PROPERTIES OF FUNCTIONS $f(x, y)$ OF BOUNDED VARIATION*

BY

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1. Introduction

In a recent paper† we investigated the relations between several definitions of bounded variation for functions $f(x, y)$ of two real variables.‡ These definitions are usually associated with the names of Vitali, Hardy, Arzelà, Pierpont, Fréchet, and Tonelli respectively; we proved the equivalence of the definition formulated by Pierpont and the modified form of it given byHahn.§

Since the several definitions were assembled in CA, it is hardly necessary to repeat them here. But we shall again denote the classes of functions satisfying the respective definitions by V, H, A, P, F, and T. In addition the class of functions continuous in (x, y) will be designated by C, the class of functions belonging to the Baire classification by|| B, the class of functions having measurable total variation functions¶ $\phi(\bar{x})$ and $\psi(\bar{y})$ by $M_{\phi,\psi}$, and the class of functions having superficial measure by M; and the common part of two or more classes will be indicated by the product of the corresponding letters. The domain of definition of $f(x, y)$ is generally to be understood as a rectangle with sides parallel to the axes** $(a \leqq x \leqq b, c \leqq y \leqq d)$; the letter R, with or without a subscript, will always stand for such a rectangle.

Functions $g(x)$ of bounded variation are of great interest and usefulness because of their valuable properties, particularly with respect to additivity, decomposability into monotone functions, continuity, differentiability, meas-

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† Clarkson and Adams, *On definitions of bounded variation for functions of two variables*, these Transactions, vol. 35 (1933), pp. 824–854. Hereafter this paper will be referred to as CA.

‡ Since the paper CA was written, our attention has been called to two additional definitions; of these the first is due to Wiener, *Laplacians and continuous linear functionals*, Acta Szeged, vol. 3 (1927), pp. 7–16. The second is that of Nalli and Andreoli, *Sull' area di una superficie, sugli integrali multipli di Stieltjes e sugli integrali multipli delle funzioni di più variabili complesse*, Accademia dei Lincei, Rendiconti, (6), vol. 5 (1927), pp. 963–966. The fact that class $T \cdot C$ contains as a proper subclass all continuous functions satisfying the definition of Nalli and Andreoli or a modified form of it has been shown by Tonelli, *Sulla definizione di funzione di due variabili a variazione limitata*, ibid., (6), vol. 7 (1928), pp. 357–363. In this sequel to CA these additional definitions will not be further considered.

§ This will be spoken of as the P_H-form of Pierpont's definition.

|| This must not be confused with B in CA, which stood for the class of bounded functions.

¶ $\phi(\bar{x})\,[\psi(\bar{y})]$ represents the total variation of $f(\bar{x}, y)\,[f(x, \bar{y})]$ in $y\,[x]$; see CA.

** For brevity we shall sometimes indicate such a closed rectangle by the notation $(a, c;\ b, d)$.

urability, integrability, etc.; and it is largely to the possession of these properties that such functions owe their important role in the study of rectifiable curves, Fourier and other series, Stieltjes and other integrals, and the calculus of variations. Proposers of definitions of bounded variation for functions $f(x, y)$ have been actuated mainly by the desire to single out for attention a class of functions having properties analogous to some *particular* properties of a function $g(x)$ of bounded variation. It has long since become apparent that to preserve properties of one sort the definition of bounded variation for $g(x)$ should be extended to $f(x, y)$ in one way, while to preserve properties of another sort a quite different extension may be needed.

It is natural that in CA the only detailed study of properties of functions $f(x, y)$ belonging to the several classes V, H, A, P, F, and T should have had to do with the nature of the total variation functions $\phi(\bar{x})$ and $\psi(\bar{y})$, since properties of this kind seemed to bear most directly upon the problem of determining relations between the classes. Properties of functions belonging to the classes V and F (and by implication H) with respect to double Stieltjes integrals of the Riemann type have recently been examined by Clarkson.* It would seem worth while to make a systematic study of the properties of additivity, decomposability, etc., enjoyed by functions belonging to each of the six classes, and it is to this object that the present paper is mainly devoted. The determination of such properties has by no means been utterly neglected by previous writers; indeed we shall state a few results that are already well known, and certain of our theorems will constitute extensions of such results.

It will appear that the aggregate of functions in class T lacks certain desirable properties because of the necessity for $\phi(\bar{x})$ and $\psi(\bar{y})$ to be measurable. And the evidence seems to indicate that the definition of Tonelli, precisely as formulated by him, may attain its greatest usefulness when applied to functions which to a certain extent are well behaved, perhaps to the extent of belonging to the Baire classification. In order that a function $f(x, y)$ may not fail to be included in the class merely because its ϕ or ψ is non-measurable, *we define the extended class T to consist of those functions f for which ϕ and ψ are respectively dominated by summable functions; this class we designate by $\overline{T}$.* Such extension of Tonelli's class has proved desirable in recent work by Gergen† and by Morrey.‡

* Clarkson, *On double Riemann-Stieltjes integrals*, Bulletin of the American Mathematical Society, vol. 39 (1933), pp. 929–936.

† Gergen, *Convergence criteria for double Fourier series*, these Transactions, vol. 35 (1933), pp. 29–63.

‡ Morrey, *A class of representations of manifolds*. I, American Journal of Mathematics, vol. 55 (1933), pp. 683–707.

Throughout this paper the difference operators Δ_{10}, Δ_{01}, and Δ_{11}, when applied to $f(x_i, y_j)$, will have the following meaning:

$$\Delta_{10}f(x_i, y_j) = f(x_{i+1}, y_j) - f(x_i, y_j),$$
$$\Delta_{01}f(x_i, y_j) = f(x_i, y_{j+1}) - f(x_i, y_j),$$
$$\Delta_{11}f(x_i, y_j) = \Delta_{10}(\Delta_{01}f(x_i, y_j)).$$

When applied to $f(x, y)$, the operators will have a similar significance, it being understood that the increments of x and y involved are greater than zero but otherwise arbitrary.

