

WARING'S PROBLEM FOR CUBIC FUNCTIONS*

BY

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1. Introduction. L. E. Dickson† has proved that all integers sufficiently large are sums of nine values of $f(x) = x + \epsilon(x^3 - x)/6$, where ϵ is prime to 3. In §6 of this paper the author considers the above function $f(x)$ with $\epsilon = 3a$. For $a \equiv 0$ or $1 \pmod{3}$ he obtains the same result as Dickson obtained for ϵ prime to 3. However, for $a \equiv 2 \pmod{3}$ it is proved that every integer sufficiently large is expressible as a sum of *ten* values of $f(x)$.

Certain classes of cubic functions with the square term present are treated in §§2-5, inclusive, the results being stated in Theorems 1, 2, and 3. These results are analogous to those stated for polynomials without square term.

In the same paper Dickson showed that *all* positive integers are sums of nine values of $f(x) = (x^3 + 2x)/3$ and stated the possibility of such a theorem for $f(x) = (x^3 + 5x)/6$. Miss Frances Baker‡ proved a universal theorem for representation of weight nine by $f(x) = (x^3 + x)/2$. The only cubic functions of the form $f(x) = x + \epsilon(x^3 - x)/6$ for which it is possible to obtain a universal theorem giving representation of weight nine are those for which ϵ takes one of the values $1, \dots, 6$. The author proves in §7 that every integer may be represented as a sum of fifteen values of $f(x) = x^3 + 3(x^2 - x)$ for values ≥ 0 of x . Since 41 requires fifteen values this is the best theorem obtainable.

2. Determination of all functions (1) having certain properties. We consider cubic functions of the form

$$(1) \quad f(x) = \frac{ax^3 + b_0x^2 + cx}{d}, \quad a > 0, \quad b_0 \neq 0,$$

where a, b_0, c and d are integers having no common divisor greater than 1. Further, in order that a true Waring's Problem be considered, it is stipulated that the coefficients of $f(x)$ must satisfy the following conditions:

(a) that the values of $f(x)$ be positive integers for all integral values ≥ 0 of x ,

(b) that the function have the value 1 for some integral value $\xi \geq 0$ of x .

The quantities $f(1), f(2)$ and $f(3)$ will be integral if d divides each of

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$a + b_0 + c$, $8a + 4b_0 + 2c$, and $27a + 9b_0 + 3c$. Eliminate c and b_0 from these three expressions; the result of this process is that d divides $6a$. Consequently d divides each of $2b_0$ and $6c$. If d has a prime factor $p > 3$, p divides each of a , b_0 , and c contrary to hypothesis. Thus the only prime factors of d are 2 and 3. Since $f(\xi) = 1$, d must satisfy

$$(2) \quad d = a\xi^3 + b_0\xi^2 + c\xi.$$

Case I, $d = 6\omega$. Since ω divides each of a , b_0 and c , then $\omega = 1$. From (2), ξ is a positive divisor of 6. Since $d = 6$ divides $2b_0$, let $b_0 = 3b$.

I_1 , $\xi = 1$. Thus, from (2), $c = 6 - a - 3b$, and

$$f(x) = \frac{a}{6}(x^3 - x) + \frac{b}{2}(x^2 - x) + x.$$

If $b < 0$, write $b = -b_1$. Then necessary and sufficient conditions that $f(x)$ satisfy property (a) above are $b > 0$, or if $b < 0$ then $0 < b_1 \leq a + 2$ if $a \geq 3$, and $0 < b_1 \leq (4a + 4)/3$ if $a = 1$ or 2.

I_2 , $\xi = 2$. Thus $c = 3 - 4a - 6b$ and

$$f(x) = \frac{a}{6}(x^3 - x) + \frac{b}{2}(x^2 - x) + \frac{1}{2}(1 - a - b)x.$$

In order that $f(x)$ be integral a and b must have different parity. From $f(1) \geq 0$, $b \leq 1 - a$ requires $b < 0$. Let $b = -b_1$; thus $b_1 \geq a - 1$. Also, $f(3) \geq 0$, $f(4) \geq 0$ and $f(5) \geq 0$ require $b_1 \leq \eta$, where $\eta = (4a + 1)/2$ if $a = 1$ and $\eta = (5a + 3)/3$ if $a > 1$. Accordingly, the conditions $a - 1 \leq b_1 \leq \eta$ and $a - b_1 \equiv 1 \pmod{2}$ are necessary and sufficient that

$$f(x) = \frac{a}{6}(x^3 - x) - \frac{b_1}{2}(x^2 - x) + \frac{1}{2}(1 - a + b_1)x$$

satisfy conditions (a) and (b).

I_3 , $\xi = 3$. With $c = 2 - 9a - 9b$ the conditions $f(1) \geq 0$, $f(2) \geq 0$, and $f(4) \geq 0$ require that $b = -b_1$, $b_1 > 0$ and $(5a - 2)/3 \leq b_1 \leq (7a + 2)/3$. The function $f(x)$ becomes

$$f(x) = \frac{a}{6}(x^3 - x) - \frac{b_1}{2}(x^2 - x) + \frac{1}{3}(1 - 4a + 3b_1)x.$$

Further, $a \equiv 1 \pmod{3}$.

I_4 , $\xi = 6$. Conditions (a) and (b) require

$$f(x) = \frac{a}{6}(x^3 - x) - \frac{b_1}{2}(x^2 - x) + \frac{1}{6}(1 - 35a + 15b_1)x,$$

where a and b_1 are such that $a \equiv 3b_1 - 1 \pmod{6}$ and $(20a - 1)/6 \leq b_1 \leq (13a + 1)/3$.

Case II, $d = 3\omega$, ω odd. As in I, $\omega = 1$ and $b_0 = 3b$. From (2), $\xi = 1$ or 3.

II₁, $\xi = 1$. With $c = 3 - a - 3b$,

$$f(x) = \frac{a}{3}(x^3 - x) + b(x^2 - x) + x.$$

Condition (a) requires that $b \geq (-3 - 8a)/6$.

II₂, $\xi = 3$. Since $c = 1 - 9a - 9b$,

$$f(x) = \frac{a}{3}(x^3 - x) - b_1(x^2 - x) + \frac{1}{3}(1 - 8a + 6b_1)x.$$

Condition (a) requires that $a \equiv 2 \pmod{3}$ and that $(5a - 1)/3 \leq b_1 \leq (7a + 1)/3$.

