CONVERGENCE PROPERTIES OF FOURIER SERIES*

BY

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I. INTRODUCTION

1. In our previous papers† we were concerned with Fourier series

\[ f(\theta) \sim \frac{a_0}{2} + \sum_{r=1}^{\infty} (a_r \cos \nu \theta + b_r \sin \nu \theta) \]

the coefficients of which satisfy the conditions

\[ \nu a_r \geq -K, \nu b_r \geq -K, \nu \geq 1, K \text{ a non-negative constant.} \]

These conditions characterize a special case of "slowly oscillating" series, and it is natural to generalize our results in this direction. In what follows the knowledge of our previous papers is not presupposed, except for the proof of Lemma 5 below.

2. We start with some preliminary notions. A series of real terms \( \sum_{r=0}^{\infty} c_r \) or the corresponding sequence of its partial sums \( \{s_n\} \), is called slowly oscillating from below if to any positive \( \epsilon \) there corresponds an integer \( N = N(\epsilon) \) and a positive number \( \delta = \delta(\epsilon) \) such that

\[ s_{n+k} - s_n > -\epsilon, \quad n > N, \quad 0 < k < \delta(n + 1). \]

It is readily seen that this condition is equivalent to

\[ \lim_{\delta \to 0} \lim_{n \to \infty} \inf_{0 < k < \delta(n + 1)} \min (s_{n+k} - s_n) \geq 0. \]

We denote by \((A)\) the class of such series, or sequences.

A series is called slowly oscillating from below in the generalized sense if there exist two positive numbers \( q \) and \( \delta \) such that

\[ s_{n+k} - s_n > -q; \quad n = 0, 1, 2, \ldots; \quad 0 < k < \delta(n + 1), \]

which is equivalent to

\[ \lim_{n \to \infty} \inf_{0 < k < \delta(n + 1)} \min (s_{n+k} - s_n) > -\infty. \]

The class of such series will be denoted by \((A)\). It is obvious that \((A) \subset (\bar{A})\).
These notions are important for converse theorems of the theory of summability.*

Let \( (A') \) be the class of series satisfying the condition \( \liminf_{n \to \infty} n c_n > -\infty \). Then we have \( (A') \subset (A) \subset (A) \). If \( \{ \delta_n \} \) is any bounded sequence of non-negative numbers and \( \sum_0^\infty c_n, c (A') \) then also \( \sum_0^\infty c_n, c (A') \). This property was used in our papers [5, 6]. An analogous but larger class of series is obtained if we observe that, in order that the series \( \sum_0^\infty c_n, c (A) \) for an arbitrary bounded and non-negative sequence \( \{ \delta_n \} \), it is necessary and sufficient that it belong to \( (A) \) in the special case

\[
\delta_n = 1, \text{ or } 0, \text{ according as } c_n < 0, \text{ or } \geq 0,
\]

in other words that the series \( \sum_0^\infty (c_n - |c_n|) c (A) \). The class of such series we denote by \( (B) \). Thus \( (B) \) is the class of series such that \( \sum_0^\infty c_n, c (A) \) whenever \( \sum_0^\infty c_n, c (A) \). We have the inclusion relation \( (A') \subset (B) \subset (A) \subset (A) \).

Similarly we say that \( \sum_0^\infty c_n, c (B) \) if \( \sum_0^\infty (c_n - |c_n|) c (A) \). The class \( (B) \) is identical to the class of series \( \sum_0^\infty c_n \) such that \( \sum_0^\infty c_n, c (A) \) for an arbitrary bounded and non-negative sequence \( \{ \delta_n \} \), whenever \( \sum_0^\infty c_n, c (A) \). It is plain that \( (A') \subset (B) \subset (B) \subset (A) \).

3. In the present paper we propose to prove the following theorems.

**Theorem 1.** Let \( \sum_1^\infty a_{r^*} \cos r \theta, 0 \leq r < 1, \) represent a harmonic function and let

(i) \[
\sum_{r=1}^\infty a_r \subset (B),
\]

(ii) \[
\sum_{r=1}^\infty a_r, c (B) = O(1), \quad 0 \leq r < 1.
\]

Then also

(iii') \[
\sum_{r=1}^n a_r = O(1) \quad (n = 1, 2, 3, \cdots).
\]

Conversely, the assumption (iii') implies (ii). If, in addition to (i) and (ii) we assume that for a fixed \( \theta = \theta_0 \)

(iii) \[
\sum_{r=1}^\infty a_{r^*} \cos r \theta_0^{*r} = O(1), \quad 0 \leq r < 1,
\]

then also

(iii') \[
\sum_{r=1}^n a_r \cos r \theta_0^{*r} = O(1) \quad (n^* = 1, 2, 3, \cdots).
\]

* Cf. [2], p. 332; [3], p. 30; [4], p. 326.
Conversely, (iii') implies (iii). Finally, if
\[
\phi(\theta) \sim \sum_{r=1}^{\infty} a_r \cos r\theta
\]
is a Fourier series and
\[
\sum_{r=1}^{n} a_r \cos r\theta = O(1) \quad (n = 1, 2, 3, \ldots),
\]
then
\[
\int_{0}^{h} \{\phi(\theta_0 + t) + \phi(\theta_0 - t)\} \, dt = O(h) \text{ as } h \to 0.
\]