2. A property of class P

THEOREM 1. *If $f(x, y)$ is in class P, $\phi(\bar{x})$ $[\psi(\bar{y})]$ is dominated by a summable function.*

For each $n \geqq 1$ let N_n designate the net of n^2 cells used in the P_H-form of the definition, and denote by $\phi_n^*(\bar{x})$ the sum of the oscillations of f in the cells of that column in whose base $\bar{x}$ lies. For definiteness we may associate $\bar{x}$, when it is the coordinate of a line of N_n other than $x = a$, with the subinterval of (a, b) whose right-hand end point is $\bar{x}$. Then $\phi_n^*(\bar{x})$ is a step-function and, if B denotes a bound for the P_H-sum, we have for each n

$$\int_a^b \phi_n^*(x)dx = \frac{b-a}{n} \sum_{\nu=1}^{n^2} \omega_\nu' \leqq B(b-a). \tag{1}$$

For each $\bar{x}$ let

$$\phi^*(\bar{x}) = \liminf_{n\to\infty} \phi_n^*(\bar{x});$$

in the light of (1) it is known† that $\phi^*(\bar{x})$ is summable in (a, b). Next let

$$\phi_n^{**}(\bar{x}) = \sum_{i=1}^{n} [\text{oscillation of } f(\bar{x}, y) \text{ in the interval } y_{i-1} \leqq y \leqq y_i]$$

for each n, and set

$$\phi^{**}(\bar{x}) = \liminf_{n\to\infty} \phi_n^{**}(\bar{x}).$$

For every $\bar{x}$ in (a, b) for which $\phi(\bar{x})$ is finite we have‡

$$\phi^{**}(\bar{x}) \geqq \phi(\bar{x})/2, \tag{2}$$

† See Schlesinger and Plessner, *Lebesguesche Integrale und Fouriersche Reihen*, Berlin, 1926, p. 91.

‡ See Hobson, *Theory of Functions of a Real Variable*, 3d edition, vol. 1, Cambridge, 1927, p. 331. It is easily proved that the total variation and total fluctuation of any function $g(x)$ are equal when both are finite, and that if either is infinite the other is likewise.

and it is easily seen that when $\phi(\bar{x})$ is infinite, $\phi^{**}(\bar{x})$ is likewise. Moreover, for each $\bar{x}$ we have

$$\phi_n^*(\bar{x}) \geqq \phi_n^{**}(\bar{x})$$

for all n, whence

$$\phi^*(\bar{x}) \geqq \phi^{**}(\bar{x}) \tag{3}$$

except when $\phi^{**}(\bar{x})$ is infinite, in which case $\phi^*(\bar{x})$ is also infinite. The theorem for $\phi(\bar{x})$ now follows from inequalities (2) and (3); a similar proof may be given for $\psi(\bar{y})$.

COROLLARY 1. *If $f(x, y)$ is in class P and $\phi(\bar{x})$ and $\psi(\bar{y})$ are measurable,*† *$f(x, y)$ is also in class T.*

The common part of the overlapping classes P and T may now be specified by the relation $P \cdot T = P \cdot M_{\phi,\psi}$.

COROLLARY 2. *If $f(x, y)$ is in class P, $\phi(\bar{x})\,[\psi(\bar{y})]$ is finite almost everywhere.*‡

That ϕ may be infinite at an everywhere dense set§ and that $\partial f/\partial x$ and $\partial f/\partial y$ may fail to exist (finite or infinite) at a set everywhere dense in the rectangle R, when f is in P, is illustrated by the following example. Let the rational points in the interval $0 \leqq x \leqq 1$ be enumerated as $x_1, x_2, \cdots$; for $x = x_n (n = 1, 2, \cdots)$ and y rational $(0 \leqq y \leqq 1)$ let $f(x, y) = 1/2^n$; elsewhere in the unit square I $(0, 0; 1, 1)$ let $f = 0$.

From Theorem 1 we have $\overline{T} \geqq P$; the relation $\overline{T} > P$ then follows from example (D) of CA. The fundamental relations of inclusiveness between the several classes are therefore

$$\overline{T} > P > A > H, \quad F > V > H, \quad \overline{T} > T > H; \tag{4}$$

and when only functions belonging to the Baire classification are admitted to consideration‖,

† Montgomery, *Properties of plane sets and functions of two variables*, to appear in the American Journal of Mathematics, Theorem 17, has shown that $f \subset B$ implies measurability of ϕ and ψ.

‡ Although Theorem 2 of CA was sufficient for the purposes of that paper, this corollary improves the result.

§ This first fact was illustrated by the example following the proof of Theorem 2 in CA, but the example given here is somewhat more easily shown to be in P.

‖ These relations are an immediate consequence of the results of CA in conjunction with Montgomery's Theorem 17, loc. cit. From the standpoint of continuity we may remark that the inclusiveness relations are like (5) when only functions possessing one of the following properties are admitted to consideration: continuity in (x, y) [see CA], continuity in x and in y, semi-continuity in (x, y), upper semi-continuity in one variable and lower semi-continuity in the other. A function having this last property belongs to Baire's class 1 at most; see Kempisty, *Sur les fonctions semicontinues par*

$$(5) \qquad T\cdot B > P\cdot B > A\cdot B > H\cdot B, \quad F\cdot B > V\cdot B > H\cdot B.$$

These are the basis for numerous statements in the following pages.

3. Closure of the several classes under arithmetic operations

THEOREM 2. *Each of the classes V, H, A, P, F, and $\overline{T}$ is closed under addition (and subtraction).* This is not true† of T.*

The first part of this theorem is an immediate consequence of the definitions. For the second part we may break up example (C) of CA into monotone components as follows: E being a linearly non-measurable set of points on the downward-sloping diagonal d of the unit square I (0, 0; 1, 1), set

$$f_1(x, y) = \begin{cases} 0 \text{ below } d, \\ 1 \text{ above } d, \\ 1 \text{ on } E, \\ 0 \text{ elsewhere on } d. \end{cases} \qquad f_2(x, y) = \begin{cases} 0 \text{ below and on } d, \\ 1 \text{ above } d. \end{cases}$$

Each of these functions is clearly in T, although $f_1 - f_2$, which is example (C) of CA, is not.

THEOREM 3. *Each of the classes H, A, and P is closed under multiplication.‡ This is not true of V, F, T, or $\overline{T}$.*

For H and A the theorem may readily be proved by aid of decomposition theorems given in §4. Since P contains only bounded functions, the proof for P flows at once from

LEMMA 1. *Let f_1 and f_2 be functions of any number of variables, defined for an arbitrary range of variation S of those variables. If f_1 and f_2 are bounded, and the least upper bound of $|f_i|$ is denoted by $B_i (i=1, 2)$, the following inequality connects the oscillations of f_1, f_2, and $f_1\cdot f_2$ over S:*

$$\operatorname{Osc}(f_1\cdot f_2) \leqq B_2 \operatorname{Osc} f_1 + B_1 \operatorname{Osc} f_2.$$

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rapport à chacune de deux variables, Fundamenta Mathematicae, vol. 14 (1929), pp. 237–241. On the other hand, when only functions upper [lower] semi-continuous in each separate variable are admitted, the relations are like (4); see example (C) of CA.

