FUCHSIAN GROUPS AND ERGODIC THEORY*†

BY
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Introduction. Let $\Omega$ be the phase space of a dynamical system. We suppose that every motion can be continued along the entire time-axis. Thus we are concerned with a steady flow in $\Omega$. The following concepts are of fundamental significance for the study of dynamical flows.

(a) There exists a curve of motion everywhere dense on $\Omega$.

The existence of such a motion is known under the name of regional transitiv-

ity. We now suppose that a measure $m$ in the sense of Lebesgue invariant
under the flow exists on $\Omega$. Such a measure is usually defined by an invariant
phase element $dm$. The following property is stronger than (a).

(b) The curves of motion not everywhere dense on $\Omega$ form a point set on $\Omega$
of $m$-measure zero.

Still stronger and more important than (b) is strict ergodicity. We suppose
$m(\Omega)$ to be finite.

(c) Let $f(P)$ be an arbitrary $m$-summable function on $\Omega$. The time-aver-

age of $f(P)$ along a curve of motion is then, in general, equal to $\int_a f(P)dm/m(\Omega)$,
the exceptional curves forming a point set on $\Omega$ of $m$-measure zero.

How these concepts are interrelated is seen most clearly if we state them in
the following way.

(a') Every open point set on $\Omega$ that is invariant under the flow is every-

where dense on $\Omega$.

(b') Every open point set on $\Omega$ that is invariant under the flow has the
measure $m(\Omega)$.

(c') Every $m$-measurable point set on $\Omega$ that is invariant under the flow has either
the $m$-measure zero or $m(\Omega)$.

The latter property of a flow is called metric transitivity.† Its importance rests

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† To August Kopff.
‡ G. D. Birkhoff and P. Smith, Structure analysis of surface transformations, Journal de Mathé-
maticques, (9), vol. 7 (1928), pp. 345–379.
in its equivalence to (c).* The problem whether a given flow is metrically transitive or not is, in general, as difficult as it is interesting. Beyond simple examples progress has recently been made in the direction of certain geodesics problems. Let Σ be a surface (two-dimensional manifold) of class $C_3$. We denote by $p$ an arbitrary point on $Σ$ and by $ϕ$ the angle measuring directions through $p$. Every geodesic on $Σ$ is supposed to be continuable indefinitely in both directions. The line elements

$$P = (p, ϕ)$$

then constitute the phase space $Ω$ associated with $Σ$. To the uniform motion along the geodesics on $Σ$ there corresponds a steady flow on $Ω$. The element of volume

$$dm = dσdϕ,$$

$dσ$ being the element of area, is well known to be invariant under the flow.

The particular surfaces† $Σ$ considered in this paper are those of constant negative curvature and of finite connectivity. Their geodesics are supposed to satisfy the above condition of unlimited continuability. Differential geometry shows that there exists a one-to-many correspondence between $Σ$ and the interior $|z| < 1$ of the unit circle such that the elements of length $ds$ and area $dσ$ go over into the NE-elements in $|z| < 1$,

$$ds = 2(1 - z\bar{z})^{-1}|dz|, \quad dσ = 4(1 - z\bar{z})^{-2}dxdy$$

respectively. The geodesics on $Σ$ go over into the arcs of orthogonal circles within $|z| < 1$ (NE-straight lines). The covering transformations are known to form a Fuchsian group $G$ of linear substitutions $S$ transforming $|z| \leq 1$ into itself. $|z| = 1$ is the principal circle of $G.$‡ A more general notion of the

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* See the literature on the ergodic theorem, viz.


‡ An elementary introduction into the theory of Fuchsian groups is found in L. R. Ford, *Automorphic Functions*, New York, 1929.
Fuchsian group will be considered here. In order to include one-sided surfaces we shall admit anti-analytic substitutions, i.e., analytic substitutions of $\bar{z}$. On identifying all points of $|z| < 1$ equivalent under any Fuchsian group $G$ one defines, conversely, an associated surface $\Sigma$.

$G$ always possesses a fundamental region $R$. It can be chosen so as to form a NE-convex polygon bounded by a finite number of segments of NE-straight lines and a finite number of arcs of $|z| = 1$. The images of $R$ under all $S$ of $G$ cover the whole of $|z| < 1$ simply.