**Theorem 2.** Let \( \sum a_r \in \mathcal{B} \) and \( \sum a_r r^s \to s \) as \( r \to 1 - 0 \). Then \( \sum a_r \) converges to \( s \). If in addition, for a fixed \( \theta = \theta_0 \),
\[
\sum_{r=1}^{\infty} a_r r^s \cos r\theta_0 \to s(\theta_0) \text{ as } r \to 1 - 0,
\]
then \( \sum a_r \cos r\theta_0 \) converges to \( s(\theta_0) \). Finally, if
\[
\phi(\theta) \sim \sum_{r=1}^{\infty} a_r \cos r\theta
\]
is a Fourier series and \( \sum a_r \cos r\theta_0 \) converges to \( s(\theta_0) \), then
\[
(2h)^{-1} \int_{0}^{h} \{\phi(\theta_0 + t) + \phi(\theta_0 - t)\} \, dt \to s(\theta_0) \text{ as } h \to 0.
\]

**Theorem 3.** Let
\[
\omega(\theta) \sim \sum_{r=1}^{\infty} b_r \sin r\theta
\]
be a Fourier series and \( \sum b_r \in \mathcal{B} \). Let
\[
\int_{0}^{h} \omega(t) \, dt = O(h) \text{ as } h \to 0.
\]
Then
\[
\sum_{r=1}^{n} v b_r = O(n) \quad (n = 1, 2, 3, \ldots).
\]
If, in addition,
\[
\sum_{r=1}^{\infty} b_r r^s \sin r\theta_0 = O(1), \quad 0 \leq r < 1,
\]

* This is a well known result of R. Schmidt.
then
\[ \sum_{r=1}^{n} b_r \sin \nu \theta_0 = O(1) \quad (n = 1, 2, 3, \ldots). \]

Conversely, if
\[ \sum_{r=1}^{n} b_r \sin \nu \theta_0 = O(1) \quad (n = 1, 2, 3, \ldots), \]
then
\[ \int_{0}^{h} \{\omega(\theta_0 + t) + \omega(\theta_0 - t)\} \, dt = O(h) \text{ as } h \to 0. \]

**Theorem 4.** Let \( \sum_{r=1}^{*} b_r \in (B) \) and
\[ \frac{2}{h} \int_{0}^{h} \omega(t) \, dt \to d \text{ as } h \to 0. \]

Then
\[ \frac{1}{n} \sum_{r=1}^{n} \nu b_r \to \frac{d}{\pi} \text{ as } n \to \infty. \]

If, in addition,
\[ \sum_{r=1}^{*} b_r r^{*} \sin \nu \theta_0 \to s(\theta_0) \text{ as } r \to 1 - 0, \]
then \( \sum_{r=1}^{*} b_r \sin \nu \theta_0 \) converges to \( s(\theta_0) \). Conversely, if \( \sum_{r=1}^{*} b_r \sin \nu \theta_0 \) converges, then
\[ \frac{2}{h} \int_{0}^{h} \{\omega(\theta_0 + t) + \omega(\theta_0 - t)\} \, dt \to s(\theta_0) \text{ as } h \to 0. \]

In proving these theorems we are using methods analogous to those of our papers \([5, 6]\). Similar theorems hold with respect to the uniform boundedness or uniform convergence in a given interval \( \alpha \leq \theta \leq \beta \). In the hypotheses of our theorems the sums \( \sum_{r=1}^{*} \frac{1}{2}(a_r - |a_r|), \sum_{r=1}^{*} \frac{1}{2}(b_r - |b_r|) \) may be replaced by the sums \( \sum_{r=1}^{*} \frac{1}{2}(-a_r - |a_r|), \sum_{r=1}^{*} \frac{1}{2}(-b_r - |b_r|) \) respectively.

Analogous theorems may also be stated for double Fourier series, as well as for almost periodic functions or for Fourier integrals.

The treatment of the cosine series is conspicuously simpler than that of the sine series, the reason being that to the value \( \theta = 0 \) in the first case there

\[ \text{* The quantity } d \text{ may be interpreted as the generalized jump of } \omega(\theta) \text{ at } \theta = 0. \]
corresponds the series \( \sum a_r \) while, in the second case, all terms of the series vanish.*

II. The cosine series

1. Proof of Theorem 1. We begin by establishing some lemmas.

**Lemma 1.** Put

\[
\sum_{r=1}^{n} c_r = s_n, \quad \sum_{r=1}^{n} \nu c_r = v_n, \quad v_0 = s_0 = c_0 = 0,
\]

and assume that there exist two numbers, \( p > 0 \) and \( \mu > 0 \), such that

\[(1) \quad s_{n+k} - s_n \geq 1 + \mu(n + 1) \quad (n = 0, 1, 2, \ldots).
\]

Then

\[(2) \quad v_{n+k} - v_n \geq -p \mu n, \quad n > 0.
\]

