^{2\pi i y})|^2 dy = \sum_{j=0}^{2n} R^2(j) \leqq \sum_{j=0}^{2n} r_{k,2}^2(j) < G_3 (2n)^{2a+\epsilon_1}.$$

Combining this with (28) we get

$$\sum_m \int_m |f^s(x)|^2 |dx| < G_{13}^{2s-4} \cdot G_3 \cdot 2^{2a+\epsilon_1} n^{(2s-4)a-(2s-4)(1-D\epsilon_1)aA+2a+\epsilon_1}.$$

The exponent of n equals

$$\begin{aligned} 2sa - 1 - ((2s-4)(1-D\epsilon_1)aA - 1 - 2a) + \epsilon_1 \\ &= 2sa - 1 - \Big((2s-4)(1-D\epsilon_1) - (k-2)K - 2k \\ &\quad - (1 + (1-a)^{s_2-2}) kK\epsilon_1\Big) aA - 2A - (1-a)^{s_2-2}\epsilon_1 \\ &\leqq 2sa - 1 - 2A - (1-a)^{s_2-2}\epsilon_1 \qquad \text{(by (2))} \\ &= 2sa - 1 - \lambda \end{aligned}$$

and the theorem follows.

THEOREM 37. (G, Theorem 12.) *We have*

$$|f(x) - \psi_\rho(x)| < G_{18} q^{1-A+AD\epsilon_1} \cdot \max(n|y|, 1),$$

where $G_{18} = (2\pi + 1)\ (2(3G_4)^A + \gamma(a))$.

As before

$$f(x) = (1 - x/\rho) \sum_{j=0}^{n-1} \tau(j)(x/\rho)^j + \tau(n)(x/\rho)^n; \tag{29}$$

$$\psi_\rho(x) = q^{-1}S_\rho\Big((1 - x/\rho) \sum_{j=0}^{n-1}(\Gamma(1 + a + j)/j!)(x/\rho)^j + (\Gamma(1 + a + n)/n!)(x/\rho)^n\Big)$$

$$= (1 - x/\rho) \sum_{j=0}^{n-1} v(j)(x/\rho)^j + v(n)(x/\rho)^n. \tag{30}$$

Each $\tau(j)$ has $[j^a]+1$ terms and so may be written as $[j^a/q]$ partial sums each equal to S_ρ and $[j^a]+1-[j^a/q]q \leqq q$ further terms. Then by Theorem 30 with $t=0$, $m \leqq q$,

$$\begin{aligned} |\tau(j) - [j^a/q]S_\rho| &\leqq G_4^A q^{(\epsilon_2+\epsilon_3)A}(q^{K-1} + q^{K-1} + q^{K-k}q)^A \\ &< (3G_4)^A q^{AD\epsilon_1+1-A} \qquad (\epsilon_2 + \epsilon_3 < \epsilon_2 + k\epsilon_3 = D\epsilon_1); \\ |\tau(j) - (j^a/q)S_\rho| &\leqq |\tau(j) - [j^a/q]S_\rho| + |S_\rho| \\ &< 2(3G_4)^A q^{1-A+AD\epsilon_1}. \end{aligned} \tag{31}$$

Since by Theorem 29

$$|\Gamma(1 + a + j)/j! - j^a| < \gamma(a)j^{a-1} \leqq \gamma(a),$$

we have

$$|v(j) - j^a q^{-1}S_\rho| = |q^{-1}S_\rho|\,|\Gamma(1 + a + j)/j! - j^a| < \gamma(a). \tag{32}$$

From (31) and (32) it follows that

$$\begin{aligned} |\tau(j) - v(j)| &= |\tau(j) - j^a q^{-1}S_\rho + j^a q^{-1}S_\rho - v(j)| \\ &\leqq |\tau(j) - j^a q^{-1}S_\rho| + |v(j) - j^a q^{-1}S_\rho| \\ &< 2(3G_4)^A q^{1-A+AD\epsilon_1} + \gamma(a) \\ &\leqq (2(3G_4)^A + \gamma(a))q^{1-A+AD\epsilon_1}. \end{aligned}$$

Then from (29) and (30)

$$\begin{aligned} |f(x) - \psi_\rho(x)| &= \Big|(1 - x/\rho) \sum_{j=0}^{n-1}(\tau(j) - v(j))(x/\rho)^j + (\tau(n) - v(n))(x/\rho)^n\Big| \\ &\leqq |1 - x/\rho| \sum_{j=0}^{n-1}|\tau(j) - v(j)| + |\tau(n) - v(n)| \\ &< (2\pi n|y| + 1)(2(3G_4)^A + \gamma(a))q^{1-A+AD\epsilon_1} \\ &< (2\pi + 1)(2(3G_4)^A + \gamma(a))q^{1-A+AD\epsilon_1}\cdot\max(n|y|, 1). \end{aligned}$$

THEOREM 38. (G, Theorem 13.) *We have*

$$\sum_{M_1} \int_{M_1} |f^s(x) - \psi_\rho^s(x)|^2 |dx| < (G_{24} + G_{25}) n^{2sa-1-\lambda},$$

where

$$G_{24} = 2^{2s+1} G_{18}^{2s} \frac{(2s+2)(2s+3-(2s+1)\theta - 2s(1-D\epsilon_1)A)}{(2s+1)(2s+2-(2s+1)\theta - 2s(1-D\epsilon_1)A)},$$

$$G_{25} = 2^{2s+1} G_{18}^2 G_9^{2s} \frac{(2sa-2a-2)(2sa-3+2(1-D\epsilon_1)A)}{(2sa-2a-3)(2sa-4+2(1-D\epsilon_1)A)}.$$

Write $\Phi_\rho(x) = f(x) - \psi_\rho(x)$. Then

$$\begin{aligned} |f^s(x) - \psi_\rho^s(x)|^2 &= |(\Phi_\rho(x) + \psi_\rho(x))^s - \psi_\rho^s(x)|^2 \\ &= |\Phi_\rho(x)|^2 |\Phi_\rho^{s-1}(x) + \cdots + s\psi_\rho^{s-1}(x)|^2 \\ &< |\Phi_\rho(x)|^2 \cdot 2^{2s}(|\Phi_\rho(x)|^{2s-2} + |\psi_\rho(x)|^{2s-2}) \\ &= 2^{2s}(|\Phi_\rho(x)|^{2s} + |\Phi_\rho(x)|^2 |\psi_\rho(x)|^{2s-2}), \end{aligned}$$

where by Theorem 37

$$|\Phi_\rho(x)| < G_{18} q^{1-A+AD\epsilon_1} \max(n|y|, 1).$$

Hence from Theorems 37 and 33 we get

$$\begin{aligned} &\int_{M_1} |f^s(x) - \psi_\rho^s(x)|^2 |dx| \\ &\quad < 2^{2s} \left(\int_{M_1} |\Phi_\rho(x)|^{2s} |dx| + \int_{M_1} |\Phi_\rho(x)|^2 |\psi_\rho(x)|^{2s-2} |dx| \right) \\ &< 2^{2s+1} G_{18}^{2s} q^{2s-2s(1-D\epsilon_1)A} \left(\int_0^{1/n} dy + n^{2s} \int_{1/n}^{1/(q^\theta n^{1-\theta a})} y^{2s} dy \right) \\ &\quad + 2^{2s+1} G_{18}^2 G_9^{2s-2} q^{2-2(1-D\epsilon_1)A-(2s-2)a} \left(n^{2sa-2a} \int_0^{1/n} dy \right. \\ &\quad \left. + n^2 \int_{1/n}^{1/(q^\theta n^{1-\theta a})} y^{2-(2s-2)a} dy \right) \\ &< 2^{2s+1} G_{18}^{2s} q^{2s-2s(1-D\epsilon_1)A} (n^{-1} + n^{2s-(2s+1)(1-\theta a)} q^{-(2s+1)\theta} (2s+1)^{-1}) \\ &\quad + 2^{2s+1} G_{18}^2 G_9^{2s-2} q^{2+2a-2sa-2(1-D\epsilon_1)A} (n^{2sa-1-2a} + n^{2+(2s-2)a-3}((2s-2)a-3)^{-1}) \\ &< G_{29} n^{(2s+1)\theta a-1} q^{2s-(2s+1)\theta-2s(1-D\epsilon_1)A} + G_{30} n^{2sa-1-2a} q^{2+2a-2sa-2(1-D\epsilon_1)A}, \end{aligned}$$

where

$$G_{29} = 2^{2s+1}G_{18}^{2s}(2s+2)(2s+1)^{-1},$$
$$G_{30} = 2^{2s+1}G_{18}^{2}G_{9}^{2s-2}(2sa-2a-2)(2sa-2a-3)^{-1}.$$

The exponent of q in the first term is

$$> 2s - (2s+1)\theta - 2sA = (1-\theta-A)2s - \theta > -\theta > -2 \qquad \text{(by (20))}.$$

The exponent of q in the second term is $<2+2a-2sa<-2$ since $2s>4k+2$. Thus

$$\sum_{M_1}\int_{M_1}|f^s(x)-\psi_\rho^s(x)|^2|dx| < G_{29}n^{(2s+1)\theta a-1}\sum_{q=1}^{n^a} q^{1+2s-(2s+1)\theta-2s(1-D\epsilon_1)A}$$
$$+ G_{30}n^{2sa-1-2a}\sum_{q=1}^{n^a} q^{3+2a-2sa-2(1-D\epsilon_1)A}$$
$$\leqq G_{29}n^{(2s+1)\theta a-1}\left(1+\int_1^{n^a} q^{1+2s-(2s+1)\theta-2s(1-D\epsilon_1)A}dq\right)$$
$$+ G_{30}n^{2sa-1-2a}\left(1+\int_1^{n^a} q^{3+2a-2sa-2(1-D\epsilon_1)A}dq\right)$$
$$< G_{29}n^{(2s+1)\theta a-1}\cdot n^{2a+2sa-(2s+1)\theta a-2s(1-D\epsilon_1)aA}$$
$$\times\left(1+(2s+2-(2s+1)\theta-2s(1-D\epsilon_1)A)^{-1}\right)$$
$$+ G_{30}n^{2sa-1-2a}(1+(2sa+2a+4-2(1-D\epsilon_1)A)^{-1})$$
$$\leqq G_{24}n^{2sa-1-(2s(1-D\epsilon_1)aA-2a)} + G_{25}n^{2sa-1-2a}.$$

In the exponents of n we have

$$2s(1-D\epsilon_1)aA - 2a > (2s-4)(1-D\epsilon_1)aA - 1 + 2a \geqq 2A + (1-a)^{s_2-2}\epsilon_1$$

as in the proof of Theorem 36; and

$$2sa - 1 - 2a \leqq 2sa - 1 - 2A - (1-a)^{s_2-2}\epsilon_1 = 2sa - 1 - \lambda.$$

This completes the proof.