We are to distinguish between two fundamentally different kinds of Fuchsian groups $G$, and, therefore, of surfaces $\Sigma$. $G$ and $\Sigma$ are of the first kind if the surface area of $\Sigma$ or, what is the same, the NE-area of the fundamental region $R$ is finite. In the opposite case, $\sigma(\Sigma) = \infty$, we speak of the groups and surfaces of the second kind.* If $G$ is of the first kind, $R$ has no arcs of $|z| = 1$ on its boundary. Its vertices lie partly in $|z| < 1$, partly on $|z| = 1$, the angle being zero in the latter case. A well known example is offered by the case where $R$ is bounded by a regular NE-polygon with the sum of the angles equal to $2\pi$. If the $4p$ sides ($p > 1$) be paired in a certain way, $\Sigma$ represents a closed two-sided surface of genus $p$. Another well known example is furnished by the modular group where $R$ is bounded by a NE-triangle with one vertex on $|z| = 1$. The surface $\Sigma$ has, in this case, a cuspidal singularity. For every group $G$ of the second kind, however, $R$ has at least one arc of $|z| = 1$ on its boundary and $\Sigma$ possesses, accordingly, at least one funnel.

For surfaces $\Sigma$ of the first kind, the regional transitivity (a) has been proved,† in various degrees of generality, by Artin, J. Nielsen, Koebe and Löbell, whereas Myrberg discovered the property (b). It is only recently that Hedlund‡ succeeded in proving the deeper property of metric transitivity of the two examples mentioned above. It is the purpose of the present paper to develop an entirely novel and simple method that yields a proof of the metric transitivity for all surfaces $\Sigma$ of the first kind.

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* This definition is readily found to be in agreement with the one usually given.
Theorem. For all surfaces $\Sigma$ of the first kind, the flow associated with the geodesics problem on $\Sigma$ is metrically transitive.

On surfaces $\Sigma$ of the second kind (treated in §5), the geodesics show an entirely different behavior. The corresponding theorem can be stated without reference to the phase space $\Omega$.

The geodesics through an arbitrarily given point $p$ of $\Sigma$ disappear, for almost all directions through $p$, into the funnels of $\Sigma$.

By this we mean that the corresponding NE-straight lines end on one of the arcs of $|z| = 1$ belonging to the boundary of $R$, or on one of the images under $G$ of those arcs. The theorem, therefore, merely states that those arcs and their images form a set on $|z| = 1$ of the same measure as the unit circle itself, which statement, in contrast to the theorem concerning surfaces of the first kind, is most readily proved.

The essential tools used in this paper are potential theory and the NE-metric in $|z| < 1$.

1. Preliminaries on Fuchsian groups. The cross ratio

$$[z_1, z_2, z_3, z_4] = \frac{z_3 - z_1}{z_3 - z_2} \frac{z_4 - z_2}{z_4 - z_1}$$

is unchanged by an analytic linear substitution, whereas it goes over into its conjugate value under an anti-analytic one. The same holds for the differentials

$$\frac{dz_2}{z_1 - z_2} \frac{z_3 - z_2}{z_3 - z_2} = - [z_1, z_2, z_3, z_2 + dz_2]$$

and

$$(z_1 - z_2)^{-2} dz_1 dz_2 = - [z_1, z_2, z_1 + dz_1, z_2 + dz_2].$$

Only those substitutions $S$ will be considered in the sequel which leave $|z| \leq 1$ invariant. The relation

$$S(1/z) = 1/S(z)$$

yields the invariant

$$\left| \frac{w - z}{1 - \bar{z}w} \right|^2 = [z, 1/z, w, 1/w],$$

and, according to (1), the differential invariant

$$\frac{dz}{1 - z^2} \frac{1 - \bar{z}w}{w - z} = [1/z, z, w, z + dz].$$
From (3) and (4) we obtain Poincaré's invariant NE-element of length

\[ ds = 2(1 - z\bar{z})^{-1} |dz| . \]