Case III, $d = 2\omega$, $(\omega, 3) = 1$. According to hypothesis $\omega = 1$; from (2) $\xi = 1$ or 2.

III₁, $\xi = 1$. Then

$$f(x) = \frac{a}{2}(x^3 - x) + \frac{b_0}{2}(x^2 - x) + x,$$

where $b_0 \geq -2 - 3a$.

III₂, $\xi = 2$. Then

$$f(x) = \frac{a}{2}(x^3 - x) + \frac{b_0}{2}(x^2 - x) + \frac{1}{2}(1 - 3a - b_0)x,$$

where $a + b_0 \equiv 1 \pmod{2}$, $b_0 < 0$ and $3a - 1 \leq -b_0 \leq 5a + 1$.

Case IV, $d = 1$. From (2), $\xi = 1$. This requires

$$f(x) = a(x^3 - x) + b_0(x^2 - x) + x,$$

where $b_0 \geq -1 - 3a$.

Consider

$$(3) \quad f(x) = \frac{p}{6}(x^3 - x) + \frac{q}{2}(x^2 - x) + ux, \quad p > 0, \quad q \neq 0,$$

where p , q and u are integers satisfying necessary and sufficient conditions, as stated in Cases I-IV, that $f(x)$ be integral and ≥ 0 for all integral values ≥ 0 of x . The substitution $x = X + t$, with $q = -tp$, transforms (3) into

$$(4) \quad f(x) = F(X) + \alpha,$$

where

$$\begin{aligned}
 F(X) &= \frac{p}{6}(X^3 - X) + gX, \\
 \alpha &= ut + \frac{1}{6}(3pt^2 - 2pt^3 - pt) \quad (\text{an integer}), \\
 g &= u - \frac{1}{2}pt(t - 1).
 \end{aligned}$$

In §§3, 4, and 5 the only values of p and q considered are those for which the above transformation is possible, i.e., values such that $q = -tp$, t an integer.

3. The functions (5) for $(p, 3) = 1$. The functions to be studied in this section are

$$(5) \quad F(X) = \frac{p}{6}(X^3 - X) + gX \quad (p > 0, (p, 3) = 1, p \text{ and } q \text{ integers}).$$

The investigation is entirely analogous to that of L. E. Dickson in his Transactions paper mentioned heretofore except that the inequalities require $X \geq |t|$. This ensures that $x \geq 0$.

We prove

THEOREM 1. *Let the triple of integers p, q, u be given satisfying the conditions stated at the end of §2, $(p, 3) = 1$, and let α be defined as under (4). Then there exist integers C and v such that every integer $\geq C \cdot 3^{3v} + 9\alpha$ is a sum of nine values of (3) for integral values ≥ 0 of x .*

Let $|t| \leq 3^\delta$, δ being an integer ≥ 0 .

The following three lemmas are necessary.

LEMMA 1. *Let the integers t and δ be defined as above. Corresponding to any positive integer s there exists an integer m' such that s is congruent to $F(3m')$ modulo 3^n , where $|t| \leq 3m' < 3^{n+1} + 3^\delta$.*

Define Δ by

$$\Delta = F(z + 3r) - F(z) = \frac{p}{2}(3rz^2 + 9r^2z + 9r^3 - r) + 3gr.$$

It may be proved by induction that $\Delta \not\equiv 0 \pmod{3^n}$ if and only if $r \not\equiv 0 \pmod{3^n}$. Let m'' be an arbitrary integer such that $0 \leq m'' < 3^n$ and let k' be an integer such that $0 \leq k' < 3^n$, $k' < m''$. Then, if m' and k are defined by $m' = m'' + 3^{\delta-1}$ and $k = k' + 3^{\delta-1}$, we obtain $m' - k = m'' - k' \not\equiv 0 \pmod{3^n}$. Use this value $m' - k$ as an r in Δ . Then

$$F(3m') - F(3k) = F\{3k + 3(m' - k)\} - F(3k) \not\equiv 0 \pmod{3^n}.$$

Since m'' ranges over a complete residue system modulo 3^n , m' does likewise, hence the same is true of $F(3m')$. From $3m' = 3m'' + 3^s$ it follows that $3^s \leq 3m' < 3^{n+1} + 3^s$. This proves the lemma.

Lemma 2 is taken directly from the Dickson Transactions paper.

LEMMA 2. *If η is an odd constant integer, $v(\eta - v)$ is even and can be made congruent to any assigned even integer modulo 2^k by choice of an integer v .*

LEMMA 3. *If $n \geq \max(3, \delta)$, $g \leq 13p + 1$, and $3m < 3^{n+1}p + 3^s$, then $F(3m) < 3^{3n}\gamma$, where*

$$(6) \quad \gamma = \frac{p}{2}(9p^3 + 9p^2 + 4p + 1).$$

Since $X^3 - X$ is monotone increasing,

$$\begin{aligned} F(3m) &= \frac{p}{6}(27m^3 - 3m) + 3gm \leq \frac{p}{2}(9m^3 - m) + 3(13p + 1)m \\ &< \frac{p}{2}(9 \cdot 3^{3n}p^3 + 9 \cdot 3^{2n+\delta}p^2 + 9 \cdot 3^{n+2\delta-1}p + 9 \cdot 3^{3\delta-3} - 3^np - 3^{\delta-1}) \\ &\quad + (13p + 1)(3^{n+1}p + 3^s) \\ &< \frac{3^{3n}p}{2}(9p^3 + 9p^2 + 4p + 1) = 3^{3n}\gamma. \end{aligned}$$

According to Lemma 1 every integer s may be written as $s = F(3m') + 3^nM'$, where M' is an integer. Substituting $z = 3m'$ and $r = 3^ny$ in Δ , and writing $\Delta = 3^nE$, we obtain

$$(7) \quad E = (p/2)(3yz^2 + 9 \cdot 3^ny^2z + 9 \cdot 3^{2n}y^3 - y) + 3gy;$$

also, with $m = m' + 3^ny$,

$$F(3m) - F(3m') = F(z + 3r) - F(z) = \Delta = 3^nE.$$

Thus

$$(8) \quad s = F(3m) + 3^nM,$$

where $M = M' - E$ is an integer. Later we will choose y such that $0 \leq y < p$. With these values of y and m' the upper bound used in Lemma 3 is obtained, $|s| \leq 3m < 3^{n+1}p + 3^s$.