THEOREM 39. (G, Theorem 14.) *On* M_2

$$|f(x)| < G_{31}n^{a-(1-aD\epsilon_1)A}q^{-A}|y|^{-A},$$

where $G_{31} = (2\pi+1)2^{1+2A+(\epsilon_2+2\epsilon_3)A}G_4{}^A$.

On M_2 we have

$$x = e^{2\pi i(b/q+y)}, \qquad 1 \leqq q \leqq n^a;$$
$$1/(2q^\theta n^{1-\theta a}) \leqq |y| < 1/(qn^{1-a}), \qquad 2/(q|y|) > 1.$$

From the theory of Farey fractions* it is known that there exist integers b_1 and q_1 and a number y_1 such that

$$b_1/q_1 + y_1 = b/q + y, \qquad 0 \leqq b_1 \leqq q_1 \leqq 2/(q|y|), \qquad (b_1, q_1) = 1;$$

$$|y_1| < q|y|/(2q_1). \tag{33}$$

It follows from (33) that $|y_1| < 1/(2q_1 n^{1-a})$ and so we cannot have $n^a < q_1 \leqq n^{1-a}$, for if this were the case x would be a point of a minor arc m. Also, $1 \leqq q_1 \leqq n^a$ is impossible when $n > 2^k$, since otherwise

$$\begin{aligned} |bq_1 - b_1 q| &= qq_1|y_1 - y| \leqq qq_1|y| + qq_1|y_1| < qq_1|y| + q^2|y| \\ &= (q + q_1)|y| < 2n^a q/(qn^{1-a}) = 2n^{2a-1} \leqq 2n^{-a} < 1, \end{aligned}$$

and therefore $bq_1 - b_1 q = 0$. Since $(b, q) = 1$, $(b_1, q_1) = 1$, we have $q_1 = q$, $b_1 = b$, $y_1 = y$, $|y_1| = q|y|/q_1 > q|y|/(2q_1)$, which contradicts (33). Hence, $q_1 > n^{1-a}$. Write $\rho_1 = e^{2\pi i b_1/q_1}$, $\tau_1(j) = \sum_{h=0}^{j^a} \rho_1^{h^k}$,

$$f(x) = (1 - x/\rho_1) \sum_{j=0}^{n-1} \tau_1(j)(x/\rho_1)^j + \tau_1(n)(x/\rho_1)^n. \tag{34}$$

By Theorem 30 with $q = q_1$, $m = [j^a] + 1$, $t = -1$,

$$\begin{aligned} |\tau_1(j)|^K &< G_4([j^a] + 1)^{\epsilon_2} q_1^{\epsilon_3}(([j^a] + 1)^{K-1} + ([j^a] + 1)^K q^{-1} + ([j^a] + 1)^{K-k} q_1) \\ &\leqq G_4(2j^a)^{\epsilon_2} q_1^{\epsilon_3}((2j^a)^{K-1} + (2j^a)^K q^{-1} + (2j^a)^{K-k} q_1) \\ &\leqq G_4(2n^a)^{\epsilon_2} q_1^{\epsilon_3}((2n^a)^{K-1} + (2n^a)^K q^{-1} + (2n^a)^{K-k} q_1) \\ &< G_4 2^{\epsilon_2} n^{a\epsilon_2} q_1^{\epsilon_3}(2^{K-1} + 2^K + 2^{K-k}) n^{aK-1} q_1 \qquad (q_1 > n^{1-a}) \\ &\leqq 2^{K+1+\epsilon_2+2\epsilon_3} G_4 n^{aK-1+aD\epsilon_1} \cdot 2q^{-1}|y|^{-1}; \end{aligned}$$

$$|\tau_1(j)| < 2^{1+2A+(\epsilon_2+2\epsilon_3)A} G_4^A n^{a-(1-aD\epsilon_1)A} q^{-A} |y|^{-A}. \tag{35}$$

Also,

$$|1 - x/\rho_1| = |e^{\pi i y_1} - e^{-\pi i y_1}| = 2 \sin \pi |y_1| \leqq 2\pi |y_1| < 2\pi q_1^{-2} < 2\pi n^{2a-2}.$$

Then from (34) and (35) we obtain

$$\begin{aligned} |f(x)| &\leqq |1 - x/\rho_1| \sum_{j=0}^{n-1} |\tau_1(j)| + |\tau_1(n)| \\ &< (2\pi n \cdot n^{2a-2} + 1) 2^{1+2A+(\epsilon_2+2\epsilon_3)A} G_4^A n^{a-(1-aD\epsilon_1)A} q^{-A} |y|^{-A} \\ &< G_{31} n^{a-(1-aD\epsilon_1)A} q^{-A} |y|^A. \end{aligned}$$

This proves the theorem when $n \geqq 2^k$. If $n < 2^k$ we have

* See the footnote at the beginning of this section.

$$\begin{aligned}|f(x)| &\leqq n^a = n^{(1-aD\epsilon_1)A-(A-aA)}n^{a-(1-aD\epsilon_1)}\cdot n^{A-aA}\\ &< 2^{k(1-aD\epsilon_1)A-k(A-aA)}n^{a-(1-aD\epsilon_1)}q^{-A}|y|^{-A}\\ &< G_{31}n^{a-(1-aD\epsilon_1)A}q^{-A}|y|^{-A}.\end{aligned}$$

THEOREM 40. (G, Theorem 15.) *We have*

$$\sum_{M_2}\int_{M_2}|f^s(x)|^2|dx| < G_{35}n^{2sa-1-\lambda},$$

where $G_{35}=2^{2sA}G_{31}^{2s}(2sA-1)^{-1}(2sA-(2sA-1)\theta+3)(2sA-(2sA-1)\theta+2)^{-1}$.

By Theorem 39

$$\begin{aligned}\int_{M_2}|f^s(x)|^2|dx| &< 2G_{31}^{2s}n^{2sa-2s(1-aD\epsilon_1)A}q^{-2sA}\int_{1/(2q^\theta n^{1-\theta a})}^{\infty}y^{-2sA}dy\\ &= 2G_{31}^{2s}n^{2sa-2s(1-aD\epsilon_1)A}q^{-2sA}2^{2sA-1}(2sA-1)^{-1}n^{(2sA-1)(1-\theta a)}q^{(2sA-1)\theta}\\ &= 2^{2sA}(2sA-1)^{-1}G_{31}^{2s}n^{2sa-1-(2sA-1)\theta a+2saAD\epsilon_1}q^{-2sA+(2sA-1)\theta}.\end{aligned}$$

Since the exponent of q is >-2 by (21), we have

$$\begin{aligned}&\sum_{M_2}\int_{M_2}|f^s(x)|^2|dx|\\ &< 2^{2sA}(2sA-1)^{-1}G_{31}^{2s}n^{2sa-1-(2sA-1)\theta a+2saAD\epsilon_1}\sum_{q=1}^{n^a}q^{1-2sA+(2sA-1)\theta}\\ &\leqq 2^{2sA}(2sA-1)^{-1}G_{31}^{2s}n^{2sa-1-(2sA-1)\theta a+2saAD\epsilon_1}\left(1+\int_1^{n^a}q^{1-2sA+(2sA-1)\theta}dq\right)\\ &< 2^{2sA}(2sA-1)^{-1}G_{31}^{2s}n^{2sa-1-(2sA-1)\theta a+2saAD\epsilon_1}\\ &\quad\times(1+((2sA-1)\theta-2sA+2)^{-1}n^{(2sA-1)\theta a-2saA+2a})\\ &\leqq 2^{2sA}(2sA-1)^{-1}(2sA-(2sA-1)\theta+3)(2sA-(2sA-1)\theta+2)^{-1}\\ &\quad\times G_{31}^{2s}n^{2sa-1-(2s(1-D\epsilon_1)aA-2a)}.\end{aligned}$$

As in Theorem 38 we have $2s(1-D\epsilon_1)aA-2a>\lambda$ and the proof is complete.