* For H this fact was observed by Hardy, *On double Fourier series, and especially those which represent the double zeta-function with real and incommensurable parameters*, Quarterly Journal of Mathematics, vol. 37 (1905), pp. 53–79.

† It is quite clear that if $f_1(x, y)$ and $f_2(x, y)$ are both in T, $f=f_1+f_2$ will fail to be in T when and only when at least one of its total variation functions is non-measurable. By Theorem 17 of Montgomery, loc. cit., this cannot happen if only functions belonging to the Baire classification are admitted to consideration.

‡ For H this fact was observed by Hardy, loc. cit.

Designating by a_i and b_i respectively the greatest lower and least upper bounds of $f_i (i=1, 2)$, one may easily construct a proof of the lemma by considering seriatim all possible cases for the relationship of the intervals (a_i, b_i) to the origin.

That the product of two functions in $V \cdot C$ may not even be in F is seen at once from the following example:

$$(6) \qquad \left.\begin{aligned} f_1(x, y) &= \begin{cases} x \sin (1/x) \text{ for } x > 0 \\ 0 \text{ for } x = 0 \end{cases} \\ f_2(x, y) &= y \end{aligned}\right\} \text{ in the unit square I.}$$

The theorem fails for T because the product of two functions in T may have a non-measurable ϕ or ψ; viz.,

$$(7) \qquad \begin{aligned} f_1(x, y) &= \begin{cases} 1 \text{ for } x \text{ in } E,\ y = 0, \\ 1 \text{ for } x \text{ in } C(E),\ y = 1, \\ 0 \text{ elsewhere,} \end{cases} \\ f_2(x, y) &= \begin{cases} 1 \text{ for } y = 0, \\ 0 \text{ for } y > 0, \end{cases} \end{aligned}$$

in I, E being a non-measurable set in the interval (0, 1) and $C(E)$ its complement. The theorem also fails for T, and likewise for $\overline{T}$, because of the well known theorem of Lebesgue: *if $g_1(x)$ is a summable function not essentially bounded, there always exists a summable function $g_2(x)$ such that $g_1 \cdot g_2$ is not summable over the interval considered.* We may consider the interval in question as (0, 1) and set in I

$$f_i(x, y) = \begin{cases} g_i(x) \text{ for } y = 0 \\ 0 \quad\ \text{ for } y > 0 \end{cases} \qquad (i = 1, 2).$$

Remarks. That the relations $f_1 \subset V \cdot C$, $f_2 \subset H \cdot C$ do not imply $f_1 \cdot f_2 \subset F$ is shown by (6). That $f_1 \subset T$, $f_2 \subset H$ do not imply $f_1 \cdot f_2 \subset T$ is apparent from (7); nevertheless one may readily show by aid of Lemma 1 and the theorem of Montgomery referred to above that if both f_1 and f_2 are in T, are bounded, and belong to the Baire classification, $f_1 \cdot f_2$ must be in T; similarly, if f_1 and f_2 are in $\overline{T}$ and are bounded, $f_1 \cdot f_2$ is in $\overline{T}$. That f_1 may be in $A \cdot C [P \cdot C$ or $T \cdot C]$ and f_2 in $H \cdot C$ without $f_1 \cdot f_2$ being in $H [A$ or P respectively] is clear from the fact that $f(x, y) \equiv 1$ is in $H \cdot C$.

THEOREM 4. *Each of the classes H, A, and P is closed under division, the denominator being assumed bounded away from zero.* This is not true of V, F, or T.*

In the light of Theorem 3 it suffices, for the first statement, to consider the case of $1/f$ for f in the class in question and $|f| \geqq m > 0$. The fact has been stated for H by Hardy, loc. cit.; a proof can be constructed by aid of a little double series technique. For A one may give a proof precisely like that of the corresponding theorem for a function $g(x)$ of bounded variation.

Proof for P. Let $\mathfrak{B}$ be a bound for the P_H-sum for f, and for each n let α_n be the number of cells, in the net of n^2 cells, in which f changes sign; then

$$\mathfrak{B} \geqq \sum_{\nu=1}^{n^2} \omega_\nu'(f)/n \geqq 2m\alpha_n/n. \tag{8}$$

Let us set

$$\sum_{\nu=1}^{n} \omega_\nu'(1/f) = \sum{}' + \sum{}'',$$

$\sum'$ representing the sum over the cells in which f changes sign and $\sum''$ the sum over the remaining cells. In each cell of the first set we have

$$\omega_\nu'(1/f) \leqq 2/m; \tag{9}$$

denoting by M_ν and m_ν respectively the least upper and greatest lower bound of $|f|$ in the νth cell, we have for each cell of the second set

$$\omega_\nu'(1/f) = 1/m_\nu - 1/M_\nu \leqq M_\nu - m_\nu)/m^2 = \omega_\nu'(f)/m^2. \tag{10}$$

From (8), (9), and (10) follows the inequality

$$\sum_{\nu=1}^{n^2} \omega_\nu'(1/f)/n \leqq 2\mathfrak{B}/m^2$$

for every n, and the proof is complete.

That f may be in V and $|f| \geqq m > 0$ without $1/f$ even being in F is seen from the following example. Let I be divided into subrectangles by the lines $x = 1 - 1/n$ $(n = 2, 3, \cdots)$. Proceeding from left to right, in the first, third, $\cdots$ rectangles let $f = 1$ except along the (closed) top and right-hand side; on the entire top and right-hand side, except at their common point where $f = 3$, let $f = 2$. At points of the even-numbered (closed) rectangles not already considered define f as 2 except along the top, where $f = 3$. For $x = 1$ and all y let $f = 1$.

If f is in T and $|f| \geqq m > 0$ one readily sees that $1/f$ can fail to be in T

* The necessity for imposing this restriction is clear, since H, A, and P contain no unbounded functions.

only if its ϕ or ψ is non-measurable. This situation occurs in the case of

$$f(x, y) = \left\{ \begin{array}{l} 3/2 \text{ for } x \text{ in } E,\ y = 0 \\ 1/2 \text{ for } x \text{ in } C(E),\ y = 1 \\ 1 \text{ otherwise} \end{array} \right\} \text{ in I},$$

E being a non-measurable set.