The corresponding NE-element of area is then

\[ d\sigma = 4(1 - z\bar{z})^{-2} dxdy . \]

The absolute value of (2) with \( z_1 = \xi, z_2 = z \), after being divided by (5), finally furnishes the invariant Poisson differential

\[ \frac{1 - z\bar{\xi}}{|\xi - z|^2} |d\xi| . \]

A group \( G \) of substitutions \( S \) preserving \( |z| < 1 \) is called Fuchsian if it is infinite and discontinuous, i.e., if it contains no infinitesimal \( S \). From now on, we suppose \( G \) to be Fuchsian. A point \( z, |z| < 1 \), and all its equivalent points \( S(z) \) determine the same point of the surface \( \Sigma \). Associated with \( G \) is a group \( \Gamma \) of contact transformations \( T \) in the space of the line elements

\[ (z, \phi), \phi = \arg (dz), \]

where \( T \) has the form

\[ (z, \phi) \to (S(z), \phi + \arg S'(z)) \]

in the case of an analytic \( S \) and a similar form when \( S \) is anti-analytic. Equivalent line elements define the same point \( P \) in the phase space \( \Omega \). As \( S \) preserves angles, the transformations \( T \) leave the volume element

\[ dm = d\sigma d\phi \]

in the space of line elements invariant.

We now introduce the new coordinates

\[ (\eta_1, \eta_2, s); |\eta_1| = |\eta_2| = 1, \eta_1 \neq \eta_2, -\infty < s < \infty , \]

in the space of line elements. \( \eta_1 \) and \( \eta_2 \) are initial and end point, respectively, of the sensed NE-straight line passing through \( (z, \phi) \). Let \( s \) denote the NE-distance of \( z \) from the point \( z_0 \) bisecting the circular arc \( (\eta_1, \eta_2) \), the sign of \( s \) being plus (minus) if \( z \) is met after (before) \( z_0 \) on \( (\eta_1, \eta_2) \). The correspondence between \( (z, \phi) \) and \( (\eta_1, \eta_2, s) \) is evidently one-to-one. In the new coordinates, the transformations \( T \) are easily seen to be of the form

\[ (\eta_1, \eta_2, s) \to (S(\eta_1), S(\eta_2), s + f_S(\eta_1, \eta_2)). \]

We now prove that the volume element (9) becomes

\[ dm = k \left| \frac{d\eta_1}{\eta_1 - \eta_2} \right| d\eta_2 \left| ds \right|, \]
$k$ being a positive constant. Indeed we can always find a linear substitution preserving $|z| = 1$ which transforms an arbitrarily given line element
\[ (z, \phi) = (\eta_1, \eta_2, s) \]
into any other one,
\[ (z', \phi') = (\eta_1', \eta_2', s'). \]
The associated contact transformation is of the form (8) and therefore leaves (9) invariant. Being of the form (10), it also leaves invariant the right-hand side of (11) in view of the invariance of (2) and (5). Hence, the two sides can differ by a constant factor only.

In the new coordinates the flow associated with the geodesics on $\Sigma$ is described by the simple formulas
\[ P = (\eta_1, \eta_2, s) \rightarrow P_t = (\eta_1, \eta_2, s + t). \]
The invariance of $dm$ under the flow is now a trivial consequence of (11) and (12).

The explicit connection between the coordinates is readily established,
\[
s = \log [\eta_1, \eta_2, z, z_0], \quad z_0 = \frac{\eta_1 + \eta_2}{2 + |\eta_1 - \eta_2|},
\]
\[
\phi = \arg \frac{(z - \eta_1)(z - \eta_2)}{\eta_1 - \eta_2},
\]
but it is not needed for our purposes.