THEOREM 41. (G, Theorem 16.) *We have*

$$\int_C\left|f^s(x)-\sum_{q,\rho}\psi_\rho^s(x)\right|^2|dx| < G_1n^{2sa-1-\lambda},$$

where $G_1=2(G_{10}+G_{15}+G_{24}+G_{25}+G_{35})$. *The summation extends over all* ρ *for which* $1\leqq q\leqq n^a$. *There are* $\sum_{q=1}^{n^a}\phi(q)$ *terms in the sum.*

We may write

$$\int_C \left| f^s(x) - \sum_{q,\rho} \psi_\rho^s(x) \right|^2 |dx| = \sum_m \int_m |f^s(x) - \sum \psi_\rho^s(x)|^2 |dx|$$
$$+ \sum_{M_2} \int_{M_2} |f^s(x) - \sum \psi_\rho^s(x)|^2 |dx|$$
$$+ \sum_{M_1} \int_{M_1} |f^s(x) - \psi_\rho^s(x) - \sum{}' \psi_\rho^s(x)|^2 |dx|,$$

the accent indicating that the term which corresponds to M_1 itself is omitted in the summation and written separately. Using the inequality

$$\left| \sum_{j=1}^N \xi_j \right|^2 \leqq N \sum |\xi_j|^2$$

we obtain

$$\int_C |f^s(x) - \sum \psi_\rho^s(x)|^2 |dx|$$
$$\leqq 2 \sum_m \int_m |f^s(x)|^2 |dx| + 2 \sum_m \int_m |\sum \psi_\rho^s(x)|^2 |dx|$$
$$+ 2 \sum_{M_2} \int_{M_2} |f^s(x)|^2 |dx| + 2 \sum_{M_2} \int_{M_2} |\sum \psi_\rho^s(x)|^2 |dx|$$
$$+ 2 \sum_{M_1} \int_{M_1} |f^s(x) - \psi_\rho^s(x)|^2 |dx|$$
$$+ 2 \sum_{M_1} \int_{M_1} |\sum{}' \psi_\rho^s(x)|^2 |dx|$$
$$\leqq 2 \sum_m \int_m |f^s(x)|^2 |dx| + 2n^{2a} \sum_m \int_m \sum |\psi_\rho^s(x)|^2 |dx|$$
$$+ 2 \sum_{M_2} \int_{M_2} |f^s(x)|^2 |dx| + 2n^{2a} \sum_{M_2} \int_{M_2} \sum |\psi_\rho^s(x)|^2 |dx|$$
$$+ 2 \sum_{M_1} \int_{M_1} |f^s(x) - \psi_\rho^s(x)|^2 |dx| + 2n^{2a} \sum_{M_1} \int_{M_1} \sum{}' |\psi_\rho^s(x)|^2 |dx|$$
$$= 2 \sum_m \int_m |f^s(x)|^2 |dx| + 2 \sum_{M_2} \int_{M_2} |f^s(x)|^2 |dx|$$
$$+ 2 \sum_{M_1} \int_{M_1} |f^s(x) - \psi_\rho^s(x)|^2 |dx| + 2n^{2a} \sum_{M_1} \int_{C-M_1} |\psi_\rho^s(x)|^2 |dx|.$$

To these terms we apply Theorems 36, 40, 35, and 34, respectively.

Then

$$\int_C \left| f^s(x) - \sum_{q,\rho} \psi_\rho^s(x) \right|^2 |dx| < 2(G_{10} + G_{15} + G_{24} + G_{25} + G_{35}) n^{2sa-1-\lambda}.$$

THEOREM 42. THE MAIN LEMMA. (G, §9.) *We have*

$$\sum_{j=1}^{n} |\sigma(j)|^2 < G_1 n^{2sa-1-\lambda}.$$

We note that

$$\begin{aligned}
\psi_\rho^s(x) &= \Gamma^s(1+a)q^{-s}S_\rho^s \times \text{the first } n+1 \text{ terms of } (1 - x/\rho)^{sa} \\
&\quad + \text{a finite number of terms with higher powers of } x/\rho \\
&= \Gamma^s(1+a)q^{-s}S_\rho^s \left(1 + \sum_{j=1}^{n} (sa(sa+1)\cdots(sa+j-1)/j!)(x/\rho)^j\right) \\
&\quad + \text{higher powers} \\
&= (\Gamma^s(1+a)/\Gamma(sa))q^{-s}S_\rho^s \sum_{j=0}^{n} (\Gamma(sa+j)/j!)(x/\rho)^j \\
&\quad + \text{higher powers;}
\end{aligned}$$

$$\begin{aligned}
\sum_{q,\rho} \psi_\rho^s(x) &= (\Gamma^s(1+a)/\Gamma(sa)) \sum_{j=0}^{n} (\Gamma(sa+j)/j!) \sum_q \sum_\rho q^{-s}S_\rho^s \rho^{-j} x^j \\
&\quad + \text{higher powers} \\
&= (\Gamma^s(1+a)/\Gamma(sa)) \sum_{j=0}^{n} (\Gamma(sa+j)/j!) \mathfrak{S}(j, k, s, n^a) x^j \\
&\quad + \text{higher powers.}
\end{aligned}$$

Also,

$$f^s(x) = 1 + \sum_{j=1}^{n} r_{k,s}(j) x^j + \text{higher powers.}$$

Hence

$$f^s(x) - \sum_{q,\rho} \psi_\rho^s(x) = \sigma(0) + \sum_{j=1}^{n} \sigma(j) x^j + \text{higher powers.}$$

By Theorem 27,

$$\int_{-1/2}^{1/2} \left| f^s(e^{2\pi i y}) - \sum \psi_\rho^s(e^{2\pi i y}) \right|^2 dy = \sum_{j=1}^{n} |\sigma(j)|^2 + \text{a positive quantity.}$$

Therefore

$$\sum_{j=1}^{n} |\sigma(j)|^2 < \int_{-1/2}^{1/2} \left| f^s(e^{2\pi i y}) - \sum \psi_\rho^s(e^{2\pi i y}) \right|^2 dy < G_1 n^{2sa-1-\lambda}$$

by Theorem 41.

COROLLARY. *Let*

$$\sigma_0(j) = (\Gamma^s(1+a)/\Gamma(sa))(\Gamma(sa+j)/j!)\mathfrak{S}(j, k, s, \infty).$$

Then

$$\sum_{j=1}^{n} |\sigma_0(j)|^2 < A_4 n^{2sa-1-\lambda},$$

where $A_4 = 2G_1 + 2(\Gamma^{2s}(1+a)/\Gamma^2(sa))(sa/(sa-1))^2(2sa/(2sa-1))b_{19}^2\gamma_2^2(sa-1)$.

By the proof of Theorem 4 we have

$$|\sigma_0(j) - \sigma(j)| = (\Gamma^s(1+a)/\Gamma(sa))(\Gamma(sa+j)/j!)|\mathfrak{S}(j, k, s, \infty) - \mathfrak{S}(j, k, s, n^a)|,$$

$$|\sigma_0(j)| < |\sigma(j)| + (\Gamma^s(1+a)/\Gamma(sa))\gamma_2(sa-1)j^{sa-1}\sum_{q>na} |A(q)|.$$

Also, by Theorem 16,

$$\sum_{q>n^a} |A(q)| < b_{19} \sum_{q>n^a} q^{1-sa} \leqq b_{19}\left(n^{a(1-sa)} + \int_{n^a}^{\infty} q^{1-sa}dq\right)$$
$$\leqq b_{19}(sa/(sa-1))n^{a(2-sa)}.$$

Hence

$$|\sigma_0(j)|^2 < 2|\sigma(j)|^2 + 2(\Gamma^{2s}(1+a)/\Gamma^2(sa))(sa/(sa-1))^2 b_{19}^2\gamma_2^2(sa-1)n^{2a(2-sa)}j^{2sa-2},$$

$$\begin{aligned}\sum_{j=1}^{n}|\sigma_0(j)|^2 &< 2G_1 n^{2sa-1-\lambda} \\ &\quad + 2(\Gamma^{2s}(1+a)/\Gamma^2(sa))(sa/(sa-1))^2 b_{19}^2\gamma_2^2(sa-1)n^{2a(2-sa)} \\ &\quad \times \left(\int_1^n j^{2sa-2}dj + n^{2sa-2}\right) \\ &< 2G_1 n^{2sa-1-\lambda} + 2(\Gamma^{2s}(1+a)/\Gamma^2(sa)) \\ &\quad \times (sa/(sa-1))^2 b_{19}^2\gamma_2^2(sa-1)(2sa/(2sa-1))n^{2sa-1-2a(sa-2)} \\ &\leqq A_4 n^{2sa-1-\lambda}.\end{aligned}$$

6. **The third and fourth Hardy-Littlewood theorems.** In this section we again follow paper L in the proof of the third and fourth Hardy-Littlewood theorems which are here Theorems 43 and 45, respectively.

THEOREM 43. (L, Theorem 346.) *Let* $H(\xi)$ *denote the number of positive integers* $j \leqq \xi$ *for which the equation*

$$j = \sum_{i=1}^{s} h_i^k, \qquad h_i \geqq 0, \tag{36}$$

is not solvable. Then

$$H(\xi) < C_{66}\xi^{1-\lambda},$$

where $C_{66}=3A_4/c_{105}$, $c_{105}=(\Gamma^{2s}(1+a)/\Gamma^2(sa))2^{2-2sa}\gamma_3{}^2b_4$.

By the Corollary to Theorem 42 we have for $\xi\geqq 2$

$$\sum_{\xi/2<j\leqq\xi}|\sigma_0(j)|^2 \leqq \sum_{j=1}^{[\xi]}|\sigma_0(j)|^2 < A_4\xi^{2sa-1-\lambda}. \tag{37}$$

In the summation $\sum_{\xi/2<j\leqq\xi}|\sigma_0(j)|^2$ there are $H(\xi)-H(\xi/2)$ terms in which $r_{k,s}(j)=0$. For these terms

$$|\sigma_0(j)|^2=|(\Gamma^s(1+a)/\Gamma(sa))(\Gamma(sa+j)/j!)\mathfrak{S}|^2 > (\Gamma^{2s}(1+a)/\Gamma^2(sa))\gamma_3^2b_4^2j^{2sa-2}$$