Remarks. Since V, F, and T contain unbounded functions, the restriction that the denominator be bounded away from zero in connection with these classes is perhaps more than would normally be expected. That $\overline{T}$ is not closed under division is apparent from the following example: $f_1(x, y) = 1/x^{1/2}$ and $f_2(x, y) = x^{1/2}$ for $y = 0$, $x > 0$; and both functions equal to 1 elsewhere in I. If consideration is restricted to bounded functions in $\overline{T}$, it is readily seen that this subclass of $\overline{T}$ is closed under division, the denominator being assumed bounded away from zero (see Remarks following Theorem 3).

4. Relationships with monotone functions; decomposition

THEOREM 5. *A necessary and sufficient condition that* $f(x, y)$ be in class V is that it be expressible as the difference between two functions, $f_1(x, y)$ and $f_2(x, y)$, satisfying the inequalities*

$$\Delta_{11} f_i(x, y) \geqq 0 \qquad (i = 1, 2).$$

The necessity has essentially been shown by Hobson†; the sufficiency is quite clear from Theorem 2.

THEOREM 6 (Hardy‡). *A necessary and sufficient condition that $f(x, y)$ be in class H is that it be expressible as the difference between two bounded functions, $f_1(x, y)$ and $f_2(x, y)$, satisfying the inequalities§*

$$\Delta_{10} f_i(x, y) \geqq 0,\ \Delta_{01} f_i(x, y) \geqq 0,\ \Delta_{11} f_i(x, y) \geqq 0 \qquad (i = 1, 2).$$

THEOREM 7 (Arzelà‖). *A necessary and sufficient condition that $f(x, y)$ be in class A is that it be expressible as the difference between two bounded functions, $f_1(x, y)$ and $f_2(x, y)$, satisfying the inequalities*

* In order that the V-definition may always have meaning it is to be understood here that f, although perhaps unbounded, is everywhere finite. The functions f_1, f_2 are of like character.

† Hobson, loc. cit., p. 345.

‡ Hardy, loc. cit.

§ Functions satisfying these inequalities have been called "monotonely monotone" by W. H. and G. C. Young, *On the discontinuities of monotone functions of several variables*, Proceedings of the London Mathematical Society, (2), vol. 22 (1923), pp. 124–142. They belong to the class of "quasi-monotone" functions as defined by Hobson, loc. cit., p. 347.

‖ Arzelà, *Sulle funzioni di due variabili a variazione limitata*, Bologna Rendiconto, (2), vol. 9 (1904–05), pp. 100–107.

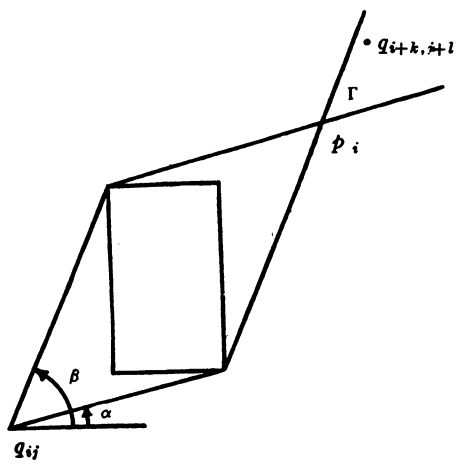
$$\Delta_{10}f_i(x, y) \geqq 0, \ \Delta_{01}f_i(x, y) \geqq 0 \qquad (i = 1, 2).$$

Remarks. Although every bounded function monotone in the sense of Hobson is in class $A \cdot T$, not all such are in H. Every function quasi-monotone in the sense of Hobson is in class V and if bounded is also in H.

THEOREM 8. *Every bounded *function non-decreasing in each of two directions is in class P.*

Let α and β respectively $(\alpha<\beta)$ be the angles made with the positive x-axis by the given directions in which f is non-decreasing. For $\alpha=0, \beta=\pi/2$ a proof has been given by Hahn†, in establishing the relation $P \geqq A$. We now prove the theorem for $0 \leqq \alpha < \beta \leqq \pi/2$; it will be clear that the method is applicable in all cases.

Using the P_H-form of the definition, let a net of n^2 cells be placed upon R, and let the columns of cells be numbered from left to right and the rows from bottom to top. Indices i, j may then be employed to designate the cell in the ith row and jth column. With this cell (for each pair of values i, j) we associate two points p_{ij} and q_{ij} defined as follows (see accompanying figure):



$p_{ij}[q_{ij}]$ is the point from which the cell is seen under the angle $\beta-\alpha$, the sides of the angle having the directions $\pi+\alpha, \pi+\beta$ $[\alpha, \beta]$. Let $q_{i+k,j+l}$ be the point (or *a* point) of the set q_{ij} lying in the closed sector marked Γ in the figure and at a minimum distance from p_{ij}. *The integers k, l are now fixed and are clearly independent of n.*

* If the directions of assumed monotonicity are axial (i.e., the function is monotone in the sense of Hobson), finiteness of the function everywhere implies boundedness; otherwise this may not be so.

† Hahn, *Theorie der Reellen Funktionen*, Berlin, 1921, p. 546.

It suffices to consider $n \geqq \max(10k, 10l)$. For such a value of n the sum of the oscillations of f in the first and last k rows and the first and last l columns of cells is $\leqq 2B \cdot 2kn + 2B \cdot 2ln = O(n)$, B being a bound for $|f|$ in R. For each remaining cell the associated points p_{ij}, q_{ij} lie within R. These remaining cells constitute a block of $(n-2k)(n-2l)$ cells, and it will simplify matters a little to regard the row indices of these cells as running from 1 to $n-2k$ and the column indices from 1 to $n-2l$. From the above choice of k and l we clearly have

$$f(p_{ij}) \leqq f(q_{i+k,j+l}) \quad (i = 1, 2, \cdots, n-3k; j = 1, 2, \cdots, n-3l).$$

Hence, for this remaining block of cells, the sum of the oscillations of f is

$$\begin{aligned}
&\leqq \sum_{i,j=1}^{n-2k,n-2l} [f(p_{ij}) - f(q_{ij})] \\
&\leqq \left[\sum_{i=1}^{n-2k} \sum_{j=n-3l+1}^{n-2l} + \sum_{i=n-3k+1}^{n-2k} \sum_{j=1}^{n-3l} \right] f(p_{ij}) \\
&\qquad - \left[\sum_{i=1}^{k} \sum_{j=1}^{n-2l} + \sum_{i=k+1}^{n-2k} \sum_{j=1}^{l} \right] f(q_{ij}) \\
&\leqq [(n-2k)l + k(n-3l) + k(n-2l) + (n-3k)l]B = O(n).
\end{aligned}$$

This completes the proof for the case considered.