Consideration of the values of p , q , and u of the different sub-cases in I and II of §2 shows that $13p + 1$ is the upper bound* of g ; this was used in Lemma 3. For example, in Case I₄,

* If universal theorems are desired it is advantageous to lower this upper bound, this being possible when a particular function is being considered. The upper bound of $F(3m_i)$ may usually be lowered by consideration apart from the general theory.

$$g = \frac{1}{6}(1 - 35p - 18q) - \frac{q^2}{2p} \leq -3q \leq 13p + 1,$$

since $1 - 35p < 0$ and $-3q \leq 13p + 1$.

The integer s lies in some interval

$$C \cdot 3^{3n} \leq s < C \cdot 3^{3n+3}$$

and thus in one of the sub-intervals

$$3^{i-1}C \cdot 3^{3n} \leq s_i < 3^i C \cdot 3^{3n} \quad (i = 1, 2, 3).$$

Since $f(x) \geq 0$, $F(X) \geq -\alpha$, and thus $-\alpha \leq F(3m_i) < 3^{3n}\gamma$. From $3^{i-1}C \cdot 3^{3n} \leq F(3m_i) + 3^n M_i < 3^i C \cdot 3^{3n}$ and the last inequality we obtain

$$(9) \quad (3^{i-1}C - \gamma)3^{3n} < M_i \leq 3^i C \cdot 3^{3n} + \frac{\alpha}{3^n}.$$

Six functional values to be used in the representation of s have the sum

$$(10) \quad T_i = \sum_{j=1}^3 \{F(3^n - X_j) + F(3^n + X_j)\} = p(3^{3n} + 3^n Q_i - 3^n) + 6g \cdot 3^n,$$

where $Q_i = \sum_{j=1}^3 X_j^2$. The two remaining values to be used are given by

$$(11) \quad \phi_i = F(v_i) + F(w_i) = (v_i + w_i) \left[\frac{p}{6}(v_i^2 - v_i w_i + w_i^2 - 1) + g \right].$$

Let $v_i + w_i = 3b_i 3^n$, where b_i is an odd positive integer. Then $\phi_i = 3^n B_i$, where

$$(12) \quad B_i = 3b_i \left[\frac{p}{6} \{9 \cdot 3^{2n} b_i^2 - 3v_i(3b_i \cdot 3^n - v_i) - 1\} + g \right].$$

A necessary and sufficient condition that there exist values of X of the forms $3m_i, 3^n - X_j, 3^n + X_j, v_i$ and w_i ($j=1, 2, 3$) for which s_i is expressible as the sum of nine values of (5) is that

$$s_i = F(3m_i) + 3^n M_i = F(3m_i) + \phi_i + T_i,$$

or

$$(13) \quad \begin{aligned} 3^n M_i &= \phi_i + T_i = 3^n B_i + p(3^{3n} + 3^n Q_i - 3^n) + 6g \cdot 3^n, \\ p Q_i &= M_i - B_i - p(3^{3n} - 1) - 6g. \end{aligned}$$

The value of Q_i as defined in (13) will be shown later to be integral.

We proceed to introduce inequalities which will enable us to choose the desired constants b_i and which will ensure that the arguments of $F(X)$ are $\geq |t|$.

Choose v_i and Q_i such that

$$(14) \quad 3^\delta \leq v_i \leq 3b_i \cdot 3^n - 3^\delta, \quad 0 \leq Q_i \leq 3^{2n-2}.$$

The first inequality of (14) implies that $3^\delta \leq w_i \leq 3b_i \cdot 3^n - 3^\delta$ and thus that $v_i \geq |t|$, $w_i \geq |t|$. The second inequality requires that $X_i \leq 3^{n-1}$; thus $3^n - X_i \geq 3^n - 3^{n-1} = 2 \cdot 3^{n-1} \geq 3^\delta \geq |t|$ if $n \geq \delta + 1$.

Let $V_i = v_i - 3b_i \cdot 3^n / 2$. From (13), $Q_i \geq 0$ if $V_i^2 \leq A_i$, where

$$(15) \quad A_i = \left\{ \frac{M_i - p(3^{2n} - 1) - 6g}{3b_i} - g \right\} \frac{2}{p} - \frac{3}{4} b_i^2 \cdot 3^{2n} + \frac{1}{3};$$

also $Q_i \leq 3^{2n-2}$ if $V_i^2 \geq G_i$, where $G_i = A_i - 2 \cdot 3^{2n-1} / b_i$.

The inequalities (14) will be satisfied if the following are satisfied:

$$(16) \quad \begin{aligned} A_i &\geq 0, & A_i^{1/2} &\geq V_i, & V_i &\geq 0, \\ G_i &\geq 0, & G_i^{1/2} &\leq V_i, & A_i &\leq \frac{3}{2} b_i \cdot 3^n - 3^\delta. \end{aligned}$$

This gives the range on v_i , viz.,

$$(17) \quad G_i^{1/2} + \frac{3}{2} b_i \cdot 3^n \leq v_i \leq A_i^{1/2} + \frac{3}{2} b_i \cdot 3^n.$$

The two inequalities $G_i \geq 0$ and $A_i^{1/2} \leq (3/2)b_i \cdot 3^n - 3^\delta$ together with (17) are sufficient that (16) be satisfied. Accordingly $G_i \geq 0$ if

$$M_i \geq \left\{ \left(\frac{3}{4} b_i^2 3^{2n} - \frac{1}{3} \right) \frac{p}{2} + g \right\} 3b_i + 6g + p(3^{2n} + 3^{2n-2} - 1) = l_i,$$

and $A_i \leq (3b_i \cdot 3^n / 2 - 3^\delta)^2$ if

$$M_i \leq \left\{ \left(3b_i^2 3^{2n} - 3b_i 3^{n+\delta} + 3^{2\delta} - \frac{1}{3} \right) \frac{p}{2} + g \right\} 3b_i + 6g + p(3^{2n} - 1) = L_i.$$

The relation $l_i \leq M_i \leq L_i$ will be satisfied if $l_i \leq$ the lower bound in (9) and $L_i \geq$ the upper bound in (9). From these last inequalities we obtain

$$(18) \quad \frac{l_i}{3^{2n}} + \gamma \leq 3^{i-1}C \leq \frac{L_i}{3^{2n+1}} - \frac{\alpha}{3^{2n+1}} \quad (i = 1, 2, 3).$$

When n is sufficiently large, i.e., $n - \delta$ is sufficiently large, certain terms of (18) having a power of 3 in the denominator are negligibly small. The constants b_1, b_2, b_3 and C are determined so as to satisfy (18) with these terms omitted; that is,

$$(19) \quad \frac{9}{8}pb_i^3 + \frac{10}{9}p + \gamma \leq 3^{i-1}C \leq \frac{3}{2}pb_i^3 + \frac{p}{3}.$$

Then for $n \geq n_1$, say, these same constants will satisfy (18). Write (19) in the form $I_i \leq 3_i^{i-1}C \leq S_i$.