Since

\[
v_n = \sum_{r=0}^{n} (s_r - s_{r+1}), \quad v_{n+k} = \sum_{r=0}^{n+k} (s_{n+k} - s_{r+1}),
\]

we have

\[
v_{n+k} - v_n = (n + 1)(s_{n+k} - s_n) + \sum_{r=n+1}^{n+k} (s_{n+k} - s_r) \geq -p(n + 1) - p(k - 1),
\]

which is the desired inequality (2). Now put

\[
n_\nu = [n(1 + \mu)^{-\nu}] \quad (\nu = 0, 1, 2, \ldots),
\]

so that

\[
v_n = \sum_{\nu=0}^{\infty} (v_\nu - v_{\nu+1}),
\]

where only a finite number of terms are different from zero. In view of the obvious inequality \( n_\nu - n_{\nu+1} < 1 + \mu(1 + n_{\nu+1}) \) we have

\[
v_n \geq -p \sum_{\nu=0}^{\infty} n_\nu \geq -p n \sum_{\nu=0}^{\infty} (1 + \mu)^{-\nu} = -p \frac{1 + \mu}{\mu} n,
\]

* The results of this paper were communicated in the author's seminar during the spring term at the Massachusetts Institute of Technology, and also at the colloquium at Brown University May 18, 1934.

† Cf. [2], p. 333; [4], pp. 326–327.
which proves (3).

Lemma 2. Under the assumptions (1) and

\[(4) \quad \left| \sum_{r=1}^{\infty} c_r x^r \right| \leq M, \quad 0 \leq x < 1,\]

we have

\[|s_n| < M(1 + 8e) \frac{2 + \mu}{\mu} + p \left(1 + 4e \frac{(2 + \mu)(1 + \mu)}{\mu^2}\right).\]

We set

\[\sum_{r=1}^{\infty} c_r x^r = P(x);\]

then from

\[\frac{v_n}{n + 1} = s_n - (n + 1)^{-1} \sum_{r=0}^{n} s_r\]

it follows that

\[\sum_{r=1}^{\infty} \frac{v_r}{\nu + 1} x^r = (1 - x)^{-1} P(x) - x^{-1} \int_0^x P(t)(1 - t)^{-2} dt,\]

and, by (4),

\[\left| \sum_{r=1}^{\infty} \frac{v_r}{\nu + 1} x^r \right| \leq 2M(1 - x)^{-1}.\]

Hence

\[\sum_{r=1}^{\infty} \left(\frac{v_r}{\nu + 1} + p \frac{1 + \mu}{\mu}\right) x^r \leq 2M(1 - x)^{-1} + p \frac{1 + \mu}{\mu} x(1 - x)^{-1}.\]

The coefficients of the power series of the left-hand member are positive, by (3). Consequently, for \(x = 1 - 1/(n + 1),\)

\[\left(1 - \frac{1}{n + 1}\right)^n \sum_{r=1}^{n} \left(\frac{v_r}{\nu + 1} + p \frac{1 + \mu}{\mu}\right)
\leq \sum_{r=1}^{n} \left(\frac{v_r}{\nu + 1} + p \frac{1 + \mu}{\mu}\right) \left(1 - \frac{1}{n + 1}\right)^r
\leq \left(2M + p \frac{1 + \mu}{\mu}\right)(n + 1),\]

whence
\[ \sum_{r=1}^{n} \frac{v_r}{n+1} < \left( 2M + \rho \frac{1+\mu}{\mu} \right) \left( 1 + \frac{1}{n} \right)^n (n+1) \]
\[ < \left( 2M + \rho \frac{1+\mu}{\mu} \right) e(n+1). \]

On the other hand, again by (3),
\[ \sum_{r=1}^{n} \frac{v_r}{n+1} > -\rho n \frac{1+\mu}{\mu}, \quad n > 0, \]
which yields the result
\[ \left| \sum_{r=1}^{n} \frac{v_r}{n+1} \right| < e \left( 2M + \rho \frac{1+\mu}{\mu} \right) (n+1), \quad n > 0. \]

Now, on setting
\[ (n+1)^{-1} \sum_{r=0}^{n} s_r = \sigma_n, \quad \sum_{r=1}^{n} \frac{v_r}{n+1} = u_n, \]
we have
\[ x^{-1} \int_{0}^{x} P(t) (1-t)^{-2} dt = \sum_{r=1}^{\infty} \sigma_r x^r, \]
\[ (1-x)x^{-1} \int_{0}^{x} P(t) (1-t)^{-2} dt = \sum_{r=1}^{\infty} \left( \sigma_r - \sigma_{r-1} \right) x^r = \sum_{r=1}^{\infty} \frac{u_r}{\nu(\nu+1)} x^r, \]
and*
\[ (1-x)x^{-1} \int_{0}^{x} P(t) (1-t)^{-2} dt - \sigma_n = (1-x) \sum_{r=1}^{\infty} \frac{u_r}{\nu+1} x^r - \frac{u_n}{n+1} \]
\[ + \sum_{r=1}^{n} \frac{u_r}{\nu(\nu+1)} (x^r - 1) + \sum_{r=n+1}^{\infty} \frac{u_r}{\nu(\nu+1)} x^r. \]

By (4)
\[ \left| (1-x)x^{-1} \int_{0}^{x} P(t) (1-t)^{-2} dt \right| \leq M. \]