It may be noted that a function non-decreasing in two directions must be non-decreasing in any third direction lying in the angle ($<\pi$) formed by the first two. Therefore, in constructing a proof for other cases, one may always reduce a case in which $\beta-\alpha$ is $>\pi/2$ to a case in which $\beta-\alpha$ is $<\pi/2$, of which that considered above is typical.

Remarks. It would be of considerable interest to determine whether a function in class P can always be decomposed into the difference between two functions each of which is bounded and monotone in two directions. If this were true it would follow at once* that every function in P has a total differential almost everywhere, settling a question left open in §6.

Lebesgue has defined a function f to be monotone if it satisfies the following condition: p being any point of the region considered and $\mathfrak{C}$ any closed curve in this region containing p in its interior, we have g.l.b. of f on $\mathfrak{C} \leqq f(p) \leqq$ l.u.b. of f on $\mathfrak{C}$. It is easily seen by examples that not all functions satisfying

* See Haslam-Jones, *Derivate planes and tangent planes of a measurable function*, Quarterly Journal of Mathematics, Oxford Series, vol. 3 (1932), pp. 120–132; or Saks, *Théorie de l'Intégrale*, Warsaw, 1933, p. 238.

this condition belong to any one of the classes V, H, A, P, F, T, and $\overline{T}$.

We conclude this Section with two quite obvious theorems concerning decomposition of a different sort, the first expressing a fact which has already been frequently observed.

THEOREM 9. *A necessary and sufficient condition that $f(x, y)$ be in class V is that $f(x, y) \equiv \bar{f}(x, y) + g(x) + h(y)$ where $\bar{f}(x, y)$ is in H.*

DEFINITION. *The subclass of F of which each function has $\phi(\bar{x})$ and $\psi(\bar{y})$ finite somewhere (and therefore finite everywhere†) will be designated by F^*.*

It should be observed that $F^* = F \cdot T$, the relationship of which to other classes was considered in CA.

THEOREM 10. *A necessary and sufficient condition that $f(x, y)$ be in class F is that $f(x, y) \equiv \bar{f}(x, y) + g(x) + h(y)$ where $\bar{f}(x, y)$ is in F^*.*

5. ADJUNCTION OR SUBDIVISION OF RECTANGLES

We state without proof two theorems.

THEOREM 11. *If a function is in any one of the several classes for each of two rectangles R_1 and R_2 whose sum is a rectangle R, it is in the same class for R.*

THEOREM 12. *If a function is in any one‡ of the classes $V, H, A, P, F, T \cdot B$, or $\overline{T}$ for a rectangle R, it is in the same class for any subrectangle R_1. This is not true of T.*

6. CONTINUITY, DIFFERENTIABILITY, MEASURABILITY, AND INTEGRABILITY OF FUNCTIONS BELONGING TO THE SEVERAL CLASSES

THEOREM 13. *If $f(x, y)$ is in class V and $f(x, \bar{y})$ $[f(\bar{x}, y)]$ for some $\bar{y}[\bar{x}]$ has only a denumerable number of discontinuities in $x[y]$, the discontinuities in $x[y]$ of $f(x, y)$ are located on a denumerable number of parallels to the y-axis [x-axis].*

Let E be the set of points at which f has a discontinuity in x and assume the existence of a non-denumerable set S of vertical lines each containing at least one point of E. Clearly only a denumerable subset of S can be made up wholly of points of E. Let the remaining lines of S constitute the subset S_1; then each line of S_1 contains at least one point of E and at least one point not

† See CA, Theorem 3.

‡ For H this fact was observed by W. H. Young, *On multiple Fourier series*, Proceedings of the London Mathematical Society, (2), vol. 11 (1912), pp. 133–184, especially p. 143. The failure of T to enjoy the property in question is illustrated by $f_1(x, y)$ in (7).

in E, and S_1 is non-denumerable. On each line of S_1 choose a point of E; at this point f has a positive saltus in x. This non-denumerable set of saltuses contains a subset whose elements are the terms of a divergent series. A net can therefore be placed upon R to yield an arbitrarily large V-sum; from this contradiction flows the theorem.

THEOREM 14. *If $f(x, y)$ is in class V, the discontinuities in (x, y) which are not discontinuities in x or in y are denumerable.*

Let the oscillation at any such discontinuity (x_1, y_1) be α; then it is clear that in every neighborhood of this point there exists a second point (x_2, y_2) such that $\Delta_{11}f$ for the cell $(x_1, y_1; x_2, y_2)$ is $>\alpha/4$. The assumption that the set of such discontinuities is non-denumerable then leads to a contradiction just as in the case of Theorem 13.

COROLLARY. *If $f(x, y)$ is in class H, the discontinuities of $f(x, y)$ are located on a denumerable number of parallels to the axes.*†

THEOREM 15. *Class V (and therefore F) contains bounded‡ functions which are everywhere discontinuous both in x and in y; it also contains bounded non-measurable functions. Class $V\cdot C$ (and therefore $F\cdot C$) contains functions of which neither first partial derivative exists (finite or infinite) anywhere.*§

Examples. The function $f(x, y)\equiv g(x)+h(y)$, where both g and h are bounded and everywhere discontinuous, has the first property specified; if g is bounded and linearly non-measurable and h is identically zero, f has the second property; if g and h are continuous but have a derivative (finite or infinite) nowhere, f has the third property.

Of course it follows that V contains functions for which the double Lebesgue integral over R fails to exist, and that $V\cdot C$ contains functions which are nowhere totally differentiable. Nevertheless, that *every function in V for which $f(x, c)$ and $f(a, y)$ have (finite) approximate derivatives almost everywhere possesses an approximate total differential almost everywhere* is a consequence of Theorems 9 and 16, in conjunction with a theorem of Stepanoff.‖

† This corollary is also a consequence of Theorems 2 and 6 and results obtained by W. H. and G. C. Young, loc. cit.

‡ Unbounded functions having the same property are included also.

§ It would be of considerable interest to determine whether the same is true of F^* (see §4), which from one point of view is the essential part of F and which bears to F a relationship similar to that of H to V, or whether functions in F^* possess properties of continuity, etc., more like those possessed by functions in H.

‖ Stepanoff, *Sur les conditions de l'existence de la différentielle totale*, Recueil de la Société Mathématique de Moscou, vol. 32 (1925), pp. 511–526; or see Saks, loc. cit., p. 228. According to Stepanoff's theorem a necessary and sufficient condition that f ($\subset M$) have an approximate total differential almost everywhere in R is that f have (finite) approximate first partial derivatives almost everywhere in R.