The method used here for the choice of b_1, b_2, b_3 and C differs somewhat from that used by Dickson. For $p=1$ and $p=2$ the following choice of these constants satisfies (19):

p	b_1	b_2	b_3	C
1	5	7	11	168
2	9	13	19	1760

Case $p \equiv 1 \pmod{3}$, $p = 3e + 1$, e an integer ≥ 0 . Take b_1, b_2 and b_3 as linear combinations of e . For the coefficient of e in b_1 choose the least even integer for which $I_1 \leq S_1$ as far as the coefficient of e^4 is concerned* and for the constant term the value of b_1 displayed in the above table. The coefficient of e in b_2 is taken as the least even integer for which the coefficient of e^4 in $S_2/3$ is \geq that in I_1 , the constant term being chosen as before. Similarly, choose b_3 such that the coefficient of e^4 in $S_3/9$ is \geq the maximum of the coefficients of e^4 in I_1 and $I_2/3$. Take C to be the quartic polynomial in e whose coefficients are integers not less than the corresponding coefficients in $I_3/9$ and differing from them by at most a quantity less than unity. The following constants satisfy (19):

$$b_1 = 8e + 5, \quad b_2 = 12e + 7, \quad b_3 = 18e + 11,$$

$$C = 2228e^4 + 4806e^3 + 3830e^2 + 1329e + 168.$$

Case $p \equiv 2 \pmod{3}$, $p = 3e + 2$, e an integer ≥ 0 . Apply the method outlined above subject to the explanation given in the footnote. Choose

$$b_1 = 10e + 9, \quad b_2 = 14e + 13, \quad b_3 = 20e + 19,$$

$$C = 3740e^4 + 12,456e^3 + 15,510e^2 + 8551e + 1760.$$

The coefficients in C were chosen as above from the corresponding coefficients in I_1 . This choice satisfies (19).

We prove that Q_i is an integer. From (13), $M_i \equiv M_i' - E \equiv B_i + 6g \pmod{p}$, and from (7), $E \equiv 3gy \pmod{p}$. The definition of g , the respective values of u in Cases I and II of §2, and $q \equiv 0 \pmod{p}$ give $g \equiv u \pmod{p}$, $(u, p) = 1$, and thus $(g, p) = 1$. Accordingly, y may be chosen such that E is congruent

* In some cases, with this choice of the coefficient of e in b_1 it is not possible to choose the coefficient of e^4 in C to satisfy $I_1 \leq C \leq S_1$ and $I_2 \leq 3C \leq S_2$. In these cases take the next even integer as this coefficient.

to any assigned integer modulo p , and thus such that $E \equiv M_i' - B_i - 6g \pmod{p}$. Hence Q_i is an integer.

The range D_i of values of v_i , from (17), is

$$D_i = A_i^{1/2} - G_i^{1/2} = A_i^{1/2} \{ 1 - (1 - \mu_i)^{1/2} \} = \frac{A_i^{1/2} \mu_i}{1 + (1 - \mu_i)^{1/2}}$$

$$= A_i^{1/2} \cdot \frac{3^{2n-1}}{b_i A_i} \cdot \frac{2}{1 + (1 - \mu_i)^{1/2}} > \frac{3^{2n-1}}{b_i A_i^{1/2}},$$

where

$$\mu_i = \frac{2 \cdot 3^{2n-1}}{b_i A_i}.$$

From (15) and (9)

$$A_i < 3^{2n} \left[\frac{2 \cdot 3^i \cdot C}{3b_i p} - \frac{2}{3b_i} - \frac{3}{4} b_i^2 + 1 \right].$$

Therefore

$$D_i > \frac{3^{n-1}}{b_i \left(\frac{2 \cdot 3^i \cdot C}{3b_i p} - \frac{2}{3b_i} - \frac{3}{4} b_i^2 + 1 \right)^{1/2}},$$

which for $n \geq n_2$, say, exceeds 8.

The quantity Q_i is representable as the sum of three integral squares. For, from (13), $2pQ_i \equiv 2M_i - 12g - 2B_i \pmod{8p}$. Take $3b_i \cdot 3^n = \eta$ in Lemma 2 and choose v_i modulo 8 such that $v_i(3b_i \cdot 3^n - v_i) \equiv 2\zeta_i \pmod{8}$, where ζ_i is an arbitrary integer. Thus, from (12),

$$2B_i \equiv -3pb_i \cdot v_i(3b_i \cdot 3^n - v_i) + 6gb_i \equiv 6gb_i - 6pb_i \zeta_i \pmod{8p}.$$

By choice of y we made $M_i \equiv 6g + 3gb_i \pmod{p}$, from which $M_i = 6g + 3gb_i + k_i p$, where k_i is an integer. Substitute these relations for M_i and $2B_i$ into the above congruence for $2pQ_i$. Thus

$$2pQ_i \equiv 2k_i p + 6pb_i \zeta_i \pmod{8p},$$

$$Q_i \equiv k_i + 3b_i \zeta_i \pmod{4}.$$

Since $(3b_i, 4) = 1$, ζ_i can be chosen such that $Q_i \equiv 1 \pmod{4}$.*

It has been shown that every integer $s \geq C \cdot 3^\nu$, where $\nu = \max(3, \delta + 1, n_1, n_2)$, is a sum of nine values of (5) for values of $X \geq |t|$, the arguments of $F(X)$ being $3m_i, 3^n - X_j, 3^n + X_j, v_i$ and w_i ($j = 1, 2, 3$). Thus every integer

* A sufficient condition that Q_i be representable as the sum of three integral squares is that Q_i be not of the form $4^a(8b+7)$, where $a \geq 0, b \geq 0, a$ and b being integers. See Landau, *Vorlesungen über Zahlentheorie*, p. 123, Theorem 187.

$s \geq C \cdot 3^{3r} + 9\alpha$ is a sum of nine values of $f(x) = F(X) + \alpha$ for $X \geq |t|$ or $x \geq 0$. Theorem 1 is immediate.