(by Theorem 25 and the Corollary to Theorem 29)

$$= c_{104}j^{2sa-2} > c_{104}(\xi/2)^{2sa-2} = c_{105}\xi^{2sa-2}.$$

In the remaining terms $|\sigma_0(j)|^2\geqq 0$. From (37) we get

$$(H(\xi)-H(\xi/2))c_{105}\xi^{2sa-2} < \sum_{\xi/2<j\leqq\xi}|\sigma_0(j)|^2 < A_4\xi^{2sa-1-\lambda},$$

$$H(\xi)-H(\xi/2) < (A_4/c_{105})\xi^{1-\lambda} = C_{68}\xi^{1-\lambda}.$$

This holds also when $0<\xi<2$ since (36) is solvable for $j=1$ and then $H(\xi)-H(\xi/2)=0$. Hence for $\xi>0$ and every integer $v\geqq 0$

$$H(\xi/2^v)-H(\xi/2^{v+1}) < C_{68}(\xi/2^v)^{1-\lambda} = C_{68}\xi^{1-\lambda}(2^v)^{-1+\lambda}$$
$$< C_{68}\xi^{1-\lambda}2^{-2v/3} \qquad (-1+\lambda<-2/3),$$

$$H(\xi) = \sum_{v=0}^{\infty}(H(\xi/2^v)-H(\xi/2^{v+1})) < C_{68}\xi^{1-\lambda}\sum_{v=0}^{\infty}2^{-2v/3}$$
$$< 3C_{68}\xi^{1-\lambda} = C_{66}\xi^{1-\lambda}.$$

THEOREM 44. (L, Theorem 348.) *Let* $L_{s_2}(n)$ *denote the number of positive integers* $j\leqq n$ *for which equation* (36) *with* $s=s_2$ *is solvable. Then*

$$L_{s_2}(n) > B_{19}n^{1-(1-2a)(1-a)s_2^{-2}-(1-a)s_2^{-2}e_1},$$

where $B_{19}=2^{2^{-s_2}}c_{111}^{s_2^{-2}}C_{71}$, $c_{111}=2^{-1-a}(2^a-1)$, $C_{71}=2^{-1-2a}/((k-1)A_1)$.

(i) Let $s_2=2$. The number of solutions of the inequalities

$$1\leqq h_1^k+h_2^k\leqq n, \qquad h_1\geqq 0,\ h_2\geqq 0, \tag{38}$$

is at least equal to the number of solutions of

$$0\leqq h_1\leqq (n/2)^a,\quad 0\leqq h_2\leqq (n/2)^a, \qquad h_1+h_2>0. \tag{39}$$

The number of solutions of (39) is

$$([(n/2)^a] + 1)^2 - 1 > (n/2)^{2a} - 1 > n^{2a} \cdot 2^{-1-2a} = c_{109} n^{2a},$$

when $n > 2^{(2a+1)/(2a)} = c_{108}$. For each positive integer $j \leqq n$ the equation $j = h_1^k + h_2^k$ has at most $(k-1)A_1 j^{e_1} \leqq (k-1)A_1 n^{e_1}$ solutions by the proof of Theorem 28. Therefore

$$\begin{aligned} L_2(n) &> c_{109} n^{2a}/((k-1)A_1 n^{e_1}) = C_{71} n^{2a-e_1} \\ &= 2^{2-2} c_{111}^{2-2} C_{71} n^{1-(1-2a)(1-a)^{2-2}-(1-a)^{2-2}e_1}. \end{aligned}$$

(ii) Let $s_2 > 2$ and assume that the theorem is true for $s_2 - 1$, i.e.,

$$L_{s_2-1}(n) > B_{20} n^{1-(1-2a)(1-a)^{s_2-3}-(1-a)^{s_2-3}e_1}, \tag{40}$$

where $B_{20} = 2^{3-s_2} c_{11}^{s_2-3} C_{71}$. Consider all integers

$$h^k + z \tag{41}$$

such that

$$\begin{gathered} h \text{ is an integer } > 0,\ z \text{ is an integer}, \\ n/2 < h^k < (h+1)^k < n, \quad 0 < z \leqq n^{1-a}, \\ z \text{ is representable in the form } z = \textstyle\sum_{i=1}^{s_2-1} h_i^k,\ h_i \geqq 0. \end{gathered} \tag{42}$$

Since $(h+1)^k - h^k > kh^{k-1} > 2h^{k-1} > 2(n/2)^{a(k-1)} = 2^a n^{1-a} > z$, we have $h^k < h^k + z < (h+1)^k$. This shows that to distinct pairs of values h, z of (42) correspond distinct integers (41). For suppose $h_1^k + z_1 = h_2^k + z_2$. Then

$$\begin{aligned} h_1^k &< h_1^k + z_1 = h_2^k + z_2 < (h_1 + 1)^k, \\ h_2^k &< h_2^k + z_2 = h_1^k + z_1 < (h_2 + 1)^k \end{aligned}$$

is impossible unless $h_1 = h_2$ and then $z_1 = z_2$. Moreover, each of the integers (41) is > 0 and $< (h+1)^k < n$. Therefore $L_{s_2}(n)$ is at least equal to the number of pairs of values of h, z of (42). Since $(n/2)^a < h < n^a - 1$ by (42), the number of values h takes is

$$\begin{aligned} [n^a] - 1 - [(n/2)^a] - 1 &> n^a - 2 - (n/2)^a - 1 = 2^{-a}(2^a - 1)n^a - 3 \\ &\geqq 2^{-1-a}(2^a - 1)n^a = c_{111} n^a \end{aligned}$$

when $n > (3 \cdot 2^{1+a}/(2^a - 1))^k = c_{110}$. The number of values z takes is $L_{s_2-1}([n^{1-a}])$. Hence from (40)

$$\begin{aligned} L_{s_2}(n) &> c_{111} n^a \cdot L_{s_2-1}([n^{1-a}]) > c_{111} n^a B_{20} (n^{1-a}/2)^{1-(1-2a)(1-a)^{s_2-3}-(1-a)^{s_2-3}e_1} \\ &> 2^{-1} c_{111} \cdot 2^{3-s_2} c_{111}^{s_2-3} C_{71} n^{1-(1-2a)(1-a)^{s_2-2}-(1-a)^{s_2-2}e_1}, \\ &= B_{19} n^{1-(1-2a)(1-a)^{s_2-2}-(1-a)^{s_2-2}e_1}. \end{aligned}$$

THEOREM 45. (L, Theorem 350.) *For s_2 as defined in §2 we have*

$$(2-2a)A > (1-2a)(1-a)^{s_2-2}.$$

THEOREM 46. (L, Theorem 349.) *For s and s_2 as defined in §2 and every integer $n > C = \max(c_{108}, c_{110}, (C_{66}/B_{19})^{k2^{k-2}})$, the equation*

$$\sum_{i=1}^{s+s_2} h_i^k = n, \qquad h_i \geqq 0, \tag{43}$$

has at least one solution. That is, every integer $>C$ is a sum of s_0 kth powers $\geqq 0$ when

$$s_0 \geqq s + s_2 = g_1(k, \epsilon_1).$$

Let n be an integer for which (43) is not solvable and write $n = n_1 + n_2$. Then, since there are $L_{s_2}(n)$ integers $n_2 \leqq n$ for which $\sum_{i=s+1}^{s+s_2} h_i^k = n_2$, $h_i \geqq 0$, is solvable, there must be $L_{s_2}(n)$ integers $n_1 = n - n_2 \leqq n$ for which

$$\sum_{i=1}^{s} h_i^k = n_1, \qquad h_i \geqq 0, \tag{44}$$

is *not* solvable. For if (44) were solvable for one of the $L_{s_2}(n)$ integers n_1, then (43) would be solvable for n contrary to our assumption. By Theorem 43 the number of positive integers $\leqq n$ for which (44) is not solvable is $< C_{66}n^{1-\lambda}$. Hence

$$L_{s_2}(n) < C_{66}n^{1-\lambda}.$$

By Theorem 44, when $n > \max(c_{108}, c_{110})$

$$L_{s_2}(n) > B_{19}n^{1-(1-2a)(1-a)^{s_2-2}-(1-a)^{s_2-2}\epsilon_1}.$$

Therefore

$$B_{19}n^{1-(1-2a)(1-a)^{s_2-2}-(1-a)^{s_2-2}\epsilon_1} < C_{66}n^{1-\lambda},$$

$$n^{\lambda-(1-2a)(1-a)^{s_2-2}-(1-a)^{s_2-2}\epsilon_1} < C_{66}/B_{19},$$

$$n^{\lambda-(2-2a)A-(1-a)^{s_2-2}\epsilon_1} < C_{66}/B_{19} \qquad \text{(Theorem 45)},$$

$$n^{2aA} < C_{66}/B_{19} \qquad (\lambda = 2A + (1-a)^{s_2-2}\epsilon_1).$$

It follows that (43) is always solvable when

$$n > \max(c_{108}, c_{110}, (C_{66}/B_{19})^{k2^{k-2}}).$$

7. **The solution of (43) for integers $<C$.** The following theorem is well known:

THEOREM 47.* *If every integer n for which $f<n\leqq h$ is a sum of $s-1$ kth powers $\geqq 0$ and if m is the greatest integer such that*

$$(m+1)^k - m^k < h - f, \tag{45}$$

then every integer n for which $f<n\leqq h+(m+1)^k$ is a sum of s kth powers $\geqq 0$.