THEOREM 16 (Burkill and Haslam-Jones*). *A function $f(x, y)$ in class A is totally differentiable almost everywhere.*

THEOREM 17. *A function $f(x, y)$ in class P is continuous in (x, y) almost everywhere.*

Assume the set E of points at which f has a saltus $\geqq \epsilon > 0$ to have exterior measure $k > 0$. Let the area of R be denoted by S and let $[kn^2/S]$ stand for the largest integer not exceeding kn^2/S. For a net of n^2 cells under the P_H-form of the definition, we see that at least $[kn^2/S]$ cells of the net must contain points of E; hence we have

$$\sum_{\nu=1}^{n} \omega_\nu' / n \geqq [kn^2/S]\epsilon/n,$$

which is unbounded unless k is zero. Therefore, if f is in P, k must vanish for every $\epsilon > 0$, and by a classical argument it follows that the discontinuities of f are a set of plane measure zero.

Of course it may be inferred that the double Riemann integral over R of a function in P always exists†; another consequence is the relation $\overline{T} \cdot M > P$.

THEOREM 18. *If $f(x, y)$ is in class $\overline{T} \cdot M$, $\partial f/\partial x[\partial f/\partial y]$ exists (finite) almost everywhere.*‡

Since f is in M, the set E at which $\partial f/\partial x$ fails to exist (finite) is measurable.§ Since f is in $\overline{T}$, E is intersected by almost every line $y = y_1$ in a set of linear measure zero. Hence, by Fubini's theorem, E is of plane measure zero.

COROLLARY 1. *A function $f(x, y)$ in class $\overline{T} \cdot M$ has an approximate total differential almost everywhere.*‖

This follows from the theorem of Stepanoff cited above.

COROLLARY 2. *If $f(x, y)$ is in class $\overline{T} \cdot M$, each first partial derivative is L-integrable*‡ *over R.*

It is worthy of notice that *the hypothesis $f \subset M$ cannot be dispensed with in Theorem* 18 *and its corollaries.* This may easily be shown by example as fol-

* Burkill and Haslam-Jones, *Notes on the differentiability of functions of two variables*, Journal of the London Mathematical Society, vol. 7 (1932), pp. 297–305; see also Haslam-Jones, loc. cit.

† See Hobson, loc. cit., p. 477.

‡ Theorem 18 and Corollary 2 are extensions of results obtained by Morrey (loc. cit., Theorem 1, §1) on the assumption $f \subset T \cdot C$. After Theorem 18 is established, his proof suffices for Corollary 2.

§ See Burkill and Haslam-Jones, loc. cit., Lemma 2.

‖ Corollary 1 constitutes an extension of a similar result obtained at the expense of considerable trouble by Burkill and Haslam-Jones, loc. cit.: they assumed f to be in $T \cdot M$ and to satisfy a further measurability condition; i.e., the condition which in §7 we shall show is satisfied by all functions in H.

lows. The existence of a bounded set which is not plane measurable and of which at most two points lie on any straight line has been proved by Sierpiński*; let E be such a set entirely contained in the rectangle $(a, c; b, d)$, where $0<a<b<1$, $0<c<d<1$. Then choose any four numbers a_1, b_1, c_1, d_1 to satisfy the inequalities $b<a_1<b_1<1$, $d<c_1<d_1<1$ and form the set E_1 by adding to E the following points: for each $x_1[y_1]$ in the interval (0, 1), if the line $x=x_1[y=y_1]$ contains only one point of E, add the point (x_1, c_1) $[(a_1, y_1)]$; if this line contains no point of E, add both the points (x_1, c_1) $[(a_1, y_1)]$ and (x_1, d_1) $[(b_1, y_1)]$. The characteristic function of E_1 is in T (as well as $\overline{T}$) but not in M, and it fails to have any of the properties specified in Theorem 18 and its corollaries.

Remarks. Example (D) of CA shows that $T\cdot M$ contains bounded functions (satisfying in addition the measurability condition considered in §7) which are everywhere discontinuous in (x, y) and hence nowhere totally differentiable. It has been proved by Saks† that there exist functions nowhere totally differentiable which are not only in $T\cdot C$ but satisfy considerably more stringent conditions. For $f\subset H$, W. H. Young‡ has shown that the two cross partial derivatives of second order also exist almost everywhere.

7. A property of class H

Let us set $V_x(x_0, y_0)=$ the total variation of $f(x, y_0)$ in x for $a\leqq x\leqq x_0$, $V_y(x_0, y_0)=$ the total variation of $f(x_0, y)$ in y for $c\leqq y\leqq y_0$; then we may formulate the

Definition. *A function $f(x, y)$ will be said to have the property M_v when and only when $V_x(x, y)$ and $V_y(x, y)$ are both measurable functions of (x, y) in R.*

Theorem 19. *A function $f(x, y)$ in class H has the property M_v.*

We give a proof for $V_y(x, y)$. Let us assume that this function is non-measurable, and in particular that α is a number such that the set $E[V_y(x, y)>\alpha]$ is non-measurable. Clearly E consists of the points on a set of inverted ordinates Ω standing on (or hanging from) the top of the rectangle R; an ordinate may consist of a single point, or, if it contains more than one point, it may or may not have a lowest point (i.e., be closed).

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* Sierpiński, *Sur un problème concernant les ensembles mesurables superficiellement*, Fundamenta Mathematicae, vol. 1 (1920), pp. 112–115.

† Saks, *On the surfaces without tangent planes*, Annals of Mathematics, (2), vol. 34 (1933), pp. 114–124.

‡ W. H. Young, *Sur la dérivation des fonctions à variation bornée*, Comptes Rendus (Paris), vol. 164 (1917), pp. 622–625.

By Theorem 13 the discontinuities in y of $f(x, y)$ lie on a denumerable number of lines $y=\bar{y}$; let E_1 designate this set of values $\bar{y}$. The feet of the ordinates Ω form a measurable set E_2, since E_2 is identical with the set of points x for which the measurable* function $\phi(x)$ is $>\alpha$. Let the lengths of the ordinates Ω define a function $g(x)$ over E_2. Since $g(x)$ is $\leqq d-c$, E can fail to be measurable only if the L-integral of $g(x)$ over E_2 fails to exist†, and this can occur only if $g(x)$ is a non-measurable function. Let $\beta(\geqq 0)$ be a number for which E_3 $[g(x)>\beta]$ is non-measurable. There is no restriction in assuming‡, as we now do, that $d-\beta$ does not belong to E_1. All ordinates Ω of length β will then be open.