4. Functions (5) with $p = 3p_1, p_1 \not\equiv 2g \pmod{3}$. The following theorem will be proved:

THEOREM 2. *Let the integers p, q and u satisfy the conditions $p = 3p_1, p_1 \not\equiv 2g \pmod{3}, q = -3p_1t, t$ an integer, $u = 1$ or $(1 - 3p_1 - q)/3$, and let α be defined by (4). Then for each such triple p, q and u there exist constants C and ν such that every integer $s \geq C \cdot 3^{3r} + 9\alpha$ is a sum of nine integral values ≥ 0 of (3) for integral values ≥ 0 of x .*

For this section the function defined in (5) becomes

$$(20) \quad G(X) = \frac{p_1}{2}(X^3 - X) + gX, \quad p_1 > 0, \quad g = u - \frac{3}{2}p_1t(t - 1).$$

Let $|t| \leq 3^\delta, \delta$ being an integer ≥ 0 .

LEMMA 4. *For each integer s there exists an integer $m', |t| \leq 3^\delta \leq m' < 3^n + 3^\delta$, such that $s \equiv G(m') \pmod{3^n}$.*

When we note that $g \equiv u \equiv 1$ or $2 \pmod{3}$ according as $G(X)$ comes under Cases III₁, IV, or III₂ of §2, and denote $m' + 3^\delta$ by $m', k' + 3^\delta$ by k , the proof of Lemma 4 is analogous to that of Lemma 1.

Consideration of the possible values of p, q , and u as noted in III₁, III₂, and IV gives $g \leq 1$.

LEMMA 5. *If $n \geq \delta + 1, g \leq 1$ and $3^\delta \leq m < 3^n(3p_1 + 1) + 3^\delta$, then $G(m) < 3^{3n}\gamma$, where*

$$\gamma = \frac{p_1}{2}[(3p_1 + 1)^3 + (3p_1 + 1)^2 + (3p_1 + 1) + 3].$$

For,

$$\begin{aligned} G(m) &< \frac{p_1}{2}[\{3^n(3p_1 + 1) + 3^\delta\}^3 - \{3^n(3p_1 + 1) + 3^\delta\}] \\ &\quad + 3^n(3p_1 + 1) + 3^\delta \\ &< \frac{3^{3n}p_1}{2}[(3p_1 + 1)^3 + (3p_1 + 1)^2 + (3p_1 + 1) + 1] + 3^n(3p_1 + 2) \\ &< 3^{3n}\gamma. \end{aligned}$$

The integer s may be written, by Lemma 4 and the method used to obtain (7) and (8), in the form

$$(21) \quad s = G(m) + 3^n M \quad (M = M' - E \text{ an integer}),$$

where

$$(22) \quad E = \frac{p_1}{2}(3yz^2 + 3 \cdot 3^n y^2 z + 3^{2n} y^3 - y) + gy.$$

The inequalities and equalities (9)–(19) inclusive along with the arguments relative to them are applicable to this section when F is replaced by G , and p by $3p_1$. The constants b_1, b_2, b_3 and C must satisfy

$$(23) \quad \frac{27}{8} p_1 b_1^3 + \frac{10}{3} p_1 + \gamma \leq 3^{i-1} C \leq \frac{9}{2} p_1 b_1^3 + p_1.$$

For $p_1 = 1, 2,$ and 3 the following values of these constants satisfy (23):

p_1	b_1	b_2	b_3	C
1	5	7	11	505
2	9	13	19	5330
3	9	13	19	9061

Case $p_1 \equiv 0 \pmod{3}$, $p_1 = 3e$, e an integer ≥ 1 . Choose

$$b_1 = 8e + 1, \quad b_2 = 12e + 1, \quad b_3 = 18e + 1,$$

$$C = 6683e^4 + 2100e^3 + 324e^2 + 30e.$$

This set of values satisfies (23). This C is obtained from the polynomials in e which represent I_1 and $I_3/9$ when the above values of b_1, b_2 and b_3 are substituted in them.

Case $p_1 \equiv 1 \pmod{3}$, $p_1 = 3e + 1$, e an integer ≥ 0 . The values

$$b_1 = 8e + 5, \quad b_2 = 12e + 7, \quad b_3 = 18e + 11,$$

$$C = 6683e^4 + 14,432e^3 + 11,505e^2 + 3992e + 505$$

satisfy (23).

Case $p_1 \equiv 2 \pmod{3}$, $p_1 = 3e + 2$, e an integer ≥ 0 . The values

$$b_1 = 8e + 9, \quad b_2 = 12e + 13, \quad b_3 = 18e + 19,$$

$$C = 6683e^4 + 25,529e^3 + 36,223e^2 + 22,575e + 5330$$

satisfy (23).

The quantity Q_i as defined by (13) with $p = 3p_1$ is an integer. For, from (22),

$$2E \equiv (2g - p_1)y \pmod{3p_1}.$$

Case p_1 is odd. From the definition of g , $2g \equiv 2u \equiv 1$ or $2 \pmod{p_1}$, and so $(2g - p_1, p_1) = 1$. This, together with $(2g - p_1, 3) = 1$, gives $(2g - p_1, 3p_1) = 1$. Accordingly, by choice of y modulo $3p_1$, $2E$ may be made congruent to any assigned integer modulo $3p_1$, from which it follows that the same is true of E .

Case p_1 is even, $p_1 = 2p_2$. Since $g - p_2 \equiv 1 \pmod{p_2}$, $(g - p_2, p_2) = 1$, and so $(g - p_2, 3p_2) = 1$. Therefore, from this and $E \equiv (g - p_2)y \pmod{3p_2}$, it follows that E may be made congruent to any assigned integer modulo $3p_2$ by choice of y modulo $3p_2$. Write $E = k + \rho \cdot 3p_2$, where k and ρ are integers, k being arbitrary and $0 \leq k < 3p_2$. Let E' be the expression E with y replaced by $y + 3p_2$. Hence

$$E' - E = 3p_2^2 [3z^2 + 3^{n+1}z(2y + 3p_2) + 3^{2n}(3y^2 + 9yp_2 + 9p_2^2) - 1] + 3gp_2.$$

When p_1 is even, g is odd, and hence $E' - E$ is an odd multiple of $3p_2$. Accordingly, if we choose y modulo $3p_1$ we obtain for each value of k two values of ρ , one even and one odd. Thus there are $3p_1$ values of $E \equiv k + \rho \cdot 3p_2 \pmod{3p_1}$, where $0 \leq k < 3p_2$, $\rho = 0, 1$, and these values are incongruent modulo $3p_1$, each to each. Hence, by choice of y modulo $3p_1$, E may be made congruent to any assigned integer modulo $3p_1$.