THEOREM 48. *For $L=(k+1)^k-k^k>k^k$ we have*

$$s_3 < 2^k + \left(\frac{3}{2}\right)^k + 2\left(\frac{4}{3}\right)^k + 2\left(\frac{2}{3}\right)^k + 2\left(\frac{1}{2}\right)^k + \frac{k(2k+7)}{9} - 9.$$

Consider any integer n such that $0<n\leqq 2^{k+1}-2$. If $n\leqq 2^k-1$ it is obviously the sum of 2^k-1 kth powers, 0 or 1. If $2^k\leqq n\leqq 2^{k+1}-2$ we write $n=2^k+x$, $0\leqq x\leqq 2^k-2$, and again n is a sum of 2^k-1 kth powers since x is a sum of 2^k-2 kth powers, 0 or 1. Hence every integer in the interval

$$0 < n \leqq 2^{k+1} - 2 = h_1$$

is a sum of $2^k-1=m_1$ kth powers $\geqq 0$. Since $2^k-1^k<h_1<3^k-2^k$, it follows from Theorem 47 with $m=1$ that every integer in the interval $0<n\leqq h_1+2^k$ is a sum of m_1+1 kth powers $\geqq 0$. We repeat this step m_2 times so that every integer in the interval $0<n\leqq h_1+m_2 2^k$ is a sum of m_1+m_2 kth powers $\geqq 0$, where

$$h_1 + (m_2 - 1)2^k \leqq 3^k - 2^k < h_1 + m_2 2^k. \tag{46}$$

We now apply Theorem 47 m_3 times with $m=2$ and conclude that every integer in the interval $0<n\leqq h_1+m_2 2^k+m_3 3^k$ is a sum of $m_1+m_2+m_3$ kth powers $\geqq 0$, where

$$h_1 + m_2 2^k + (m_3 - 1)3^k \leqq 4^k - 3^k < h_1 + m_2 2^k + m_3 3^k. \tag{47}$$

In general every integer n such that

$$0 < n \leqq h_1 + \sum_{j=2}^{t} m_j j^k$$

is a sum of $\sum_{j=1}^{t} m_j$ kth powers $\geqq 0$, where

$$h_1 + \sum_{j=2}^{t-1} m_j j^k + (m_t - 1)t^k \leqq (t+1)^k - t^k < h_1 + \sum_{j=2}^{t} m_j j^k. \tag{48}$$

From (47) and (46) we get

$$m_3 3^k \leqq 4^k - m_2 2^k - h_1 < 4^k - 3^k + 2^k,$$

and in general from (48) when $t\geqq 3$,

* L. E. Dickson, loc. cit.

$$m_t t^k \leqq (t+1)^k - \sum_{j=2}^{t-1} m_j j^k - h_1 < (t+1)^k - t^k + (t-1)^k,$$

$$m_t < (1+t^{-1})^k - 1 + (1-t^{-1})^k. \tag{49}$$

Hence

$$\begin{aligned}
\sum_{j=1}^{t} m_j &< m_1 + m_2 + \sum_{j=3}^{t} ((1+j^{-1})^k - 1 + (1-j^{-1})^k) \\
&= m_1 + m_2 - (t-2) + 2\sum_{j=3}^{t}\left(1 + \binom{k}{2}\frac{1}{j^2} + \binom{k}{4}\frac{1}{j^4} + \cdots\right) \\
&\leqq m_1 + m_2 - (t-2) + \left(\frac{4}{3}\right)^k + \left(\frac{2}{3}\right)^k \\
&\quad + 2\int_3^t \left(1 + \binom{k}{2}\frac{1}{j^2} + \cdots\right) dj \\
&< m_1 + m_2 - (t-2) + \left(\frac{4}{3}\right)^k + \left(\frac{2}{3}\right)^k \\
&\quad + 2(t-3) + 2\left(\binom{k}{2}\frac{1}{3} + \binom{k}{4}\frac{1}{3\cdot 3^3} + \cdots\right) \\
&< m_1 + m_2 + t - 4 + \left(\frac{4}{3}\right)^k + \left(\frac{2}{3}\right)^k \\
&\quad + 2\left(\binom{k}{2}\frac{2}{3^2} - 1 + 1 + \binom{k}{2}\frac{1}{3^2} + \binom{k}{4}\frac{1}{3^4} + \cdots\right) \\
&= m_1 + m_2 + t - 4 + \left(\frac{4}{3}\right)^k + \left(\frac{2}{3}\right)^k \\
&\quad + \frac{2k(k-1)}{9} - 2 + \left(\frac{4}{3}\right)^k + \left(\frac{2}{3}\right)^k.
\end{aligned}$$

From (46), $m_2 2^k \leqq 3^k - h_1 = 3^k - 2^{k+1} + 2$, and hence when $t = k$ we get $L = (k+1)^k - k^k$ and

$$\begin{aligned}
s_3 = \sum_{j=1}^{k} m_j &< 2^k - 1 + \left(\frac{3}{2}\right)^k - 2 + 2\left(\frac{1}{2}\right)^k \\
&\quad + k - 4 + 2\left(\frac{4}{3}\right)^k + 2\left(\frac{2}{3}\right)^k + \frac{2k(k-1)}{9} - 2 \\
&= 2^k + \left(\frac{3}{2}\right)^k + 2\left(\frac{4}{3}\right)^k + 2\left(\frac{2}{3}\right)^k + 2\left(\frac{1}{2}\right)^k + \frac{k(2k+7)}{9} - 9.
\end{aligned}$$

THEOREM 49. *If every positive integer* $\leqq L$ *is a sum of* $s-1$ *kth powers* $\geqq 0$, *then every positive integer* $\leqq (L/k)^{k/(k-1)}$ *is a sum of* s *kth powers* $\geqq 0$.

Since $(L/k)^{k/(k-1)}-((L/k)^{1/(k-1)}-1)^k \leqq k(L/k)=L$, we may apply Theorem 47 with $m+1=[(L/k)^{1/(k-1)}]$. Thus every positive integer $\leqq L+[(L/k)^{1/(k-1)}]^k$ is a sum of s kth powers $\geqq 0$, and $L+[(L/k)^{1/(k-1)}]^k \geqq L+((L/k)^{1/(k-1)}-1)^k \geqq (L/k)^{k/(k-1)}$.

THEOREM 50. *If every positive integer* $\leqq L$, *where* $L>k^k$, *is a sum of* s_3 *kth powers* $\geqq 0$, *then every positive integer* $\leqq C$ *is a sum of* s_3+s_4 *kth powers* $\geqq 0$, *where*

$$s_4 = \left[\frac{\log\log C - \log(\log L - k\log k)}{\log k - \log(k-1)}\right] + 1.$$

That is, every integer $\leqq C$ *is a sum of* s_0 *kth powers* $\geqq 0$ *when*

$$s_0 \geqq s_3 + s_4 = g_2(k, \epsilon_1).$$

By Theorem 49 every positive integer $\leqq (L/k)^{k/(k-1)}$ is a sum of s_3+1 kth powers $\geqq 0$. Write $L_1=(L/k)^{k/(k-1)}$ and apply Theorem 49 again. Thus every positive integer $\leqq (L_1/k)^{k/(k-1)}=L_2$ is a sum of s_3+2 kth powers $\geqq 0$. Also,

$$L_2 = \left(\frac{L_1}{k}\right)^{k/(k-1)} = \left(\frac{L^{k/(k-1)}}{k^{k/(k-1)+1}}\right)^{k/(k-1)},$$

$$\log L_2 = (k/(k-1))^2 \log L - \left(k/(k-1) + (k/(k-1))^2\right)\log k.$$

In general, every positive integer $\leqq L_{s_4}$ is a sum of s_3+s_4 kth powers $\geqq 0$, where

$$\begin{aligned}\log L_{s_4} &= (k/(k-1))^{s_4}\log L - (k/(k-1) + \cdots + (k/(k-1))^{s_4})\log k \\ &= (k/(k-1))^{s_4}\log L - k((k/(k-1))^{s_4}-1)\log k \\ &> (k/(k-1))^{s_4}(\log L - k\log k).\end{aligned}$$

This expression is $\geqq \log C$ when

$$s_4 \geqq \frac{\log\log C - \log(\log L - k\log k)}{\log k - \log(k-1)}.$$

8. **Evaluation of the constants.** We first prove three lemmas.

LEMMA 1. *For* $w \geqq 5$ *we have*

$$\sum_{j=1}^{n}(1+j^{-1})^w < 2^{w+1} + n - 1 + (w+1)\log n.$$

Let $t=[w]+1$. Then

$$\sum_{j=1}^{n}(1+j^{-1})^w \leqq 2^w + \int_1^n (1+j^{-1})^w dj < 2^w + \int_1^n (1+j^{-1})^t dj$$

$$= 2^w + \int_1^n \left(1 + \binom{t}{1}\frac{1}{j} + \binom{t}{2}\frac{1}{j^2} + \cdots + \frac{1}{j^t}\right) dj$$

$$= 2^w + \left(j + t\log j - \binom{t}{2}\frac{1}{j} - \binom{t}{3}\frac{1}{2j^2} - \cdots - \frac{1}{(t-1)j^{t-1}}\right)\Bigg|_1^n$$

$$< 2^w + n - 1 + t\log n + \binom{t}{2} + \binom{t}{3}\frac{1}{2} + \binom{t}{4}\frac{1}{3} + \cdots + \frac{1}{t-1}$$

$$= 2^w + n - 1 + t\log n + \frac{1}{2} + \binom{t}{1}\frac{1}{2} + \binom{t}{2}\frac{1}{2}$$

$$+ \binom{t}{3}\frac{1}{2} + \binom{t}{4}\frac{1}{2} + \cdots + \frac{1}{2}$$

$$- \frac{1}{2} - \binom{t}{1}\frac{1}{2} + \binom{t}{2}\frac{1}{2} - \binom{t}{4}\frac{1}{6}$$

$$- \binom{t}{5}\frac{1}{4} - \cdots - \left(\frac{1}{2} - \frac{1}{t-1}\right).$$

Since

$$\frac{1}{2} + \binom{t}{1}\frac{1}{2} - \binom{t}{2}\frac{1}{2} + \binom{t}{4}\frac{1}{6} + \binom{t}{5}\frac{1}{4} > 0$$

when $t=[w]+1 \geqq 6$, we have

$$\begin{aligned}\sum_{j=1}^{n}(1+j^{-1})^w &< 2^w + n - 1 + t\log n + 2^{t-1} \\ &= 2^w + n - 1 + ([w]+1)\log n + 2^{[w]} \\ &\leqq 2^{w+1} + n - 1 + (w+1)\log n.\end{aligned}$$

LEMMA 2. *For $x \geqq 0$ we have*

$$(2^x - 1)^{-1} \leqq (x\log 2)^{-1}.$$

Consider the function

$$y = x(2^x - 1)^{-1}, \quad y' = (2^x - 1 - x2^x\log 2)(2^x - 1)^{-2}.$$

We have $y' \leqq 0$ when $x \geqq 0$. Hence y attains its maximum value when $x=0$. That is, $\max y = 1/\log 2$ and the desired result follows.