Let E_4 be the projection of E_3 on the line $y=d-\beta$, and let $C(E_4)$ represent its complement with respect to the interval $a\leqq x\leqq b$ on this line. At each point of E_4 *which is a limit point of* $C(E_4)$, $V_y(x, d-\beta)$ is manifestly discontinuous in x; these points constitute a set E_5. Since $C(E_4)$ is non-measurable, E_5 must be likewise. Hence $m_e E_5$, and therefore the exterior measure of the set of points at which $V_y(x, d-\beta)$ has a discontinuity in x, is positive. On the other hand, by Theorem 12 above and Theorem 1 of CA, the discontinuities of $V_y(x, d-\beta)$ are denumerable. From this contradiction we infer the theorem.

Remarks. Example (C) of CA is a function in A which does not have the property M_v; hence Theorem 19 fails for A, P, and T. Examples of a function (either measurable or non-measurable) in T but without the property M_v may readily be constructed. We think it probable that Theorem 19 fails for V and F, but an example to show this does not immediately suggest itself. It should be observed that M_v is not an additive property, as is illustrated by the example following Theorem 2. Nevertheless, f being in V, V_y for f is identical with V_y for $\bar{f}$, where $\bar{f}(x, y)=f(x, y)-f(x, \bar{y})$ and $\bar{y}$ is any fixed value in the interval (c, d); and $\bar{f}(x, y)=\bar{f}(x, y)+f(\bar{x}, y)$, $\bar{x}$ being any fixed value in (a, b), can have no discontinuity in y where $\bar{f}(x, y)$ $(\subset H)$ and $f(\bar{x}, y)$ are both continuous in y. Therefore the above proof of Theorem 19 can be used to establish the following assertion: if $f(x, y)$ is in V and there exists an $\bar{x}[\bar{y}]$ in (a, b) $[(c, d)]$ for which $f(\bar{x}, y)$ $[f(x, \bar{y})]$ is continuous almost everywhere, $V_y(x, y)$ $[V_x(x, y)]$ is measurable.

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* See CA, Theorem 1.

† See Carathéodory, *Vorlesungen über Reelle Funktionen*, Berlin, 1918, p. 419; and Schlesinger and Plessner, loc. cit., p. 78.

‡ See Saks, *Théorie de l'Intégrale*, loc. cit., p. 37, where it is shown that measurability of the set $E[g(x)>\alpha]$ *for every rational* α is sufficient to insure measurability of $g(x)$. It is clear that the proof remains valid if we assume $E[g(x)>\alpha]$ measurable for any set of values α which is everywhere dense.

8. The effect of Lipschitz conditions

It is clear that the satisfaction of a Lipschitz condition,

$$|f(x+\Delta x, y+\Delta y) - f(x, y)| \leqq k(\overline{\Delta x}^2 + \overline{\Delta y}^2)^{1/2} \quad (k = \text{constant}),$$

is sufficient to place f in class $A \cdot C$ (and therefore $P \cdot C$ and $T \cdot C$). At the same time it is insufficient to put f in H, V, or F, as the following example shows. Divide the unit rectangle I into columns by the lines $1-1/2^n (n=1, 2, \cdots)$; proceeding from left to right, divide the nth column into 2^n squares. On each square define f by the height of a regular pyramid with that square as base and with altitude equal to a side of the square, and let $f(1, y) \equiv 0$.

Fréchet* has observed that the satisfaction of a Lipschitz condition in terms of area,

$$|\Delta_{11} f(x, y)| \leqq k|\Delta x \cdot \Delta y| \qquad (k = \text{constant}),$$

is sufficient to insure that f be in V (and therefore F); that it does not suffice to put f in any of the other classes may readily be seen by examples.

9. Dependence upon axes

It is quite clear that a function in class V, H, A, or F may fail to remain in that class when the x, y axes are rotated through a suitably chosen angle. On the other hand, definition P may easily be proved independent of the axes, and $T \cdot C$ is manifestly independent of the axes because of its geometric significance.† The question for T (or $\overline{T}$) is not so easily answered, and we shall construct an example to show that T (and $\overline{T}$) is *not* independent of the axes.‡

Let $E_x[E_y]$ be the set of numbers in the interval (0, 1) which have a triadic representation free from the digit 2 [1], and define $f(x, y)$ as the characteristic function of the set E of points (x, y) for which x is in E_x and y is in E_y. Since E_x and E_y are Cantor sets of measure zero, we have $f \subset T$. It will be shown that f does not remain in T when the axes are rotated through the angle $\pi/4$.

The equation of the perpendicular to $y=x$ at (x_0, x_0) is $x+y=2x_0$, and it is apparent that for any x_0 in the interval $(0, \frac{1}{2})$ this line contains at least

* Fréchet, *Extension au cas des intégrales multiples d'une définition de l'intégrale due à Stieltjes*, Nouvelles Annales de Mathématiques, (4), vol. 10 (1910), pp. 241–256.

† That is, a necessary and sufficient condition that a continuous surface $z=f(x, y)$ have area in the Lebesgue sense is $f(x, y) \subset T$.

‡ We are indebted to Dr. W. C. Randels for suggesting this example. It is probable that our purpose would also be served by the characteristic function of some of the sets constructed by Mazurkiewicz and Saks, *Sur les projections d'un ensemble fermé*, Fundamenta Mathematicae, vol. 8 (1926), pp. 109–113, but the example given here seems somewhat easier to discuss.

one point of E. For, $2x_0$ being given in triadic form, corresponding digits in x_1 and y_1 such that $x_1+y_1=2x_0$ can be chosen as follows: in $x_1[y_1]$ put a 0 wherever a 0 or 2[1] occurs in $2x_0$, and put a 1[2] wherever a 1[2] occurs in $2x_0$. Such a choice of digits for x_1, y_1 may be said to be "according to rule." Let $\phi_1(\bar{x})$ be the ϕ-function for the new x-axis (i.e., the line $y=x$ in the original coordinate system).