2. Another formulation of the theorem. A Fuchsian group of the first kind possesses a fundamental region $\mathcal{R}$ of finite NE-area. To the subdivision of $|z| < 1$ into the NE-congruent parts $S(\mathcal{R})$ there corresponds a subdivision of the $(\eta_1, \eta_2, s)$ space into cells congruent to each other under the transformations $T$ of $\Gamma$. Each of these cells is a representative of $\Omega$, with
\[ m(\Omega) = 2\pi\sigma(\Sigma) < \infty. \]

For the proof of the announced theorem it is sufficient to prove

**Theorem A.** A point set $A$ on the $(\eta_1, \eta_2)$ torus which is measurable in the sense of ordinary Lebesgue measure, for which
\[ \int \int_A |d\eta_1| |d\eta_2| > 0 \]
and which is invariant under the simultaneous substitutions $S(\eta_1), S(\eta_2)$ of $\mathcal{G}$, has the measure of the entire torus.
Suppose Theorem A to be proved. We start with a point set on \( \Omega \) satisfying the hypothesis of the theorem announced in the introduction. According to (12), this set represents, in the \((\eta_1, \eta_2, s)\) space, a cylindrical set, i.e., a set on the \((\eta_1, \eta_2)\) torus. According to (11), the part of this set within each of the above cells is of positive measure

\[
\int \int \left| \frac{d\eta_1}{|\eta_1 - \eta_2|^2} \right|, \]

and, therefore, of positive torus measure. The sum of these parts obviously represents a set \( A \) in the sense of Theorem A. Hence its complement has the torus measure zero. Regarded as a cylindrical set in the \((\eta_1, \eta_2, s)\) space the latter set is, necessarily, of \( m \)-measure zero in that space and, therefore, in \( \Omega \).

From now on we may without loss of generality confine ourselves to two-sided surfaces \( \Sigma \), i.e., to the case where \( G \) contains only analytic substitutions \( S \). For, let \( S \) denote the analytic and \( \bar{S} \) the anti-analytic substitutions of \( G \). The \( S \)'s form a subgroup \( g \) of \( G \) and each \( \bar{S} \) can be written in the form \( \bar{S} = SS_0 \) where \( S_0 \) is a fixed \( \bar{S} \). As

\[
R + S_0(R)
\]
is evidently a fundamental region for \( g \), this group is seen to be again a Fuchsian group of the first kind. If Theorem A holds for \( S \), it holds also for \( G \).

Let now \( U(\eta_1, \eta_2) \) be the function on the torus which equals zero on the set \( A \) of Theorem A and one elsewhere. \( U \) is measurable and invariant under \( G \),

\[
(13) \quad U(S(\eta_1), S(\eta_2)) = U(\eta_1, \eta_2).
\]

It is to be proved that \( U = 0 \) except on a torus set of measure zero. We transform our problem once more by introducing harmonic functions. The Poisson integral

\[
(14) \quad U(z, \gamma) = \frac{1}{2\pi} \int_{|\gamma| = 1} U(\xi, \gamma) \frac{1 - z\bar{\xi}}{|\xi - z|^2} |d\xi|
\]

represents, for most all \( \gamma \) on \( |\gamma| = 1 \), a harmonic function of \( z \) in \( |z| = 1 \) and, for every such \( z \), a bounded and measurable function of \( \gamma \) on \( |\gamma| = 1 \). Furthermore, the function

\[
(14') \quad U(z, w) = \frac{1}{2\pi} \int_{|\gamma| = 1} U(z, \gamma) \frac{1 - w\bar{\gamma}}{|\gamma - w|^2} |d\gamma|
\]
is, for \( |z| < 1 \) and \( |w| < 1 \), harmonic in \( z \) as well as in \( w \). (14'), combined with (14), could, of course, be written as one double Poisson integral. In view of
the invariance of the Poisson differentials, the invariance (13) under $G$ of the “torus values” of $U(z, w)$ implies the invariance of the function itself,
\begin{equation}
U(S(z), S(w)) = U(z, w),
\end{equation}
for all $S$ of $G$.

Formulas (14) and (14') show that $U(z, w) = 0$ implies the vanishing of the torus values $U(\xi, \gamma)$ up to a torus set of measure zero. To prove Theorem A it therefore suffices to prove

**Theorem B.** Suppose that $U(z, w) \geq 0$ is bounded and harmonic in $z$ as well as in $w$, $|z| < 1$, $|w| < 1$, and that $U$ satisfies (15) for all $S$ of $G$. If the torus values of $U$ vanish on a set of positive torus measure then $U$ vanishes identically.