LEMMA 3. *Let t be an integer $\geqq 0$. Then*

$$\log (t!) \leqq (t+1) \log t - t + 1.$$

We have

$$\log (t!) = \sum_{n=1}^{t} \log n \leqq \int_1^t \log x \, dx + \log t = (t+1) \log t - t + 1.$$

The constants are now evaluated as follows.

(α_2)	$\alpha_2 \leqq 7 \log 2$	(Theorem 2).
$(\gamma(\beta))$	$\gamma(\beta) = 4\beta(2^\beta + 1)e^\beta \Gamma(\beta + 1)$	(Theorem 29).
$(\gamma_2(\beta))$	$\gamma_2(\beta) = e^\beta \Gamma(\beta + 1)$	(Theorem 29).
(γ_3)	$\gamma_3 = (1 + 12e)^{-1}$	(Theorem 29).
(c_{15})	$\log c_{15} = (k-1) \log k + \alpha_2(k-2)k^{2k/(k-2)}/(2k)$	(Theorem 15).
(b_{18})	$b_{18} = (1 + k^s)^{2/(s-5)}$	(Theorem 24).
$(b(p))$	$b(p) > \begin{cases} 2^{-k(4k-1)}, & p = 2, \\ p^{-k(2k-1)}, & p > 2 \end{cases}$	(Theorem 10).

Proof: By the proof of Theorem 10, $r \leqq 4k-1$ when $p=2$ and $r \leqq 2k-1$ when $p>2$. Also, $\gamma \leqq k$, since for $p>2$

$$\gamma = \Theta + 1 \leqq 2^\Theta \leqq p^\Theta \leqq k;$$

for $p=2$, $\Theta > 1$,

$$\gamma = \Theta + 2 \leqq 2^\Theta \leqq k;$$

and for $p=2$, $\Theta \leqq 1$,

$$\gamma = \Theta + 2 \leqq 3 < k.$$

Hence

$$b(p) = P^{-r} = p^{-\gamma r} > \begin{cases} 2^{-k(4k-1)}, & p = 2, \\ p^{-k(2k-1)}, & p > 2. \end{cases}$$

(b_4)	$\log (1/b_4) < 2k^5$	(Theorem 25).

Proof: From Theorem 25

$$b_4 = \prod_{p \leqq b_{18}} b(p) \prod_{p > b_{18}} (1 - p^{-3/2}).$$

Let

$$\prod\nolimits_1 = 1/\Big(\prod_{p \leqq b_{18}} b(p) \Big), \qquad \prod\nolimits_2 = 1/\Big(\prod_{p > b_{18}} (1 - p^{-3/2}) \Big).$$

Then

$$\prod_1 \leqq 2^{k(4k-1)} \prod_{3 \leqq p \leqq b_{18}} p^{k(2k-1)},$$

$$\begin{aligned}
\log \prod_1 &\leqq k(4k-1)\log 2 + k(2k-1)\vartheta(b_{18}) - k(2k-1)\log 2 \\
&< 2k^2 \log 2 + k(2k-1)\left(\frac{6}{5} c b_{18} + 3\log^2 b_{18} + 8\log b_{18} + 5\right) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(Theorem 3)} \\
&= 2k^2 \log 2 + k(2k-1)\left(\frac{6}{5} c(1+k^s)^{2/(s-5)}\right. \\
&\qquad \left. + 3(2/(s-5))^2 \log^2(1+k^s) + 8(2/(s-5))\log(1+k^s) + 5\right) \\
&\leqq 2k^2 \log 2 + k(2k-1)\left(\frac{6}{5} c 2^{2/65} k^{28/13} + 3(2/65)^2 \log^2(2k^{70})\right. \\
&\qquad \left. + 8(2/65)\log(2k^{70}) + 5\right) \qquad \text{(since } s \geqq 70 \text{ by (2))} \\
&< 2k^5 - \log 3 \qquad (k \geqq 6).
\end{aligned}$$

Also,

$$\prod_2 \leqq 1\Big/\left(\prod_p (1 - p^{-3/2})\right) = \sum_{n=1}^{\infty} n^{-3/2} \leqq 1 + \int_1^{\infty} n^{-3/2} dn = 3.$$

Therefore

$$\log(1/b_4) = \log \prod_1 + \log \prod_2 < 2k^5 - \log 3 + \log 3 = 2k^5.$$

(A_1) $$\log A_1 < 9\epsilon_1 2^{\eta_1} \qquad \text{(Theorem 1)}.$$

Proof: We have

$$\begin{aligned}
\log A_1 = {} & \pi(2^{\eta_1})\log 2 + \pi((3/2)^{\eta_1})\log(3/2) \\
& + \cdots - \epsilon_1(\vartheta(2^{\eta_1}) + \vartheta((3/2)^{\eta_1}) + \cdots).
\end{aligned}$$

Since $\pi((1+j^{-1})^{\eta_1}) = 0$ and $\vartheta((1+j^{-1})^{\eta_1}) = 0$ when $(1+j^{-1})^{\eta_1} < 2$, that is, when $j > 1/(2^{\epsilon_1} - 1)$, we may write

$$\log A_1 = \sum_{j=1}^{n} \pi((1+j^{-1})^{\eta_1})\log(1+j^{-1}) - \epsilon_1 \sum_{j=1}^{n} \vartheta((1+j^{-1})^{\eta_1}),$$

where $n = [1/(2^{\epsilon_1} - 1)] + 1$. By Theorems 2 and 3

$$\begin{aligned}
\log A_1 < \alpha_2 \sum_{j=1}^{n} \frac{(1+j^{-1})^{\eta_1}}{\eta_1 \log(1+j^{-1})} \log(1+j^{-1}) - \epsilon_1 \Bigg(c \sum_{j=1}^{n} (1+j^{-1})^{\eta_1} \\
- \frac{12}{5} c \sum_{j=1}^{n} (1+j^{-1})^{\eta_1/2} - \frac{3}{2} \sum_{j=1}^{n} \log^2(1+j^{-1}) \\
- 13 \sum_{j=1}^{n} \log(1+j^{-1}) - 15(n-1) \Bigg) \\
< (\alpha_2 - c)\epsilon_1(2^{\eta_1+1} + n - 1 + (\eta_1+1)\log n) \\
+ \frac{12}{5} c\epsilon_1(2^{(\eta_1+2)/2} + n - 1 + (\eta_1/2+1)\log n) \\
+ \frac{3}{2}\epsilon_1(n-1)\log^2 2 + 13\epsilon_1 \log(n+1) + 15\epsilon_1(n-1) \qquad \text{(Lemma 1)} \\
< (\alpha_2 - c)(\epsilon_1 2^{\eta_1+1} + \epsilon_1(2^{\epsilon_1}-1)^{-1} + (1+\epsilon_1)\log((2^{\epsilon_1}-1)^{-1}+1)) \\
+ \frac{12}{5}(\epsilon_1 2^{(\eta_1+2)/2} + \epsilon_1(2^{\epsilon_1}-1)^{-1} + (1+\epsilon_1)\log((2^{\epsilon_1}-1)^{-1}+1)) \\
+ \frac{3}{2}\epsilon_1(2^{\epsilon_1}-1)^{-1} + 13\epsilon_1 \log((2^{\epsilon_1}-1)^{-1}+2) + 15\epsilon_1(2^{\epsilon_1}-1)^{-1} \\
\leqq (\alpha_2 - c)(\epsilon_1 2^{\eta_1+1} + 1/\log 2 + (1+\epsilon_1)\log((\epsilon_1 \log 2)^{-1}+1)) \\
+ \frac{12}{5}(\epsilon_1 2^{(\eta_1+2)/2} + 1/\log 2 + (1+\epsilon_1)\log((\epsilon_1 \log 2)^{-1}+1)) \\
+ 17\epsilon_1(\epsilon_1 \log 2)^{-1} + 13\epsilon_1 \log((\epsilon_1 \log 2)^{-1}+1) \qquad \text{(Lemma 2)} \\
< (\alpha_2 - c)(\epsilon_1 2^{\eta_1+1} + 1/\log 2 + (1+\epsilon_1)\log 2\eta_1) \\
+ \frac{12}{5}(\epsilon_1 2^{(\eta_1+2)/2} + 1/\log 2 + (1+\epsilon_1)\log 2\eta_1) \\
+ 17/\log 2 + 13\epsilon_1 \log 3\eta_1.
\end{aligned}$$

Since $\alpha_2 - c < 4$ this expression is $< 9\epsilon_1 2^{\eta_1}$ when $\eta_1 \geqq 12$. By (12)

$$\eta_1 \geqq 17 > 12 \text{ when } k = 6,$$

$$\eta_1 \geqq D + 2^{k-3} > 20 > 12 \text{ when } k > 6,$$

and hence $\log A_1 < 9\epsilon_1 2^{\eta_1}$.

(A_2) $\qquad \log A_2 = (k-2)\log A_1 \qquad$ (Theorem 5).

(A_3) $\qquad \log A_3 < \log \eta_1 + \log 10k \qquad$ (Theorem 7).

Proof:

$$\log A_3 = \log (1/(\epsilon_3 e^{1-\epsilon_3})) < \log (1/\epsilon_3) = \log (10k\eta_1)$$

since $10k\epsilon_3 = \epsilon_1$.

(C_{15}) $\log C_{15} = (k-2)\log A_1 + K\log 4$ (Theorem 6).

(G_3) $\log G_3 = \log A_1 + \log (4k-4)$ (Theorem 28).

(G_4) $\log G_4 < (k-2)\log A_1 + \log \eta_1 + K\log 4 + (k+2)\log k - k + 1 + \log 80$ (Theorem 30).