On combining this with the preceding inequality we have
\[ |\sigma_n| < M + 2e \left( 2M + \rho \frac{1+\mu}{\mu} \right) \]
\[ + e \left( 2M + \rho \frac{1+\mu}{\mu} \right) \left\{ (n(1-x) + \frac{1}{n+1} \frac{x^{n+1}}{1-x} \right\}, \]

* Cf. [1], Theorem 1, with \( \nu a_r \) replaced by \( v_r/(\nu+1) \).
whence, for \( x = 1 - 1/(n+1) \),

\[
|\sigma_n| < M + 4\varepsilon \left( 2M + \frac{1 + \mu}{\mu} \right) = M_1, \quad n \geq 1.
\]

Consequently

\[
s_n = \frac{v_n}{n+1} + \sigma_n > -\frac{1 + \mu}{\mu} - M_1, \quad n \geq 1.
\]

On the other hand we have*

\[
s_n = \sigma_{n+k} + \frac{n+1}{k} (\sigma_{n+k} - \sigma_n) - \frac{1}{k} \sum_{r=1}^{k} (s_{n+r} - s_n),
\]

whence, for \( \mu(n+1) \leq k < 1 + \mu(n+1) \),

\[
s_n < M_1 + 2M_1/\mu + \rho.
\]

Lemma 2 follows at once by combining these two inequalities for \( s_n \).

We now pass on to the proof of Theorem 1. Under the hypotheses (i) and (ii) Lemma 2 implies

\[
s_n = \sum_{r=1}^{n} a_r = O(1) \quad (n = 1, 2, 3, \ldots).
\]

Conversely, if this condition is satisfied then

\[
\sum_{r=1}^{\infty} a_r x^r = (1 - x) \sum_{r=1}^{\infty} s_r x^r = O(1), \quad 0 \leq x < 1.
\]

Furthermore, the series \( \sum_{r=1}^{\infty} a_r (1 - \cos \nu \theta_0) \) also \( c(A) \). Then, if (iii) is satisfied, by Lemma 2,

\[
\sum_{r=1}^{n} a_r (1 - \cos \nu \theta_0) = O(1), \quad \text{and} \quad \sum_{r=1}^{n} a_r \cos \nu \theta_0 = O(1) \quad (n = 1, 2, 3, \ldots).
\]

The converse is proved by the same argument as before.

To prove the last statement of the theorem we observe that

\[
(2h)^{-1} \int_{0}^{h} \{ \phi(\theta_0 + t) + \phi(\theta_0 - t) \} \, dt = \sum_{r=1}^{\infty} a_r \cos \nu \theta_0 (\sin \nu h)/(\nu h).
\]

By assumption \( \sum_{r=1}^{\infty} (a_r - |a_r|) \in c(A) \), and we have proved that \( \sum_{r=1}^{n} a_r = O(1) \).

Hence the series \( -\sum_{r=1}^{\infty} a_r \) as well as \( -\sum_{r=1}^{\infty} |a_r| \in c(A) \). This means that, for suitably chosen \( \rho \) and \( \mu \),

\[
\sum_{r=n+1}^{n+k} |a_r| \leq \rho, \quad 1 \leq k < 1 + \mu(n+1), \quad n \geq 0.
\]

* Cf. [3], p. 31.
Then, by Lemma 1,
\[ \sum_{\nu=1}^{n} \nu | a_{\nu} | < n \phi \frac{1 + \mu}{\mu}, \quad n > 0. \]

Now put
\[ \sum_{\nu=1}^{\infty} a_{\nu} \cos \nu \theta / (\nu h) = \sum_{\nu=1}^{n} a_{\nu} \cos \nu \theta + \sum_{\nu=1}^{n} a_{\nu} \cos \nu \theta / (\nu h) - 1 \]
\[ + \sum_{\nu=1}^{\infty} a_{\nu} \cos \nu \theta / (\nu h) \equiv S_1 + S_2 + S_3. \]

In the previous argument we have proved the existence of a constant $G$ such that
\[ |S_1| \leq G. \]

From
\[ 0 \leq 1 - \frac{(\sin \nu h)}{(\nu h)} \leq \frac{\nu^2 h^2}{h^2} \]
we have
\[ |S_2| \leq \frac{h^2}{h^2} \sum_{\nu=1}^{n} \nu^2 | a_{\nu} | \leq \frac{h^2}{6} \sum_{\nu=1}^{n} \nu | a_{\nu} | < \phi \frac{1 + \mu}{6 \mu} n^2 h^2. \]

Finally, on writing $\tau_{n+k} = \sum_{n+1}^{n+k} | a_{\nu} | / \nu$ we have
\[ \tau_{n+k} \leq (n+1)^{-1} \sum_{r=n+1}^{n+k} | a_{\nu} | \leq \phi / (n+1), \quad 1 \leq k < 1 + \mu(n + 1). \]

Let $n_0$, $n_1$, \ldots be a sequence of integers such that
\[ (1 + \mu)n_{r+1} + \mu \leq n_r < (1 + \mu)(1 + n_r), \quad n_0 = n. \]