We have to specify in what sense the torus values $U(\xi, \gamma)$ may be regarded as limit values of $U(z, w)$. If $u(z)$ is a bounded harmonic function of a single point $z$, $|z| < 1$, we have
\begin{equation}
l.i.m. u(\rho \xi) = u(\xi)
\end{equation}
on $|\xi| = 1$. An analogue for harmonic functions of two points is quite similarly proved,
\begin{equation}
l.i.m. \left. \frac{U(r \xi, \rho \gamma)}{r, \rho \rightarrow 1} \right| \frac{U(\xi, \gamma)}{on the torus $|\xi| = |\gamma| = 1$.

3. Auxiliary theorems. In the sequel we denote by $K_i$ the interior of the circle about $z = 0$ with the NE-radius $l$. A simple computation of the NE-area yields the formula
\begin{equation}
\sigma(K_i) = \pi(e^l + e^{-l} - 2).
\end{equation}
For the validity of Lemma 1 we assume that $z = 0$ is interior to some $R$.†

**Lemma 1.** If a set $B$ in $|z| < 1$ is invariant under all $S$ of $G$, and if $R$ is a fundamental region for $G$, then, for $l$ sufficiently large,
\begin{equation}
\frac{\sigma(BK_i)}{\sigma(K_i)} < a\sigma(BR),
\end{equation}
a being a positive constant depending only on $G$.

This lemma is not quite trivial in the case where $R$ has vertices on $|z| = 1$ and is used mainly to take care of the slight complications arising from this

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* l.i.m. means limit in the mean of order two.
† If it is not, by a suitable linear substitution, we can always move a given interior point of $R$ into the origin. It may well be mentioned that the origin plays here a mere auxiliary role.
case in the proof of Theorem B. The proofs of this and of the following lemmas 2 and 3 are given in the next §4.

**Lemma 2.** Let \( u(z) \geq 0 \) be bounded and harmonic in \( |z| < 1 \), and let
\[
u(S(z)) = u(z)
\]
hold for all \( S \) of \( G \). If the boundary values of \( u \) on \( |z| = 1 \) vanish on a set of positive measure, \( u(z) \) vanishes identically.

This is a very simple special case of Theorem B. The principal difficulty in the proof of that theorem is surmounted by the main

**Lemma 3.** If \( U(z, w) \) satisfies all the hypotheses of Theorem B, the measure on \( |\gamma| = 1 \) of the set where \( U(0, \gamma) = 0 \) is necessarily positive.

**Proof of Theorem B.** The set \( E \) on \( |\gamma| = 1 \) where \( U(0, \gamma) = 0 \) is of positive measure according to Lemma 3. We now make use of Harnack’s inequalities for a non-negative harmonic function \( u(z), |z| < 1 \),
\[
e^{-s(z, z')} u(z) \leq u(z') \leq e^{s(z, z')} u(z),
\]
where \( s(z, z') \) denotes the NE-distance of the two points. These inequalities being applied to \( U(z, w) \) yield
\[
(18) \quad e^{-s(0, z)} U(0, w) \leq U(z, w) \leq e^{s(0, z)} U(0, w),
\]
which shows that, for a fixed \( z \), the set where the boundary values \( U(z, \gamma) \) of \( U(z, w) \) on \( |w| = 1 \) vanish is independent of \( z \); in fact it coincides with \( E \) except for a set of measure zero. Now according to (15) we have, for an arbitrary \( S \) of \( G \),
\[
U(S(0), w) = U(0, S^{-1}(w)).
\]
On replacing \( w \) by \( S^{-1}(w) \) and \( \gamma \) by \( S^{-1}(\gamma) \) in (14') and on taking into account the invariance of the Poisson differential, we infer from (14') that
\[
U(0, S^{-1}(w)) = \frac{1}{2\pi} \int_{|\gamma| = 1} U(0, S^{-1}(\gamma)) \frac{1 - w\bar{\gamma}}{|\gamma - w|^2} \, d\gamma.
\]
Hence \( U(0, S^{-1}(\gamma)) \) are the boundary values of \( U(0, S^{-1}(w)) \). Since \( S(E) \) is the set where these boundary values vanish, the equation \( S(E) = E \) holds for all \( S \) of \( G \) apart from a null set on the unit circle. Considering, in the same way as before, the harmonic function \( u(z), |z| < 1 \), whose boundary values are zero on \( E \) and one elsewhere, we infer that \( u \) satisfies the hypothesis of Lemma 2 and, therefore, that \( u \equiv 0 \), i.e., that \( E \) has the measure of the entire unit circle. It then follows from the definition of \( E \) that the boundary values
of $U(0, w)$ vanish almost everywhere. Hence $U(0, w) = 0$ and, according to (18), $U(z, w) = 0$, which is the desired result.