Proof:

$$\log G_4 = \log (8k!C_{15}A_3) = \log 8 + \log (k!) + \log C_{15} + \log A_3$$
$$< \log 8 + (k+1)\log k - k + 1 + (k-2)\log A_1 + K\log 4 + \log \eta_1 + \log 10k \quad \text{(Lemma 3)}.$$

(G_5) $G_5 < 3c_{15}$ (Theorem 31).

Proof:

$$G_5 = c_{15}(\Gamma(1+a) + 2a\gamma(a)(1-a^2)^{-1} + (2a+1)(a+1)^{-1}) < 3c_{15}.$$

(G_7) $G_7 < c_{15}$ (Theorem 32).

(G_8) $G_8 < c_{15}$ (Theorem 32).

(G_9) $G_9 = \max (G_5, G_7 + G_8) < 3c_{15}$ (Theorem 33).

(G_{10}) $G_{10} < 3\cdot 2^{2sa+k-1}G_9^{2s}$ (Theorem 34).

Proof: We have

$$\frac{2sa-1-(2sa-1)\theta}{2sa-2-(2sa-1)\theta} \leqq \frac{2sa-1+2K^2}{2sa-1} \quad \text{(by (19))}$$
$$\leqq \frac{(k-2)K+k+2kK^2}{(k-2)K+k} < 3K \quad \text{(by (2))}.$$

Hence

$$G_{10} = 2^{2sa}(2sa-1)^{-1}(2sa-1-(2sa-1)\theta)(2sa-2-(2sa-1)\theta)^{-1}G_9^{2s}$$
$$< 3\cdot 2^{2sa+k-1}G_9^{2s}.$$

(G_{13}) $G_{13} < 15G_4^A$ (Theorem 35).

Proof:

$$G_{13} = (2\pi+1)2^{1+A}G_4^A \leqq (2\pi+1)2^{33/32}G_4^A < 15G_4^A.$$

(G_{15}) $\log G_{15} < ((2s-4)(k-3/2)A+1)\log A_1$ (Theorem 36).

Proof:

$$\begin{aligned}\log G_{15} &= (2s-4)\log G_{13} + \log G_3 + (2a+\epsilon_1)\log 2\\ &= (2s-4)A\log G_4 + (2s-4)\log 15 + \log G_3 + (2a+\epsilon_1)\log 2\\ &< ((2s-4)(k-2)A+1)\log A_1 + (2s-4)A(\log\eta_1 + K\log 4\\ &\quad + (k+2)\log 2 - k + 1 + \log 80)\\ &\quad + \log(4k-4) + (2s-4)\log 15 + (2a+\epsilon_1)\log 2\\ &= ((2s-4)(k-2)A+1)\log A_1 + (s-2)A\log A_1\left(\frac{2\log\eta_1}{\log A_1} + \frac{2K\log 4}{\log A_1}\right.\\ &\quad + \frac{2(k+2)\log k}{\log A_1} - \frac{2k-2}{\log A_1} + \frac{2\log 80}{\log A_1} + \frac{\log(4k-4)}{(s-2)A\log A_1}\\ &\quad \left. + \frac{2\log 15}{A\log A_1} + \frac{(2a+\epsilon_1)\log 2}{(s-2)A\log A_1}\right).\end{aligned}$$

Each of the positive terms in the coefficient of $(s-2)A\log A_1$ is $<1/7$ when $\eta_1 \geqq 2k$ and thus

$$\log G_{15} < ((2s-4)(k-2)A+1)\log A_1 + (s-2)A\log A_1$$

since $\eta_1 > 17 > 12 = 2k$ when $k=6$ and $\eta_1 > 2^{k-3} > 2k$ when $k \geqq 7$.

(G_{18}) $$G_{18} < 25G_4^A \qquad \text{(Theorem 37)}.$$

Proof:

$$\begin{aligned}G_{18} &= (2\pi+1)(2(3G_4)^A + \gamma(a)) \leqq (2\pi+1)(2\cdot 3^{1/32}G_4^A + \gamma(1/6))\\ &\leqq (2\pi+1)(2\cdot 3^{1/32} + \gamma(1/6))G_4^A < 25G_4^A.\end{aligned}$$

(G_{24}) $$\log G_{24} < 2s(k-3/2)A\log A_1 \qquad \text{(Theorem 38)}.$$

Proof: We have

$$\begin{aligned}\frac{2s+2}{2s+1}\,\frac{2s+3-(2s+1)\theta-2s(1-D\epsilon_1)A}{2s+2-(2s+1)\theta-2s(1-D\epsilon_1)A} &< \frac{2s+2}{2s+1}\,\frac{2s(1-A-\theta)+3-\theta}{2s(1-A-\theta)+2-\theta}\\ &< \frac{2s+2}{2s+1}\,\frac{3-\theta}{2-\theta} < 2 \qquad \text{(by (20) and (2))}.\end{aligned}$$

Hence

$$G_{24} = 2^{2s+1}\frac{(2s+2)(2s+3-(2s+1)\theta-2s(1-D\epsilon_1)A)}{(2s+1)(2s+2-(2s+1)\theta-2s(1-D\epsilon_1)A)}G_{18}^{2s} < 2^{2s+2}G_{18}^{2s},$$

$$\log G_{24} < 2s \log G_{18} + (2s+2)\log 2 < 2sA \log G_4 + 2s \log 25 + (2s+2)\log 2$$
$$= 2s(k-2)A \log A_1 + 2sA(\log \eta_1 + K \log 4 + (k+2)\log k$$
$$- k + 1 + \log 80) + (2s+2)\log 2 + 2s \log 25$$
$$= 2s(k-2)A \log A_1 + sA \log A_1 \left(\frac{2\log \eta_1}{\log A_1} + \frac{2K \log 4}{\log A_1} + \frac{2(k+2)\log k}{\log A_1} \right.$$
$$\left. - \frac{2k-2}{\log A_1} + \frac{2 \log 80}{\log A_1} + \frac{(2s+2)\log 2}{sA \log A_1} + \frac{2 \log 25}{A \log A_1} \right).$$

As before the coefficient of $sA \log A_1$ is <1 and thus

$$\log G_{24} < 2s(k-2)A \log A_1 + sA \log A_1.$$

(G_{25}) $$\log G_{25} < 2s(k-3/2)A \log A_1 \qquad \text{(Theorem 38)}.$$

Proof:

$$\frac{(2sa-2-2a)(2sa-3)}{(2sa-3-2a)(2sa-4)} < \frac{((k-2)K-2)((k-2)K-k)}{((k-2)K-k-2)((k-2)K-2k)} < 2 \text{ (by (2))}.$$

Also

$$\log G_9 < \log c_{15} + \log 3 = (k-1)\log k + \alpha_2(k-2)k^{2k/(k-2)}/(2k) + \log 3$$
$$< (k-1)\log k + 7(\log 2)k^3/3 + \log 3$$
$$< (k-2)A \log A_1 \qquad (\eta_1 \geqq 2k)$$
$$< A \log G_4 < \log G_{18}.$$

Therefore

$$\log G_{25} < (2s+2)\log 2 + (2s-2)\log G_9 + 2 \log G_{18}$$
$$< (2s+2)\log 2 + 2s \log G_{18}$$
$$< 2s(k-3/2)A \log A_1$$

as in (G_{24}).

(G_{31}) $$G_{31} < 25 G_4^A \qquad \text{(Theorem 39)}.$$

Proof:

$$G_{31} = (2\pi+1)2^{1+2A+(\epsilon_2+2\epsilon_3)A} G_4^A \leqq (2\pi+1)2^{35/32} G_4^A < 25 G_4^A.$$

(G_{35}) $$\log G_{35} < 2s(k-3/2)A \log A_1 \qquad \text{(Theorem 40)}.$$

Proof:

$$\frac{(2s-K)\theta-(2s-3K)}{(2s-K)\theta-(2s-2K)} \leqq \frac{2s-K+2K^3}{2s-K} \qquad \text{(by (21))}$$

$$\leqq \frac{(k-2)K+2k-K+2K^3}{(k-2)K+2k-K} < \frac{(k-3)K+2K^3}{(k-3)K} < \frac{3+2K^2}{3} < 2K^2 = 2^{2k-1}.$$

Hence

$$G_{35} < 2^{2sA+1+2k-1}G_{31}^{2s},$$

$$\begin{aligned}\log G_{35} &< 2s \log G_{31} + (2sA+2k)\log 2\\ &< 2sA \log G_4 + 2s \log G_{25} + (2sA+2k)\log 2\\ &< 2s(k-3/2)A \log A_1,\end{aligned}$$

as in (G_{24}).

(G_1) $\qquad \log G_1 < ((2s-4)(k-3/2)A+1)\log A_1 + \log 10$ (Theorem 41).

Proof: Consider first G_{10}. We have

$$\begin{aligned}\log G_{10} &< 2s \log G_9 + (2sa+k-1)\log 2 + \log 3\\ &< 2sA \log G_4 + (2sa+k-1)\log 2 + \log 3\end{aligned}$$

by the proof for (G_{25}), and then as before

$$\log G_{10} < 2s(k-3/2)A \log A_1.$$

Also,

$$(2s-4)(k-3/2)A+1 = 2s(k-3/2)A+1-4A(k-3/2) > 2s(k-3/2)A.$$

Hence

$$\begin{aligned}G_1 &= 2(G_{10}+G_{15}+G_{24}+G_{25}+G_{35})\\ &\leqq 10 \max (G_{10}, G_{15}, G_{24}, G_{25}, G_{35}),\\ \log G_1 &< ((2s-4)(k-3/2)A+1)\log A_1 + \log 10.\end{aligned}$$

(A_4) $\qquad \log A_4 < ((2s-4)(k-3/2)A+1)\log A_1 + \log 22$ (Theorem 41).