Then
\[ \sum_{r=n+1}^{n+k} | a_{\nu} | / \nu = \sum_{r=n+1}^{n} + \cdots + \sum_{r=n+k-1}^{n+k} \leq \phi \sum_{r=0}^{k-1} (n_r + 1)^{-1} \]
\[ \leq \frac{\phi}{n+1} \sum_{r=0}^{k-1} (1 + \mu)^{-r} < \frac{\phi(1 + \mu)}{(n+1)\mu}, \]
since
\[ n_r + 1 \geq (1 + \mu)(n_{r-1} + 1) \geq (1 + \mu)^r(n + 1). \]

Thus
\[ \sum_{r=n+1}^{\infty} | a_{\nu} | / \nu \leq \frac{\phi(1 + \mu)}{(n+1)\mu} \]
and

\[ |S_3| \leq h^{-1} \sum_{r=n+1}^{\infty} |a_r| /\nu \leq \frac{p(1 + \mu)}{h(n + 1)\mu}. \]

On choosing \( n = \lceil 1/h \rceil, \ h < 1 \), we finally obtain

\[ \left| (2h)^{-1} \int_0^h \{ \phi(\theta_0 + t) + \phi(\theta_0 - t) \} dt \right| < G + \frac{7}{6} \frac{1 + \mu}{\mu}. \]

2. Proof of Theorem 2. By hypothesis \( \sum_{r=1}^{\omega} (a_r - |a_r|) \in \mathcal{A} \). Since \( \sum_{r=1}^{\omega} a_r \) converges, \( -\sum_{r=1}^{\omega} |a_r| \) also \( \in \mathcal{A} \). Hence \( \sum_{r=1}^{\omega} a_r \cos \nu \theta \in \mathcal{A} \) for every \( \theta \). If now

\[ \sum_{r=1}^{\omega} a_r r^n \cos \nu \theta_0 \to s(\theta_0) \text{ as } r \to 1 - 0, \]

the theorem of R. Schmidt mentioned above implies

\[ \sum_{r=1}^{\omega} a_r \cos \nu \theta_0 = s(\theta_0). \]

It remains to prove the last statement of Theorem 2. We write

\[
(2h^{-1}) \int_0^h \{ \phi(\theta_0 + t) + \phi(\theta_0 - t) \} dt - s(\theta_0)
\]

\[
= \left( \sum_{r=1}^{n} a_r \cos \nu \theta_0 - s(\theta_0) \right) + \sum_{r=1}^{n} a_r \cos \nu \theta_0 \left( \frac{\sin \nu h}{\nu h} - 1 \right)
\]

\[
+ \sum_{r=n+1}^{\lambda_n} a_r \cos \nu \theta_0 \frac{\sin \nu h}{\nu h} + \sum_{r=\lambda_n+1}^{\omega} a_r \cos \nu \theta_0 \frac{\sin \nu h}{\nu h}
\]

\[ = Z_0 + Z_1 + Z_2 + Z_3, \]

where \( \lambda_n > n \) will be fixed later. For a given \( \epsilon, 0 < \epsilon < 1 \), choose \( N = N(\epsilon) \) so that

\[ |Z_0| < \epsilon^2, \quad n > N. \]

From (5) and (6) we have

\[ |Z_1| < \rho \frac{1 + \mu}{\mu} n^2 h^2, \quad |Z_3| \leq \frac{p(1 + \mu)}{h(\lambda_n + 1)\mu}. \]

Let \( h_0 = h_0(\epsilon) \) be so small that

\[ h_0 < \epsilon, \quad [\epsilon/h_0] > N, \]

and put

\[ n = [\epsilon/h], \quad \lambda_n = \lceil \pi/(\epsilon h) \rceil, \quad h < h_0, \]
so that
\[ nh \leq h, \quad 1 + \lambda_n > \pi/(\epsilon h). \]

Then
\[ |Z_1| < \frac{1 + \mu}{\mu} \epsilon^2, \quad |Z_3| < \frac{\mu(1 + \mu)}{\pi \mu} \epsilon. \]

To estimate \( Z_2 \) subdivide the range \( 0 < v h \leq \lambda_n h < \pi/e \) into subintervals in each of which \( (\sin v h)/v h \) is monotone (as function of \( v \)), the number of these subintervals being not greater than \( 1 + [1/e] \). An easy application of the partial summation formula together with (7) will show that
\[ |Z_2| < (2/e)2\epsilon^2 = 4\epsilon. \]

To complete the proof of Theorem 2 it remains to allow \( \epsilon \rightarrow 0 \).

III. The sine series

1. Proof of Theorem 3. We shall need some additional lemmas.

**Lemma 3.** Let
\[ \omega(\theta) \sim \sum_{r=1}^{\infty} b_r \sin r\theta. \]

If
\[ \int_0^h \omega(t) dt = O(h) \text{ as } h \rightarrow 0, \]

then
\[ \sum_{r=1}^{\infty} vb_r r^* = O((1 - r)^{-1}) \text{ as } r \rightarrow 1 - 0. \]

We have*
\[ \sum_{r=1}^{\infty} vb_r r^* = - \pi^{-1} \int_0^{\pi} \omega(t) \frac{d}{dt} \varphi(r, t) dt, \quad 0 \leq r < 1, \]

where
\[ \varphi(r, t) = (1 - r^2)/\Delta, \quad \Delta = 1 - 2r \cos t + r^2 = (1 - r)^2 + 4r \sin^2 \frac{t}{2}, \]
\[ \frac{d}{dt} \varphi(r, t) = - 2r(1 - r^2) \sin t/\Delta^2. \]