Choose y such that $E \equiv M_i' - B_i - 6g \pmod{3p_1}$. This choice, according to (13), makes Q_i an integer.

As in §3, for n sufficiently large, $n \geq n_2$ say, $D_i > 8$. From (13) $6p_1Q_i \equiv 2M_i - 12g - 2B_i \pmod{8p_1}$. Using Lemma 2 we may make $v_i(3b_i \cdot 3^n - v_i) \equiv 2\zeta_i \pmod{8}$, where ζ_i is arbitrary. Accordingly, $2B_i \equiv 6gb_i - 18p_1b_i\zeta_i \pmod{8p_1}$. By the above choice of y , $M_i \equiv B_i + 6g \equiv 3gb_i + 6g \pmod{3p_1}$, and hence $M_i = 6g + 3gb_i + h_i \cdot 3p_1$, where h_i is an integer. Substituting these expressions for $2B_i$ and M_i in the above congruence involving $6p_1Q_i$, we obtain

$$6p_1Q_i \equiv 6h_i p_1 + 18p_1 b_i \zeta_i \pmod{8p_1},$$

$$Q_i \equiv h_i + 3b_i \zeta_i \pmod{4}.$$

Choose v_i such that the corresponding value of ζ_i makes $Q_i \equiv 1 \pmod{4}$. Accordingly, Q_i is representable as the sum of three integral squares.

This completes the proof that every integer $s \geq C \cdot 3^{3\nu}$, where $\nu = \max(\delta + 1, n_1, n_2)$; and C has been determined, is a sum of nine values of $G(X)$ for $X \geq |t|$, the arguments of $G(X)$ being $m_i, 3^n - X_j, 3^n + X_j, v_i$ and w_i ($j = 1, 2, 3$). Hence every $s \geq C \cdot 3^{3\nu} + 9\alpha$ is a sum of nine positive integral values of $f(x)$ given by (3) with $p = 3p_1$, the arguments of the functions $f(x)$ being derived from those above by means of $x = X + t$. Theorem 2 is immediate.

5. Functions (5) for $p = 3p_1, p_1 \equiv 2g \pmod{3}$. This section deals with functions of the form (20) where the restrictions on p_1 are not as strong as

those stated in Theorem 2. The results of this section include those of the last* but since the weight of the representation of integers sufficiently large has to be increased to ten, §4 gives better results for the special functions considered there.

We prove

THEOREM 3. *Let integers p_1, q and u be given satisfying the conditions $p_1 \equiv 2g \pmod{3}$, $q = -3p_1t$, t an integer, u as in Theorem 2, and let α be defined by (4). Then there exist integers C and v such that every integer $\geq C \cdot 5^{3v} + 10\alpha$ is a sum of ten values of the function (3) with this triple p_1, q and u as its coefficients, for positive integral values of x .*

The theory in this section differs from that in the preceding sections in three main particulars:

- (1) two values of the function $G(X)$, instead of one, are subtracted initially from the integer (see Lemma 7),
- (2) the prime 5 is used instead of 3,
- (3) the interval in which s lies is divided into five sub-intervals instead of three.

The fact that a lemma analogous to Lemma 1 cannot be obtained for the functions considered here, even with the modulus changed to any prime up to 23 inclusive, necessitates the first change noted above.

LEMMA 6. *The positive integers s and n being given, there exist integers k_1, k_2 and τ such that $s \equiv G(k_1 + 2\tau \cdot 5^n) + G(k_2 + 2\tau \cdot 5^n) \pmod{5^n}$, where $0 \leq k_1 < 5^n$, $0 \leq k_2 < 5^n$, $0 \leq \tau < 5$.*

This lemma is proved by induction on n . For $n=1$ we considered all possible combinations of values of p_1 and q modulo 5 and showed in each case that it is possible to choose integers k_1 and k_2 for which $s \equiv G(k_1) + G(k_2) \pmod{5}$,

$$p_1(3k_1^2 + 3k_2^2 - 2) + 4g \not\equiv 0 \pmod{5}, \quad 0 \leq k_1 < 5 \quad \text{and} \quad 0 \leq k_2 < 5.$$

For example, if $p_1 \equiv 1, g \equiv 1 \pmod{5}$, then

$$\begin{aligned} 0 &\equiv G(0) + G(0), & 1 &\equiv G(1) + G(2), & 2 &\equiv G(1) + G(1), \\ 3 &\equiv G(4) + G(4), & 4 &\equiv G(4) + G(2) \pmod{5}; \end{aligned}$$

these values of k_1 and k_2 satisfy the conditions stated. Now, as the induction hypothesis, let the integers k_1 and k_2 exist such that $s = G(k_1) + G(k_2) + k \cdot 5^n$,

* The constants b_i ($i=1, \dots, 5$) and C are not calculated here for $p_1 \equiv 0 \pmod{3}$. This would have to be done before the results of Theorem 3 could be applied to all functions considered in §4.

k being an integer, where $0 \leq k_1 < 5^n$, $0 \leq k_2 < 5^n$, and $p_1(3k_1^2 + 3k_2^2 - 2) + 4g \not\equiv 0 \pmod{5}$. Then, if τ is an integer,

$$(24) \quad \begin{aligned} G(k_1 + 2\tau \cdot 5^n) + G(k_2 + 2\tau \cdot 5^n) \\ \equiv s + 5^n [\{p_1(3k_1^2 + 3k_2^2 - 2) + 4g\} \tau - k] \pmod{5^{n+1}}. \end{aligned}$$

Since, by the hypothesis for the induction, the coefficient of τ in the square bracket of (24) is prime to 5, we may choose τ modulo 5 such that the coefficient of 5^n is congruent to zero modulo 5. Thus

$$s \equiv G(k_1 + 2\tau \cdot 5^n) + G(k_2 + 2\tau \cdot 5^n) \pmod{5^{n+1}}.$$

The induction is complete.