Proof:

$$\begin{aligned}A_4 &= 2G_1 + 2(\Gamma^{2s}(1+a)/\Gamma^2(sa))(sa/(sa-1))^2(2sa/(2sa-1))^2 c_{15}^{2s} e^{2sa-2}\Gamma^2(sa)\\ &< 2G_1 + 8e^{2sa-2}c_{15}^{2s} < 2G_1 + 8e^{2sa-2}(G_{18}/3)^{2s} \qquad \text{(by } (G_{25}))\\ &< 2(G_1+G_{18}^{2s}) < 2(G_1+G_{24}) \leqq 2(10 \max (G_{10}, \cdots, G_{35}) + G_{24})\\ &< 22 \max (G_{10}, \cdots, G_{35}),\end{aligned}$$

$$\log A_4 < ((2s-4)(k-3/2)A+1)\log A_1 + \log 22.$$

(c_{104}) $$c_{104} = (\Gamma^{2s}(1+a)/\Gamma^2(sa))b_4^2\gamma_3^2 \qquad \text{(Theorem 43).}$$

(c_{105}) $$c_{105} = c_{104}2^{2-2sa} \qquad \text{(Theorem 43).}$$

(C_{68}) $$\log C_{68} = \log A_4 + \log(1/c_{105}) \qquad \text{(Theorem 43).}$$

(C_{66}) $$\log C_{66} < ((2s-4)(k-1)A+1)\log A_1 \qquad \text{(Theorem 43).}$$

Proof:

$$\begin{aligned}\log C_{66} &= \log C_{68} + \log 3 = \log A_4 + \log(1/c_{105}) + \log 3\\ &< ((2s-4)(k-3/2)A+1)\log A_1 + (2sa-2)\log 2 + 2\log\Gamma(sa)\\ &\quad - 2s\log\Gamma(1+a) + 2\log(1/b_4) + 2\log(1/\gamma_3) + \log 66\\ &< ((2s-4)(k-3/2)A+1)\log A_1 + (s-2)A\log A_1\left(\frac{(2sa-2)\log 2}{(s-2)A\log A_1}\right.\\ &\quad + \frac{2\log\Gamma(sa)}{(s-2)A\log A_1} + \frac{2s\log 2}{(s-2)A\log A_1}\\ &\quad \left. + \frac{2\log(1+12e)}{(s-2)A\log A_1} + \frac{\log 66}{(s-2)A\log A_1}\right).\end{aligned}$$

As before each of the terms of the coefficient of $(s-2)A\log A_1$ is $<1/6$ when $\eta_1 > 2k+5$. Since

$$\eta_1 > 17 = 2k+5 \text{ when } k=6, \quad \eta_1 > 33 > 19 = 2k+5 \text{ when } k=7,$$
$$\eta_1 > 2^{k-3} > 2k+5 \qquad \text{when } k \geqq 8,$$

we have

$$\log C_{66} < ((2s-4)(k-3/2)A+1)\log A_1 + (s-2)A\log A_1.$$

(c_{109}) $$c_{109} = 2^{-2a-1} \qquad \text{(Theorem 44).}$$

(c_{110}) $$c_{110} = 3^k\cdot 2^{k+1}(2^a-1)^{-k} \qquad \text{(Theorem 44).}$$

(c_{111}) $$c_{111} = 2^{-1-a}(2^a-1) \qquad \text{(Theorem 44).}$$

(C_{71}) $$\log(1/C_{71}) = (k-1)\log A_1 + \log(1/c_{109}) \qquad \text{(Theorem 44).}$$

(B_{19}) $$\log(1/B_{19}) < (2s-4)A\log A_1 + (k-1)\log A_1 \qquad \text{(Theorem 44).}$$

Proof:

$$\begin{aligned}\log(1/B_{19}) &= (s-2)\log(2/c_{111}) + \log(1/C_{71})\\ &= (k-1)\log A_1 + (s-2)\log(2^{2+a}/(2^a-1)) + (2a+1)\log 2\\ &= (k-1)\log A_1 + (2s-4)A\log A_1\left(\frac{(s-2)\log(2^{2+a}/(2^a-1))}{(2s-4)A\log A_1}\right.\\ &\quad \left. + \frac{(2a+1)\log 2}{(2s-4)A\log A_1}\right).\end{aligned}$$

As before the coefficient of $(2s-4)A \log A_1$ is <1 and thus

$$\log (1/B_{19}) < (2s-4)A \log A_1 + (k-1) \log A_1.$$

$$(C) \qquad \log C < 20k^3 2^{\eta_1} \qquad \text{(Theorem 46)}.$$

Proof:

$$\begin{aligned} \log C &= k2^{k-2}(\log C_{66} + \log (1/B_{19})) \\ &< k2^{k-2}((2s-4)kA + k) \log A_1 \\ &< 9k2^{k-2}((2s-4)kA + k)\epsilon_1 2^{\eta_1} && \text{(by } (A_1)) \\ &< 20k^3 2^{\eta_1} && \text{(by (2) and (17))}. \end{aligned}$$

9. **Proof of the main theorem.** We prove the following

THEOREM. *We have*

$$g(k) \leqq \left[\frac{1}{2}\left(H + FD + Q + E + \left((H + FD + Q - E)^2 + 4F(ED + R)\right)^{1/2}\right)\right] + 1,$$

$$\overline{\lim_{k=\infty}} \frac{g(k)}{k \cdot 2^{k-1}} \leqq \frac{1}{2}.$$

By Theorem 46 every integer $\geqq C$ is a sum of s_0 kth powers when

$$\begin{aligned} s_0 &\geqq s + s_2, \\ s_0 &\geqq (H + (1 + (1-a)^{s_2-2})k2^{k-2}\epsilon_1)(1 - D\epsilon_1)^{-1} + 2 + 4 + \zeta_k \\ &= (H\eta_1 + (1 + (1-a)^{s_2-2})k2^{k-2} + (\eta_1 - D)(6 + \zeta_k))(\eta_1 - D)^{-1}, \end{aligned}$$

$$(50) \qquad s_0 \geqq ((H + Q)\eta_1 + R)(\eta_1 - D)^{-1}.$$

Also, by Theorem 50 every integer $<C$ is a sum of s_0 kth powers if

$$\begin{aligned} s_0 &\geqq s_3 + s_4, \\ s_0 &\geqq s_3 + (\log \log C - \log (\log L - k \log k))(\log k - \log (k-1))^{-1} \\ &= s_3 + (3 \log k + \log 20 + \eta_1 \log 2 - \log (\log L - k \log k)) \\ &\qquad \times (\log k - \log (k-1))^{-1} && \text{(by } (C)), \end{aligned}$$

$$(51) \qquad s_0 \geqq F\eta_1 + E.$$

The right members of (50) and (51) are equal when

$$(52) \qquad \eta_1 = \left(H + FD + Q - E + \left((H + FD + Q - E)^2 + 4F(ED + R)\right)^{1/2}\right)(2F)^{-1}$$

and then *every* integer is a sum of s_0 kth powers $\geqq 0$ when

$$s_0 > \frac{1}{2}\left(H + FD + Q + E + \left((H + FD + Q - E)^2 + 4F(ED + R)\right)^{1/2}\right).$$

It remains to show that this choice of η_1 satisfies the condition (17). Since $E > 2^k$ we have $ED + R > 0$ and thus

$$\eta_1 > (H + FD + Q - E)F^{-1},$$

$$\eta_1 - D > (H + Q - E)F^{-1}. \tag{53}$$

Also,

$$\begin{aligned} H + Q - E &= (k-2)2^{k-2} + k + 6 + \zeta_k - s_3 \\ &\quad - (3\log k + \log 20 - \log(\log L - k\log k))(\log k - \log(k-1))^{-1} \\ &> (k-2)2^{k-2} + k + 6 + \zeta_k - 2^k \\ &\quad - \left(\left(\frac{3}{2}\right)^k + 2\left(\frac{4}{3}\right)^k + 2\left(\frac{2}{3}\right)^k + 2\left(\frac{1}{2}\right)^k + \frac{k(2k+7)}{9} - 9\right) \\ &\quad - (3\log k + \log 20 - \log(\log L - k\log k))(\log k - \log(k-1))^{-1} \\ &= (k-6)2^{k-2} + k + 6 + \zeta_k \\ &\quad - \left(\left(\frac{3}{2}\right)^k + 2\left(\frac{4}{3}\right)^k + 2\left(\frac{2}{3}\right)^k + 2\left(\frac{1}{2}\right)^k + \frac{k(2k+7)}{9} - 9\right) \\ &\quad - (3\log k + \log 20 - \log(\log L - k\log k))(\log k - \log(k-1))^{-1}; \end{aligned}$$

and

$$F = \log 2(\log k - \log(k-1))^{-1}.$$

Hence $(H+Q-E)F^{-1} - 2^{k-3}$ is an increasing function of k which is positive when $k=7$ so that

$$(H + Q - E)F^{-1} - 2^{k-3} \geqq 0$$

for all $k \geqq 7$. Then from (53)

$$\eta_1 - D \geqq 2^{k-3}$$

for all $k \geqq 7$. When $k=6$ direct substitution in (52) yields $\eta_1 > 17$.

To obtain the values of $g(k)$ which are given in the introduction we require the following:

Every integer from	to	is a sum of
$11\cdot 2^6$	$12\cdot 2^6$	39 6th powers
$25\cdot 2^7+6\cdot 3^7$	$26\cdot 2^7+6\cdot 3^7$	58 7th powers
$25\cdot 2^8+9\cdot 3^8$	$26\cdot 2^8+9\cdot 3^8$	120 8th powers
$38\cdot 2^9$	$39\cdot 2^9$	285 9th powers
$57\cdot 2^{10}$	$58\cdot 2^{10}$	737 10th powers

By repeated application of Theorem 47 as indicated in the proof of Theorem 48 we obtain the following values for L and s_3.

Every positive integer $\leqq$	is a sum of
$10^{17\cdot 98}$	73 6th powers
10^{1948}	143 7th powers
$10^{3920000}$	279 8th powers
$10^{10^{9\cdot 7}}$	548 9th powers
$10^{10^{11\cdot 1}}$	1079 10th powers

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