On setting
\[ \psi(t) = \int_0^t \omega(\tau) d\tau \]

* Cf. [5], formula (21).
and integrating by parts we have
\[
\sum_{r=1}^{\infty} \nu_b r^s = -\pi^{-1}2r(1 - r^2) \int_0^\infty \psi(t) \frac{d}{dt} (\sin t \Delta^{-2}) dt.
\]
In view of (8) an easy calculation yields
\[
\sum_{r=1}^{\infty} \nu_b r^s = O\left\{ (1 - r)^2 \int_0^\infty t \{ (1 - r)^2 + t^2 \}^{-2} dt \right\} = O((1 - r)^{-1}).
\]

**Lemma 4.** If (8) holds and \(\sum_1^{\infty} b, c (A)\), then
\[
v_n = \sum_{r=1}^{n} \nu_b r = O(n) \quad (n = 1, 2, 3, \ldots).
\]

By Lemma 3
\[
\sum_{r=1}^{\infty} \nu_b r^s = O((1 - r)^{-1}),
\]
while, by Lemma 1,
\[
(3') \quad v_r + p \frac{1 + \mu}{\mu} v > 0 \quad (\nu = 1, 2, 3, \ldots),
\]
with suitably chosen \(p\) and \(\mu\). Hence
\[
\sum_{r=1}^{\infty} \left( v_r + p \frac{1 + \mu}{\mu} v \right) r^s = O((1 - r)^{-2}),
\]
\[
\sum_{r=1}^{n} \left( v_r + p \frac{1 + \mu}{\mu} v \right) = O\left\{ \sum_{r=1}^{n} \left( v_r + p \frac{1 + \mu}{\mu} v \right) \left( 1 - \frac{1}{n} \right)^r \right\} = O(n^2).
\]
Thus
\[
V_n = \sum_{r=1}^{n} v_r = O(n^2) \quad (n = 1, 2, 3, \ldots).
\]

On the other hand the relation
\[
k v_n = \frac{kV_{n+k}}{n + k + 1} + (n + 1) \left( \frac{V_{n+k}}{n + k + 1} - \frac{V_n}{n + 1} \right) - \sum_{r=1}^{k} (v_{n+r} - v_n)
\]
gives, in view of (2),
\[
v_n < Cn,
\]
where \(C\) is a generic notation for a constant, not necessarily the same in all formulas where it occurs. On combining this with (3') we obtain a proof of Lemma 4, and also of the first statement of Theorem 3.
Now, if $\sum_{i} b_{i} \in (B)$, then $\sum_{i} b_{i}(1 - \sin \nu \theta) \in (A)$. If, in addition,

$$\sum_{r=1}^{\infty} b_{r} r^{r} \sin \nu \theta_{0} = O(1), \quad 0 \leq r < 1,$$

then, as in the proof of Lemma 2,

$$\sum_{r=1}^{\infty} (\nu + 1)^{-r}(b_{1} \sin \theta_{0} + \cdots + \nu b_{r} \sin \nu \theta_{0}) r^{r} = O((1 - r)^{-1}),$$

while, by Lemma 4,

$$\sum_{r=1}^{\infty} (\nu + 1)^{-r}(b_{1} + \cdots + \nu b_{r}) r^{r} = O((1 - r)^{-2}),$$

and

$$\sum_{r=1}^{\infty} (\nu + 1)^{-r}(b_{1} + \cdots + \nu b_{r}) r^{r} = O((1 - r)^{-1}).$$

Thus

$$\sum_{r=1}^{\infty} (\nu + 1)^{-r}\{b_{1}(1 - \sin \theta_{0}) + \cdots + \nu b_{r}(1 - \sin \nu \theta_{0})\} r^{r} = O((1 - r)^{-1}).$$

We can now apply Lemma 1 obtaining

$$S_{n} = \sum_{r=1}^{n} \nu b_{r}(1 - \sin \nu \theta_{0}) > -Cn, \quad n \geq 1.$$

Using the same argument as in the proof of Lemma 4 we get

$$R_{n} = \sum_{r=1}^{n} S_{r}/(\nu + 1) = O(n) \quad (n = 1, 2, 3, \cdots).$$

Writing for simplicity $R_{0}=0$ we have

$$\sum_{r=1}^{n} S_{r} = \sum_{r=1}^{n} (\nu + 1) S_{r}/(\nu + 1) = \sum_{r=1}^{n} (\nu + 1)(R_{r} - R_{r-1})$$

$$= (n + 1)R_{n} - \sum_{r=1}^{n-1} R_{r} = O(n^{2}),$$

whence, again as in Lemma 4,

$$S_{n} < Cn.$$

On combining these results we have
\[ \sum_{r=1}^{n} v b_r \sin v \theta_0 = O(n) \quad (n = 1, 2, 3, \ldots). \]

This, together with
\[ \sum_{r=1}^{\infty} b_r r^r \sin v \theta_0 = O(1), \quad 0 \leq r < 1, \]
gives
\[ \sum_{r=1}^{n} b_r \sin v \theta_0 = O(1) \quad (n = 1, 2, 3, \ldots), \]
by an argument analogous to that used in the proof of Lemma 2.