Substitute $h_1 = k_1 + 2\tau \cdot 5^n$ and $h_2 = k_2 + 2\tau \cdot 5^n$ in Lemma 6. We obtain

LEMMA 7. *For any given integers s and n there exist integers h_1 and h_2 such that $s \equiv G(h_1) + G(h_2) \pmod{5^n}$, where $0 \leq h_1 < 9 \cdot 5^n$ and $0 \leq h_2 < 9 \cdot 5^n$.*

Choose δ and n such that $|t| \leq 5^\delta$ and $n \geq \delta$. Let $m_1 = h_1 + 5^n$ and $m_2 = h_2 + y \cdot 5^n$, where $1 \leq y \leq 3p_1$. Substitution of m_1 and m_2 into $G(X)$ gives $G(m_1) \equiv G(h_1) \pmod{5^n}$ and $G(m_2) = G(h_2) + 5^n E$, where

$$(25) \quad E = \frac{p_1}{2}(3h_2^2 y + 3 \cdot 5^n h_2 y^2 + 5^{2n} y^3 - y) + gy.$$

Combining these results with Lemma 7 we obtain

$$(26) \quad \begin{aligned} s &\equiv G(m_1) + G(h_2) \pmod{5^n}, \\ s &= G(m_1) + G(h_2) + 5^n M' = G(m_1) + G(m_2) + 5^n M, \end{aligned}$$

where M' is an integer, $M = M' - E$, and $|t| \leq 5^n \leq m_1 < 10 \cdot 5^n$, $|t| \leq 5^n \leq m_2 < (3p_1 + 9)5^n$.

LEMMA 8. *If $n \geq 1$, $g \leq 1$, $5^n \leq m_1 < 10 \cdot 5^n$, $5^n \leq m_2 < (3p_1 + 9)5^n$, then $-2\alpha \leq G(m_1) + G(m_2) < 5^{3n}\gamma$, where*

$$(27) \quad \gamma = \frac{p_1}{2} [(3p_1 + 9)^3 + 1002].$$

The statement $g \leq 1$ in §4 holds here. Substitution of the upper bounds for m_1 and m_2 into $G(m_1)$ and $G(m_2)$ gives

$$G(m_1) < \frac{1001}{2} p_1 \cdot 5^{3n}, \quad G(m_2) < \frac{p_1}{2} [(3p_1 + 9)^3 + 1] 5^{3n}.$$

Since $f(x) = G(X) + \alpha \geq 0$, then $G(X) \geq -\alpha$. The statement of the lemma follows.

As stated formerly, the interval $C \cdot 5^{3n} \leq s < C \cdot 5^{3n+3}$ is subdivided such that

$$(28) \quad 3^{i-1}C \cdot 5^{3n} \leq s_i < 3^iC \cdot 5^{3n} \quad (i = 1, 2, 3, 4, 5).$$

The following replacements will transform the relations used in §3 into those used here. Replace 3^n by 5^n and p by $3p_1$ throughout; in (9) replace α by 2α ; replace $v_i + w_i = 3b_i \cdot 3^n$ by $v_i + w_i = 5b_i \cdot 5^n$; in (14) replace 3 by 5; let $V_i = v_i - 5b_i \cdot 5^n/2$. As a result we obtain inequalities corresponding to (19),

$$(29) \quad \frac{125}{8}p_1b_i^3 + \frac{78}{25}p_1 + \gamma \leq 3^{i-1}C \leq \frac{125}{6}p_1b_i^3 + p_1 \quad (i = 1, \dots, 5).$$

The constants b_i ($i = 1, \dots, 5$) and C are chosen in accordance with (29) as follows:

$$p_1 \equiv 1 \pmod{3},$$

$$b_1 = 8e + 9, \quad b_2 = 12e + 13, \quad b_3 = 18e + 19, \quad b_4 = 24e + 27, \quad b_5 = 36e + 39,$$

$$C = 30,497e^4 + 106,839e^3 + 134,404e^2 + 70,596e + 12,759;$$

$$p_1 \equiv 2 \pmod{3}, \text{ the same } b_i \text{ as for } p_1 \equiv 1 \pmod{3}, \text{ and}$$

$$C = 30,497e^4 + 117,126e^3 + 167,074e^2 + 107,572e + 27,165.$$

The quantity Q_i is an integer. For, from (25), $E \equiv gy \pmod{3p_1}$. Also $g \equiv 1 \pmod{3p_1}$ or $2g \equiv 1 \pmod{3p_1}$ according as the cases being considered are III₁ and IV or III₂; thus $(g, 3p_1) = 1$. Accordingly we may choose y such that E is congruent to any assigned integer modulo $3p_1$; choose y such that $E \equiv M_i' - B_i - 6g \pmod{3p_1}$. From (13) with the above replacements we see that Q_i is an integer.

To prove that Q_i is representable as the sum of three integral squares, we proceed as follows. Equation (13) with $p = 3p_1$ and 3 replaced by 5 gives $6p_1Q_i \equiv 2M_i - 12g - 2B_i \pmod{8p_1}$. The replacements described above give

$$B_i = 5b_i \left[\frac{p_1}{2} \{ 25b_i^2 \cdot 5^{2n} - 3v_i(5b_i \cdot 5^n - v_i) - 1 \} + g \right],$$

and thus

$$2B_i \equiv 10gb_i - 30p_1b_i\zeta_i \pmod{8p_1},$$

where ζ_i arises from the use of Lemma 2, as described in §3, and is arbitrary. Also,

$$B_i \equiv \frac{5b_i p_1}{2}(b_i^2 - 1) + 5b_i g \pmod{3p_1}.$$

Since, according to the last paragraph, $M_i \equiv B_i + 6g \pmod{3p_1}$, then

$$M_i = 6g + 5b_i g + \frac{5b_i p_1}{2}(b_i^2 - 1) + 3k_i p_1 \quad (k_i \text{ integral}).$$

This gives

$$\begin{aligned} 6p_1 Q_i &\equiv 30p_1 b_i \zeta_i + 6k_i p_1 && \pmod{8p_1}, \\ Q_i &\equiv b_i \zeta_i + k_i && \pmod{4}. \end{aligned}$$

Choose ζ_i such that $Q_i \equiv 1 \pmod{4}$.

It has been shown that every integer $\geq C \cdot 5^{3\nu}$, where C and ν have been determined, is a sum of ten positive integral values of the function (20), p_1 and g being given, for integral values of $X \geq |t|$. The statement of the theorem follows.

6. Cubic functions without square term. L. E. Dickson, in his Transactions paper, did not consider the Waring problem for functions $f(x) = x + \epsilon(x^3 - x)/6$ where ϵ is a multiple of 3. Frances Baker* considered the problem for functions of the above form where $\epsilon = 3a$, a odd and $a \equiv 1 \pmod{3}$.