Consider first any $2x_0$ of the form $.10\cdots$, the remaining digits being arbitrary. The points (x_0, x_0) corresponding to these numbers fill an interval I_1 on $y=x$ of length $1/(3^2 2^{1/2})$. To each such number we have

$$\begin{cases} x_1 = .10\cdots, \\ y_1 = .00\cdots, \end{cases} \qquad \begin{cases} x_1 = .01\cdots, \\ y_1 = .02\cdots, \end{cases} \tag{11}$$

the remaining digits in all cases being chosen according to rule. We then have

$$\int_{I_1} \phi_1 > 2/(3^2 2^{1/2}).$$

Next consider $2x_0 = .ab10\cdots$, the subsequent digits being arbitrary and a, b anything except 1, 0. The points (x_0, x_0) corresponding to these numbers fill 3^2-1 intervals each of length $1/(3^4 2^{1/2})$; this set of intervals we may call I_2. To each number $2x_0$ of the present form we may choose the third and fourth digits as the first two were chosen in (11) and choose the rest according to rule. We obtain

$$\int_{I_2} \phi_1 > 2(3^2 - 1)/(3^4 2^{1/2}).$$

Continuing in this manner we find

$$\int \phi_1 > \frac{2}{2^{1/2}} \sum_{n=0}^{3^2-1} \frac{3^2 - n}{3^4} > 1/2^{1/2}.$$

Repeating this process using blocks of $2p$ digits $1010\cdots 10$, to each of which there correspond 2^p choices instead of the two in (11), we obtain

$$\int \phi_1 > \frac{2^p}{2^{1/2}} \sum_{n=0}^{3^{2p}-1} \frac{3^{2p} - n}{3^{4p}} > 2^{p-1}/2^{1/2}.$$

Since p is arbitrary, $\int \phi_1$ does not exist and our assertion is proved.

10. Factorable functions belonging to the several classes

For our present purposes a function $f(x, y)$ will be called *factorable* if and only if we have in R

$$f(x, y) \equiv g(x)h(y),$$

with neither g nor h identically zero.* The verification of the following equations is then immediate:

$$\Delta_{11} f(x, y) = \Delta\, g(x)\, \Delta\, h(y),$$

and for each net

$$\begin{aligned}\max \sum_{i,j} \epsilon_i \bar{\epsilon}_j \Delta_{11} f(x_i, y_j) &= \max \left[\sum_i \epsilon_i \Delta g(x_i) \sum_j \bar{\epsilon}_j \Delta h(y_j) \right] \\ &= \sum_i | \Delta g(x_i) | \sum_j | \Delta h(y_j) | \\ &= \sum_{i,j} | \Delta_{11} f(x_i, y_j) | .\end{aligned}$$

Conclusions may be drawn as follows.

THEOREM 20. *A necessary and sufficient condition that a factorable function be in class H is that each factor be of bounded variation. A factorable function, with one factor of unbounded variation and the other a constant, is in V and F but not in A, P, T, or $\overline{T}$. A factorable function, with one factor of unbounded variation and the other not a constant, is not in V, F, or A; it is not in P, T, or $\overline{T}$ unless the latter factor vanishes almost everywhere, and even then it may not be.*

COROLLARY. *Class A contains no factorable functions save those in H; F contains no factorable functions save those in V; but each of the classes V, P, and T contains factorable functions which are not in H. A factorable function in T but not in H must vanish almost everywhere in R.*

11. THE "VARIATION" OF FUNCTIONS BELONGING TO THE SEVERAL CLASSES

It is our object here to direct attention to two things: (i) the fact that a function belongs to one of the several classes conveys, in most cases, comparatively little idea of the extent to which the function fluctuates in R; and (ii) the difficulty of associating with a function belonging to any one of the several classes, by means of the definition of that class, a number which conveys any precise notion of the amount of fluctuation of the functional values.

Let us first consider the classes V, F, and T (or $\overline{T}$). It has already been remarked in CA that the V-sum and the maximum F-sum for a given net N are never decreased when new horizontal or vertical lines are added to form a net N'. Therefore it might be considered natural to define the total variation of a function in V or F as the least upper bound of the respective V- or F-sum. For a function in T the quantities

* In the contrary case f is obviously in H.

$$\frac{1}{b-a}\int_a^b \phi(x)dx, \quad \frac{1}{d-c}\int_c^d \psi(y)dy \tag{12}$$

are the average total variations respectively of $f(\bar{x}, y)$ in y and $f(x, \bar{y})$ in x. One might therefore consider it desirable to define the total variation of a function in T as the larger of the numbers (12), or perhaps some linear combination of them. Under such definitions each of the classes V, F, and T contains* functions *with an arbitrarily large saltus at every point* in R whose total variation is zero! The reader inclined to be critical of our point of view may aver that it should not matter much what values a function $f(x, y)$ has on a set of plane measure zero, and it is true that a function in T whose total variation is zero according to the definition suggested above is "almost a constant"; nevertheless we are inclined to insist that when the total variation of $g(x)$ is in question, it matters a great deal what values $g(x)$ assumes on a set of linear measure zero.

If when f is in H one were to define the total variation as the least upper bound of the V-sum, very little notion of the amount of fluctuation would be conveyed; for every function $f(x, y) \equiv g(x)$, where $g(x)$ is of bounded variation, would have total variation zero as a function of two variables, independently of the value of the total variation of $g(x)$ and although in general $f(x, y)$ is not even "almost a constant".

If a function is in A, it would be natural to define its total variation as the least upper bound of the A-sum. This procedure, however, has several disadvantages, including the fact that the total variation of a function in R would not in general be the sum of its total variations in the two rectangles into which R is divided by a vertical (or horizontal) line.

It is quite clear that except when $f(x, y)$ is a constant, the total variation of a function in P, defined as the least upper bound of the P-sum, would depend upon the value of the fixed upper bound for D, the side of the square cells employed. One would naturally turn, therefore, to the P_H-form of the definition. Since the P_H-sum may decrease as n increases, it might be preferable to define the total variation, not as the least upper bound of the P_H-sum, but as the $\overline{\lim}_{n\to\infty}$ of this sum. Whichever choice were made, the definition would be open to the objection that the total variation of such a function as

$$f(x, y) = \left\{ \begin{array}{l} 1 \text{ for } x = \bar{x} \\ 0 \text{ for } x \neq \bar{x} \end{array} \right\} \text{ in I}$$

would be different for $\bar{x}$ rational and for $\bar{x}$ irrational. This objection can only

* This is clearly indicated by examples given above in §6 and example (D) of CA.

be met by insisting that the oscillations ω_r' in the n^2 cells be computed for cells so defined that no two have points in common; yet if this were done, the total variation in R would not in general be equal to the sum of the total variations in the two rectangles into which R is divided by a vertical (or horizontal) line.

For the reasons described above *it would seem desirable to regard the several definitions of bounded variation for functions of two variables purely as formal generalizations of analytic conditions in common use in the theory of functions of a single variable or as conditions which single out for consideration some class of functions having one or more properties like those of a function $g(x)$ of bounded variation,* and rather completely to disassociate the term "function of bounded variation" from any notion of the amount which the function $f(x, y)$ fluctuates in the rectangle R.*

* See certain remarks in CA, pp. 826–827.

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