A NEW METHOD FOR WARING THEOREMS WITH POLYNOMIAL SUMMANDS; II*

BY

L. E. DICKSON

1. In a paper† with the same title, I showed how to deduce instantaneously a Waring theorem for an even polynomial \( f(x) \) of degree \( 2n \) from a known Waring theorem for a polynomial \( q(x) \) of degree \( n \). Here I extend the method to the new case in which \( f(x) \) contains also a term in \( x \).

2. First, let \( n = 2 \) and

\[
(1) \quad f(x) = u x^4 + v x^2 - w x + k, \quad q(x) = u x^2 + v x + 2k.
\]

We have the identity in \( a, b, c, d, u, v, w, k \)

\[
(2) \quad 6q(s) = \sum f(z), \quad s = a^2 + b^2 + c^2 + d^2,
\]

in which \( z \) takes the following twelve values:

\[
(3) \quad b \pm a, \quad c \pm a, \quad d \pm a, \quad \pm b - c, \quad \pm d - b, \quad \pm c - d,
\]

whose sum is zero. Since some of the numbers (3) are negative, we impose the condition

\[
(4) \quad f(x) \text{ is an integer } \geq 0 \text{ for all integers } x.
\]

But when \( x \) ranges over all integers (positive, negative, or zero), evidently \( f(-x) \) takes the same values as \( f(x) \). Without loss of generality we may therefore take \( w \geq 0 \). Since \( f(-x) = f(x) + 2wx \), \( f(-x) \) will be \( \geq 0 \) for all integers \( x \geq 0 \) if the same is true for \( f(x) \). Hence (4) follows from

\[
(5) \quad f(x) \text{ is an integer } \geq 0 \text{ for every integer } x \geq 0.
\]

Since \( u > 0 \), only a limited number of integers \( x \) yield negative values of \( f(x) - k \). If one of these values is \( -P \), while all the remaining are \( \geq -P \), then (5) holds if and only if \( k \geq P \). In brief, we need only take \( k \) sufficiently large in (1).

Consider triangular, pyramidal, and figurate numbers

\[
(6) \quad T(x) = (x^2 - x)/2, \quad P(x) = (x^3 - x)/6, \quad F(x) = (x + 2)(x + 1)x(x - 1)/24 = (x^4 + 2x^3 - x^2 - 2x)/24.
\]

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* Presented to the Society, November 29, 1935; received by the editors June 14, 1935.
† These Transactions, vol. 36 (1934), pp. 731–748. In (36) read \( \pm d \) for \( +d \). In (38) delete exponent 6.

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Any quartic function with rational coefficients can evidently be expressed in the form

\[ f = AF + gP + hT + tx + k, \]

where \( A, \ldots, k \) are rational numbers. Let (5) hold. Taking \( x = 0, \ldots, 4 \), we see that

\[
\begin{align*}
k, & \quad t + k, & \quad A + g + h + 2t + k, & \quad 5A + 4g + 3h + 3t + k, \\
& \quad 15A + 10g + 6h + 4t + k
\end{align*}
\]

must be integers. Hence \( k, t, A, g, h \) are integers. The coefficients of \( x^3 \) and \( x \) in (7) are

\[
\frac{A}{12} + \frac{g}{6}, \quad -\frac{A}{12} + \frac{g}{6}, \quad -\frac{A}{2} + t.
\]

These must be 0 and \( \leq 0 \) respectively if (7) shall be of the form (1) with \( w \geq 0 \). Hence

\[ A + 2g = 0, \quad h \geq 2t. \quad (8) \]

3. Of special interest are functions \( f \) for which

\[ \text{Every integer } \geq 0 \text{ is a sum of } 0 \text{ values of } f(x) \quad (9) \]

for integers \( x \). The smaller \( A \) is, the more slowly will \( f \) increase with \( x \), and the smaller \( V \) will be in general. Hence we give to \( A \) its minimum (even) value 2. Evidently (9) requires that

\[ f(y) = 0, \quad f(z) = 1 \text{ for certain integers } y, z. \quad (10) \]

The functions (7) satisfying (4), (8), and (10) and having \( A = 2, t \geq -5 \), are found to be those with

\[
\begin{array}{c|cccccccccccc}
t & 1 & 0 & 0 & -1 & -1 & -2 & -2 & -2 & -3 & -3 & -3 \\
h & 2 & 0 & 1 & -1 & \geq 1 & -2 & -1 & 0 & 2 & -4 & 1 & 3 \\
k & 0 & 0 & 0 & 2 & 1 & 6 & 4 & 3 & 2 & 16 & 4 & 3
\end{array}
\]

\[
\begin{array}{c|cccccccccccc}
t & -4 & -4 & -4 & -4 & -4 & -5 & -5 & -5 \\
h & -3 & -1 & 0 & 2 & 4 & -6 & 3 & 5 \\
k & 15 & 9 & 7 & 5 & 4 & 36 & 6 & 5
\end{array}
\]