The work contained in Chapter I of Miss Baker's thesis, with two or three minor changes, holds equally well when the only restriction on a is $a \not\equiv 2 \pmod{3}$. For $a = 3e$ the following constants b_1, b_2, b_3 and C satisfy the inequalities in her paper corresponding to (19) of this paper:

$$\begin{aligned} b_1 &= 14e + 9, & b_2 &= 20e + 13, & b_3 &= 30e + 19, \\ C &= 30,497e^4 + 57,713e^3 + 36,552e^2 + 7719e + 14. \end{aligned}$$

The proofs which it is necessary to change are contained on pages 12 and 13 of her paper, these changes being in accordance with similar work contained in the previous part of this paper. This results in the following theorem, Miss Baker's results being included:

THEOREM 4. *To each positive integer $a, a \not\equiv 2 \pmod{3}$, there correspond positive integers C and ν such that every integer $\geq C \cdot 3^{3\nu}$ is a sum of nine values of*

$$(30) \quad f(x) = x + \frac{a}{2}(x^3 - x)$$

for integral values ≥ 0 of x .

Consider the problem for the functions (30) with $a \equiv 2 \pmod{3}$. The results are stated in

THEOREM 5. *To each positive integer $a, a \equiv 2 \pmod{3}$, there correspond positive integers C and ν such that every integer $\geq C \cdot 5^{3\nu}$ is a sum of ten values of (30) for integral values ≥ 0 of x .*

* *A Contribution to the Waring Problem for Cubic Functions*, Doctoral Dissertation, University of Chicago, 1934.

The proof of Theorem 5 parallels that of Theorem 3, the major changes being necessitated by the requirement $x \geq 0$ instead of $X \geq |t|$ as heretofore required in this paper. This is due to the fact that it is not necessary to transform linearly our original function into another without square term before the theory is applied. By choosing $m_1 = h_1$ instead of $m_1 = h_1 + 5^n$ we lower the upper bound for $G(m_1)$ as contained in Lemma 7. The inequalities corresponding to (14) would be

$$0 \leq v_i \leq 5b_i \cdot 5^n \quad \text{and} \quad 0 \leq Q_i \leq 5^{2n}.$$

Choose the constants b_i ($i = 1, \dots, 5$) and C so that

$$\frac{125}{8}ab_i^3 + \gamma + 6a \leq 3^{i-1}C \leq \frac{125}{6}ab_i^3 + a \quad (i = 1, \dots, 5)$$

are satisfied, the choice being

$$b_1 = 8e + 5, \quad b_2 = 12e + 7, \quad b_3 = 16e + 11, \quad b_4 = 24e + 15,$$

$$b_5 = 34e + 21,$$

$$C = 27,365e^4 + 66,465e^3 + 62,661e^2 + 27,136e + 4681.$$

The remainder of the proof is so similar to that of Theorem 3 that it is needless to repeat it here.

Theorems 4 and 5 complete the general theory for cubic functions without square term.

7. Universal theorems. A universal theorem for weight nine is possible for only two functions of the type considered in Theorems 4 and 5. The function (30) has the values

$$f(0) = 0, \quad f(1) = 1, \quad f(2) = 2 + 3a.$$

If $f(2) \geq 12$, the integer 11 has a representation of weight eleven as a sum of functions (30), i.e., $11 = 11f(1)$. Accordingly, $a = 1, 2$, and 3 are the only values of a for which universal theorems of weight ten are possible. The case $a = 1$ was considered by Miss Baker* and a universal theorem of weight nine was obtained. The case $a = 2$ reduces to the problem of cubes† for which the result is well known. For $a = 3$, $f(2) = 11$, and $21 = f(2) + 10f(1)$, a representation of weight eleven.

We prove

* Loc. cit.

† L. E. Dickson, *Simpler proofs of Waring's theorem on cubes, with various generalizations*, these Transactions, vol. 30 (1928), pp. 1-18.

THEOREM 6. *Every integer ≥ 0 is a sum of fifteen values of*

$$(31) \quad f(x) = x^3 + 3(x^2 - x)$$

for integral values ≥ 0 of x .

This function (31) is the function (3) for $p = 3p_1$, $p_1 = 2$, $q = 6$, $u = 1$. These values of p and p_1 satisfy the hypothesis of Theorem 3, and so, for ν sufficiently large, every integer $\geq C \cdot 5^{3\nu} + 50$ is a sum of ten values of (31). The integers C and ν calculated from the theory are $C = 27,165$ and $\nu = 8$. We have to prove that all integers $< 27,165 \times 5^{24} + 50$ can be represented as a sum of fifteen values of (31).

A table of minimum weights of the representations of integers 1–1000 shows that integers 298–1000, inclusive, have weight 8; 169, 83 and 41 are the largest integers of weights 11, 13 and 15, respectively.

Apply the following theorem* to the data given above:

THEOREM 7. *Let a polynomial $f(x)$ take integral values ≥ 0 for all integers $x \geq 0$; let $f(x+1) - f(x)$ increase with x . Suppose that every integer n for which $l < n \leq g + f(0)$ is a sum of $k - 1$ values of $f(x)$ for integers $x \geq 0$. Let m be the maximum integer for which $f(m+1) - f(m) < g - l$. Then every integer N for which $l + f(0) < N \leq g + f(m+1)$ is a sum of k values of $f(x)$ for integers $x \geq 0$.*

Seven applications of Theorem 7 lead to the result that all integers \leq a constant greater than $27,165 \times 5^{24} + 50$ are sums of fifteen values of (31). The proof of Theorem 6 is complete.

8. **Generalization.**† In this section we show that the theory contained in §§3, 4 and 5 holds for values of the parameters p , q and u of (3) subject only to the condition $q = -tp$. We shall consider §3. If, in Lemma 3, we use $g \leq |g|$ in place of $g \leq 13p + 1$, the only effect is to alter certain terms of (18). When we pass to (19) these terms drop out, so the same set of b_1 , b_2 , b_3 and C will suffice. In the proof that Q_i is an integer there is the restriction $(g, p) = \theta = 1$. However, if $\theta > 1$, $F(X)$, and likewise any sum of values of $F(X)$, would be a multiple of θ . As stated at the beginning of §2, functions of this type do not enter into the theory. Accordingly, the parameter g is arbitrary and the result stated above follows for §3. Similar results for §§4 and 5 are immediate.

* L. E. Dickson, *Waring's problem for cubic functions*, these Transactions, vol. 36 (1934), pp. 1–12. This is Dickson's Theorem 3.

† Section appended June 30, 1934. This is analogous to theory recently obtained by L. E. Dickson.