Conversely, assume
\[ \sum_{r=1}^{n} b_r \sin v \theta_0 = O(1) \quad (n = 1, 2, 3, \ldots), \]
and write
\[
(2h)^{-1} \int_{0}^{h} \{ \omega(\theta_0 + t) + \omega(\theta_0 - t) \} dt = \sum_{r=1}^{\infty} b_r \sin \frac{v \theta_0}{\nu h} \\
= \sum_{r=1}^{n} b_r \sin v \theta_0 + \sum_{r=1}^{n} b_r \sin v \theta_0 \left( \frac{\sin \frac{v \theta_0}{\nu h}}{\nu h} - 1 \right) + \sum_{r=n+1}^{\infty} b_r \sin v \theta_0 \frac{\sin \frac{v \theta_0}{\nu h}}{\nu h} \\
= J_1 + J_2 + J_3.
\]

Then, by assumption,
\[ J_1 = O(1). \]

For \( J_2 \) we obtain the estimate
\[ |J_2| \leq \frac{1}{h^2} \sum_{r=1}^{n} v^2 |b_r| \leq \frac{1}{h^2} n h^2 \sum_{r=1}^{n} v |b_r|. \]

But, by Lemma 4, \( \sum_{r=1}^{n} v b_r = O(n) \) and, since \( \sum_{r=1}^{n} b_r \subset (B) \), by Lemma 1, \( \sum_{r=1}^{n} v (b_r - |b_r|) > -Cn \). Thus
\[ g_n = \sum_{r=1}^{n} v |b_r| = O(n) \]
and
\[ J_2 = h^2 O(n^2). \]

Finally, on putting \( t_{nk} = \sum_{n+1}^{n+k} b_r / v \) we have
\[ t_n = \sum_{r=n+1}^{n+k} (g_r - g_{r-1}) \nu^{-2} \]
\[ = -g_n(n + 1)^{-2} + g_{n+k}(n + k)^{-2} + \sum_{r=n+1}^{n+k-1} g_r\nu^{-2} - (\nu + 1)^{-2} \]
\[ = O\left(\frac{1}{n}\right). \]

Hence \( \sum_{n=1}^{\infty} |b_n|/\nu = O(1/n) \) and
\[ |J_3| \leq h^{-1} \sum_{n=1}^{\infty} |b_n|/\nu = h^{-1} O\left(\frac{1}{n}\right). \]

On choosing \( n = \lfloor 1/h \rfloor, h < 1 \), we get immediately
\[ \int_0^h \{\omega(\theta_0 + t) + \omega(\theta_0 - t)\} dt = O(h). \]

2. Proof of Theorem 4. We first state two additional lemmas.

**Lemma 5.** Let
\[ \omega(\theta) \sim \sum_{r=1}^{\infty} b_r \sin \nu \theta. \]
If the limit
\[ \lim_{h \to 0} \frac{2}{h} \int_0^h \omega(t) dt = d \]
exists, then
\[ (1 - r) \sum_{r=1}^{\infty} vb_r r^r \rightarrow \frac{d}{\pi} \text{ as } r \to 1 - 0. * \]

**Lemma 6.** If (10) holds and \( \sum_1^\infty b_r \in (A) \), then
\[ \frac{v_n}{n} = n^{-1} \sum_{r=1}^{n} vb_r \rightarrow \frac{d}{\pi} \text{ as } n \to \infty. \]

By Lemma 5,
\[ (1 - r) \sum_{r=1}^{\infty} vb_r r^r \rightarrow \frac{d}{\pi} \text{ as } r \to 1, \]
while, by Lemma 1,
\[ v_\nu + \frac{1}{\mu} \nu > 0 \quad (\nu = 1, 2, 3, \ldots), \]

* This was proved in our paper [5], Theorem 5.
with suitably chosen \( p \) and \( \mu \). Hence

\[
(1 - r)^2 \sum_{r=1}^{\infty} v_r r^s \to \frac{d}{\pi},
\]

\[
(1 - r)^2 \sum_{r=1}^{\infty} \left( v_r + \frac{p}{\mu} \frac{1 + \mu}{v} \right) r^s \to \frac{d}{\pi} + \frac{p}{\mu} \frac{1 + \mu}{v}, \quad \text{as } r \to 1.
\]

By a theorem of Hardy and Littlewood,

\[
\sum_{r=1}^{n} \left( v_r + \frac{p}{\mu} \frac{1 + \mu}{v} \right) \sim \left( \frac{d}{\pi} + \frac{p}{\mu} \frac{1 + \mu}{v} \right) \frac{n^2}{2},
\]

so that

\[
V_n = \sum_{r=1}^{n} v_r \sim \frac{dn^2}{(2\pi)} \quad \text{as } n \to \infty.
\]