4. **Proof of Lemmas 1 and 2.** We denote by $N(z, l)$ the number of points $S(z)$ congruent to $z$ which lie in $K_i$. We first show that, for $l$ sufficiently large,

$$ \frac{N(z, l)}{a(K_i)} < a, \tag{19} $$

where $a > 0$ depends on $G$ only. $N(z, l)$ is the number of points $S(z)$ whose NE-distance from the origin does not exceed $l$, $s(0, S(z)) \leq l$.

Since

$$ s(0, S(z)) = s(S^{-1}(0), z), $$

$N(z, l)$ is not greater than the number of points $S^{-1}(0)$ congruent to the origin with a NE-distance $\leq l$ from $z$, provided that the points $S^{-1}(0)$ are different for different substitutions $S$. This is the case, as the origin is interior to $R$ and as an interior point of $R$ cannot be a fixed point for any $S$ except for the identity transformation. We furthermore know that

$$ s(S(0), 0) > b, \quad S(0) \neq 0, $$

holds, where $b > 0$ depends on $G$ only. Therefore a circle of NE-radius $b$ about any point congruent to the origin contains no other such point. This implies that the number of the different points $S(0)$ with a NE-distance $\leq l$ from $z$ is less than the number of mutually exclusive circles of NE-radius $b$ which can be placed within a circle of NE-radius $l + b$. Hence

$$ N(z, l) < \frac{\sigma(K_{i+b})}{\sigma(K_b)} = \frac{\sigma(K_{i+b})}{\sigma(K_i)\sigma(K_b)} \sigma(K_i). $$

Here the first factor tends to $\pi^{-1}(1 + e^{-2b} - 2e^{-b})^{-1}$ as $l \to \infty$, which proves (19).

We now return to the set $B$ of Lemma 1. By means of the function

$$ \phi(z) = \begin{cases} 1 \text{ in } K_i, \\ 0 \text{ elsewhere,} \end{cases} $$

we obtain

$$ N(z, l) = \sum_s \phi(S(z)), $$

and therefore

$$ \int \int_{RB} N(z, l) d\sigma_z = \sum_s \int \int_{RB} \phi(S(z)) d\sigma_z = \sum_s \int \int_{S^{-1}(RB)} \phi(z) d\sigma_z. \tag{20} $$
Since $B = S(B)$ for all $S$ of $\mathcal{G}$,
\[ \sum_S S^{-1}(RB) = \sum_S BS^{-1}(R) = B \sum_S S(R) = B, \]
and the right hand side of (20) equals
\[ (20') \int \int_B \phi(z) d\sigma_z = \sigma(BK_i). \]

Lemma 1 obviously follows from (19), (20) and (20').