Hence each of these functions represents 0 and 1, and is an integer \( \geq 0 \) for every integer \( x \). The general theory therefore yields a value of \( V \) in (9) and hence a universal Waring theorem for summands \( f(x) \).
4. Consider, for example, the seventh set \( t = -2, h = -1, k = 4 \). Then

\[
\begin{align*}
f(x) &= 2F - P - T - 2x + 4 = (x^4 - 7x^2)/12 - \frac{x}{3} + 4, \\
Q &= 6q = H + 42, \\
H &= \frac{1}{3}(x - 3)(x - 4).
\end{align*}
\]

Every integer \( \geq 0 \) is a sum of three values of the triangular number \( H \) for integers \( x \geq 4 \). Thus every integer \( \geq 126 \) is a sum of three values of \( Q \). Hence by (2), every integer \( \geq 126 \) is a sum of 36 values of \( f(x) \). We next verify this fact also for positive integers \( < 126 \) and indicate a probable reduction from 36 to 5. By a table to 1000 of sums of three values of \( f(x) \), we find that 415, 734, and 749 are the only positive integers \( < 1000 \) which are not sums of four values. We find at once that all integers \( \leq 5114 \) are sums of five values of \( f(x) \) for integers \( x \) (positive, negative, or zero).

5. Waring theorem for sextic polynomials. Let

\[
\begin{align*}
f(x) &= ux^6 + vx^4 + wx^2 - hx + k, \\
q(x) &= 120ux^3 + 72vx^2 + 60wx + 108k.
\end{align*}
\]

Then a like generalization of (38) of the former paper gives

\[
q(x) = \sum f(y) + 8 \sum f(z) + f(2a) + f(2b) + f(2c) + f(2d),
\]

where \( z \) ranges over the twelve values (3), and \( y \) ranges over the eight values

\[
\begin{align*}
- a - b - c \pm d, & \quad \pm a - b + c - d, & \quad \mp a + b - c - d, \\
\pm (a - b - c) + d,
\end{align*}
\]

whose sum is \(-2a - 2b - 2c - 2d\). Hence the sum of the 108 arguments of \( f \) in (13) is zero. In case \( f(x) \) is an integer \( \geq 0 \) for every integer \( x \) (which is true when \( k \) is sufficiently large), a Waring theorem for \( q \) leads instantly to one for \( f \). This condition (4) holds if \( f(x) \) is

\[
x^6 + x^2 - x, \quad H(x) = x^6 + 3x^2 - x,
\]

each of which represents 0 and 1. Hence each yields at once a universal Waring theorem.

6. Quartics with property (4). Replacing \( x \) by \(-x-1\) in (7), we get

\[
AF(x) - gP(x) + (h - g)T(x) + (2h - g - t)x + h - t + k.
\]

Hence (7) remains unaltered if and only if

\[
g = 0, \quad h = t.
\]

In this case the values of \( f(x) \) for negative integers coincide with its values for integers \( x \geq 0 \). Such unfavorable cases are

\[
f = F(x), \quad F - T - x + 2, \quad F - 6T - 6x + 56.
\]
whose values are $\geq 0$ for an integer $x \geq 0$ and hence for all integers. By tables to 1000, all integers from 0 to 3366 inclusive are sums of 7 values of $F(x)$ except only 64, 99, 119, 189, 314, 774. Hence all $\leq 23841$ are sums of 8.

Since $T(-x) = T(x+1)$, $F+T$ is $\geq 0$ for all integers. By a table to 1000 of sums by three, it was verified that all integers from 0 to 3900 are sums of four values of $F+T$.

Miss H. Rees found that (7) has property (4) if

$$A = 1, \quad g = -p - 2, \quad h = p + 1 + \frac{1}{6}p(2p + 1) + m, \quad t = 1, \quad k = 0,$$

$$p \geq 0, \quad m \geq 0.$$

To obtain an integer $h$ take $m$ to be the sum of an integer $\geq 0$ and $\frac{1}{3}$ if $p \equiv 1$ or 3 (mod 6); 0 if $p \equiv 0$ or 4; $\frac{1}{6}$ if $p \equiv 2$; $\frac{2}{3}$ if $p \equiv 5$ (mod 6). We may remove the term involving $P$ by the transformation $x = y + p + 2$. We get

$$F + (m - r)T + \{1 - r + m(p + 2)\}y + (p + 2)\{1 + (p + 1)J\},$$

$$(19) \quad r = \frac{1}{6}p(p + 2), \quad J = (p + 1)(p - 12)/24 + \frac{1}{3}(m + p + 1).$$

When $p = 1, m = \frac{1}{2}, (19)$ is $f = F + 2y + 5$. By a table of sums of three values from 0 to 3000 and from 9000 to 11000, it was found that every integer $\leq 16151$, except only 11784, is a sum of four (positive) values of (16) for positive and negative integers $x$. It follows that all $\leq 210739$ are sums of five such values.

Another favorable function $f = F + y + 2$ is the case $p = m = 0$ of (19); it represents 0, 1, 2, 3, 5, 10, 12, 21, etc.

University of Chicago,
Chicago, Ill.