On the other hand the relation

\[
v_n/n = (m - n)^{-1}(V_m/m - V_n/n) + V_m/(mn) - n^{-1}(m - n)^{-1} \sum_{r=n+1}^{m} (v_r - v_n),
\]

\[
m > n > 0,
\]

combined with

\[
V_n = n^2(d/(2\pi) + \epsilon_n), \quad \epsilon_n \to 0,
\]

gives

\[
v_n/n = (m - n)^{-1} \left\{ (m - n)d/(2\pi) + m\epsilon_m - n\epsilon_n \right\} + m(d/(2\pi) + \epsilon_m)/n - \sum_{r=n+1}^{m} (v_r - v_n)/(n(m - n))\]

\[
= (1 + m/n)d/(2\pi) + \epsilon_m m^2/(n(m - n)) - \epsilon_n n/(m - n) - \sum_{r=n+1}^{m} (v_r - v_n)/(n(m - n)).
\]

Here we may use inequality (2) of Lemma 1 with

\[
p = \epsilon, \quad \mu(n + 1) + n \leq m < (\mu + 1)(n + 1), \quad \mu = \mu(\epsilon) \to 0 \quad \text{as } \epsilon \to 0,
\]

which gives

\[
\limsup_{n \to \infty} v_n/n \leq (2 + \mu)d/(2\pi) + (1 + \mu)\epsilon.
\]

On allowing here \( \epsilon \to 0 \) and \( \mu \to 0 \), we get

\[
\limsup_{n \to \infty} v_n/n \leq d/\pi.
\]
In a similar fashion the relation

\[ \frac{v_m}{m} = (m - n)^{-1}(V_m/m - V_n/n) + V_n/(nm) + m^{-1}(m - n)^{-1} \sum_{r=n+1}^{m} (v_m - v_r) \]

yields

\[ \lim \inf \frac{v_m}{m} \geq \frac{d}{\pi}, \]

whence

\[ \lim \frac{v_n}{n} = \frac{d}{\pi}. \]

This proves Lemma 6, and also the first statement of Theorem 4. Assume now

\[ \sum_{r=1}^{\infty} b_r r^\nu \sin \nu \theta_0 \to s(\theta_0) \text{ as } r \to 1 - 0. \]

Then it is readily seen that

\[ (1 - r) \sum_{r=1}^{\infty} (\nu + 1)^{-1}(b_1 \sin \theta_0 + \cdots + \nu b_r \sin \nu \theta_0)r^\nu \to 0 \text{ as } r \to 1; \]

hence

\[ (1 - r) \sum_{r=1}^{\infty} (\nu + 1)^{-1}(b_1(1 - \sin \theta_0) + \cdots + \nu b_r(1 - \sin \nu \theta_0))r^\nu \to \frac{d}{\pi} \text{ as } r \to 1. \]

Being combined with (9) this yields

\[ R_n = \sum_{r=1}^{n} S_r/(\nu + 1) \sim dn/\pi. \]

Taking into account that

\[ \sum_{r=1}^{\infty} b_r(1-\sin \nu \theta_0) \in (A) \]

and using the same argument as in the proof of Lemma 6 we conclude

\[ S_n \to \frac{d}{\pi}, \text{ or } n^{-1} \sum_{r=1}^{n} \nu b_r \sin \nu \theta_0 \to 0 \text{ as } n \to \infty. \]

The relation

\[ \sum_{r=1}^{\infty} b_r \sin \nu \theta_0 = s(\theta_0) \]

now follows by the classical theorem of Tauber.

Conversely, assuming that this relation is satisfied, we have
\[
\frac{2}{h} \int_{0}^{h} \{ \omega(\theta_0 + t) + \omega(\theta_0 - t) \} \, dt - s(\theta_0)
\]
\[
= \left( \sum_{r=1}^{n} b_r \sin \nu \theta_0 - s(\theta_0) \right) + \sum_{r=1}^{n} b_r \sin \nu \theta_0 \left( \frac{\sin \nu h}{\nu h} - 1 \right)
\]
\[
+ \sum_{r=n+1}^{\lambda_n} b_r \sin \nu \theta_0 \frac{\sin \nu h}{\nu h} + \sum_{r=\lambda_n+1}^{\infty} b_r \sin \nu \theta_0 \frac{\sin \nu h}{\nu h}
\]
\[
= U_0 + U_1 + U_2 + U_3.
\]

Given \(0 < \varepsilon < 1\), we first choose \(N = N(\varepsilon)\) so that
\[
| U_0 | < \varepsilon^2, \quad n > N.
\]

We also have
\[
| U_1 | \leq \frac{1}{2} n h^2 \sum_{r=1}^{n} | b_r | < C n^2 h^2,
\]
\[
| U_3 | \leq h^{-1} \sum_{r=\lambda_n+1}^{\infty} | b_r | / \nu < C/(\nu \lambda_n).
\]

Now choose \(n = [\varepsilon/h], \lambda_n = [\pi/(\varepsilon h)]\). An estimate for \(U_2\) and the final result
\[
\frac{2}{h} \int_{0}^{h} \{ \omega(\theta_0 + t) + \omega(\theta_0 - t) \} \, dt \to s(\theta_0) \text{ as } h \to 0
\]
is obtained by precisely the same argument as in the proof of Theorem 2.

REFERENCES TO PREVIOUS PAPERS BY THE AUTHOR


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