Proof of Lemma 2. In the particular case where the boundary of $R$ lies within $|z| < 1$ the lemma is obvious, since then $u(z)$ attains its extrema at some points of $|z| < 1$, i.e., at interior points of $|z| < 1$. If $R$ has vertices on $|z| = 1$ an elementary proof could still be given. We prefer, however, to use tools which seemed unavoidable in the further course of the proof of Theorem B. The auxiliary function
\[ h_{\varepsilon}(t) = \begin{cases} (1 - t/\varepsilon)^3, & 0 \leq t < \varepsilon, \\ 0, & \varepsilon \leq t, \end{cases} \]
is concave and possesses a continuous second derivative, $t \geq 0$. We first show that
\[ \lim_{l \to \infty} \frac{1}{\sigma(K_i)} \int \int_{K_i} h_{\varepsilon}(u(z)) d\sigma_z = \frac{1}{2\pi} \int \int_{|\zeta| = 1} h_{\varepsilon}(u(\zeta)) |d\zeta| \]
and that
\[ \lim_{\varepsilon \to 0} \int \int_{|\zeta| = 1} h_{\varepsilon}(u(\zeta)) |d\zeta| = \text{meas} \{u(\zeta) = 0\}. \]
The integral average on the left of (22) is, evidently, an average of
\[ \frac{1}{2\pi} \int \int_{|\zeta| = 1} h_{\varepsilon}(u(r\zeta)) |d\zeta| \]
over a certain range of $r$ corresponding to the range from 0 to $l$ for the NE-radius. In this average large values of $l$, i.e., values of $r l$ near one, are of dominating weight. Since, according to (16), (24) tends to the right-hand side of (22) as $r \to 1$ it follows that (22) must also be true. Finally, (23) follows from the obvious inequalities
\[ \text{meas} \{u(\zeta) = 0\} \leq \int \int_{|\zeta| = 1} h_{\varepsilon}(u(\zeta)) |d\zeta| \leq \text{meas} \{u(\zeta) < \varepsilon\}. \]
For later purposes, we need the analogous relations resulting from (16'). On setting
\[ M_x(l) = \frac{1}{\sigma^2(K_l)} \int \int_{K_l} \int \int_{K_l} h_x(U(z, w))d\sigma_xd\sigma_w, \]
we obtain quite similarly
\[ \lim_{\epsilon \to 0} \lim_{l \to \infty} M_x(l) = (4\pi^2)^{-1} \text{meas} \{ U(z, \gamma) \} = 0. \]

Returning to the proof of Lemma 2 we note that the integral average on the left in (22) is less than
\[ \frac{\sigma(B, K_l)}{\sigma(K_l)}, \quad l \text{ sufficiently large}, \]
where \( B \) is the set of all \( z \) in \( |z| < 1 \) satisfying \( u(z) < \epsilon \). The invariance under \( G \) of the function \( u(z) \) implies that of the point set \( B \). By Lemma 1, (27) and therefore the left-hand average in (22) does not exceed the value
\[ a \sigma(B, R). \]
From the hypothesis of the present lemma, viz. that the right-hand side of (23) is positive, it then follows that (28) remains, for all \( \epsilon > 0 \), above a positive constant. In particular, the set common to all \( B \) is not empty. There exists therefore a point in \( |z| \leq 1 \) where \( u = 0 \), i.e., where \( u \geq 0 \) attains its minimum, which completes the proof of the lemma.

5. Proof of Lemma 3. We first confine ourselves to the simpler case where the boundary of the fundamental region \( R \) of \( G \) lies entirely in \( |z| < 1 \). Of all the images \( S(R) \) we call \( R_0 \) the particular one that contains the origin in its interior or on its boundary,
\[ z = 0 \in R_0. \]
\( R_0 \) and all its images have the same finite NE-diameter \( D \). We enumerate all these congruent parts of \( |z| < 1 \) in an arbitrary way, \( R_0, R_1, R_2, \ldots \), and we call \( S \), the substitution of \( G \) that transforms \( R \) into \( R_0 \),
\[ S_v(R_v) = R_0. \]
We now consider all \( R_v \) lying entirely in the closed circular disc \( K_l \). They evidently cover the whole of \( K_{l-D} \). From (25) we then obtain
\[ M_x(l - D) \leq q(l) \frac{1}{\sigma^2(K_l)} \sum_{\gamma, \mu} \int \int_{R_v} \int \int_{R_v} h_x(U(z, w))d\sigma_xd\sigma_w, \]
where
\[ q(l) = \left[ \frac{\sigma(K_l)}{\sigma(K_{l-D})} \right]^{-2}. \]
and where integration and summation are carried out so as to satisfy the conditions

\[(31'') \quad z \subset R, \subset K_1, \quad w \subset R, \subset K_1\]

in all possible ways. By (29), (30), and (31''),

\[s(0, S_*(z)) \leq D,\]

whence, by Harnack's inequality,

\[U(z, w) = U(S_*(z), S_*(w)) \geq e^{-D} U(0, S_*(w)),\]

and, as \(h_*(t)\) nowhere increases,

\[(32) \quad h_*(U(z, w)) \leq h_*(\{e^{-D} U(0, S_*(w))\}, z \subset R).\]

Now, the function of \(w\)

\[(33) \quad h_*(e^{-D} U(0, w))\]

is a concave function of a harmonic function and, therefore, subharmonic in \(|w| < 1\). This is most easily proved by verifying the non-negativeness of the Laplacian.* Hence, (33) nowhere exceeds the function \(V_*(w) = V_*(w)\) which is harmonic in \(|w| < 1\) and which possesses, on \(|w| = 1\), the same boundary values. The slight difficulty brought about by the fact that these boundary values are merely measurable is readily surmounted by considering first smaller circles and by proceeding then to the limit as \(r \to 1\). We note that

\[(34) \quad V(0) = \frac{1}{2\pi} \int_{|\gamma| = 1} h_*(\{e^{-D} U(0, \gamma)\}) \, d\gamma,\]

and that, by (32),

\[(35) \quad h_*(U(z, w)) \leq V_*(w).\]

From (31), (31''), and (35) we obtain, with regard to \(V_* \geq 0,\)

\[M_*(l - D) \leq g(l) \frac{1}{\sigma^2(K_1)} \sum_{r, u} \int_{K_1} \int_{R_u} \int_{K_1} V_*(w) \, d\sigma_u \, d\sigma_w \]

\[\leq g(l) \frac{1}{\sigma(K_1)} \sum_{r} \int_{K_1} \int_{R_u} \left\{ \frac{1}{\sigma(K_1)} \int_{K_1} V_*(w) \, d\sigma_w \right\} \, d\sigma_u.\]

Since \(V_*(w)\) is harmonic in \(|w| < 1\), we have by Gauss's mean-value theorem,

\[\frac{1}{\sigma(K_1)} \int_{K_1} V_*(w) \, d\sigma_w = V_*(0),\]

* This is where the existence and continuity of \(h_''\) is used.
whence
\[ M_*(l - D) \leq q(l) \frac{1}{\sigma(K_1)} \sum_\nu \sigma(R_\nu) V(S_\nu(0)) \]
\[ = q(l) \sigma(K_0) \sum_\nu V(S_\nu(0)), \]
the summation being extended over all \( \nu \) for which \( R_\nu \subset K_1 \). On setting
\[ S_\nu(R_0) = R_\nu', \]
we infer from (29) that
\[ S_\nu(0) \subset R_\nu'. \]
For two different \( R \), the corresponding \( R' \) are obviously different. Furthermore, it follows from (30) and (37) that the NE-distance of \( R' \) from \( R_0 \) is the same as that of \( R \) from \( R_0 \). According to (31”) the NE-distance of \( R \) from \( R_0 \) is less than \( l \). Hence all regions \( R' \) considered here must lie within the circle \( K_{l+2D} \). Thus (36) can be written as
\[ M_*(l - D) \leq \frac{q(l)}{\sigma(K_1)} \sum_\nu \int \int_{R_\nu} V(S_\nu(0)) d\sigma_z, \]
where
\[ S_\nu(0) \subset R_\nu' \subset K_{l+2D}. \]
By (38’) on applying Harnack’s inequality to \( V \geq 0 \) we have
\[ V(S_\nu(0)) \leq e^{D} V(z), \quad z \subset R_\nu', \]
which being combined with (38) yields
\[ M_*(l - D) \leq \frac{q(l)}{\sigma(K_1)} e^{D} \sum_\nu \int \int_{R_\nu} V(z) d\sigma_z \]
\[ \leq \frac{e^{D} q(l)}{\sigma(K_1)} \int \int_{K_{l+2D}} V(z) d\sigma_z = q(l) e^{D} \frac{\sigma(K_{l+2D})}{\sigma(K_1)} V(0), \]
and, according to (17),
\[ \lim_{l \to \infty} M_*(l) \leq e^{D} V(0). \]
On taking account of (26) and (34) we infer from this inequality that
\[ (4\pi^2)^{-1} \operatorname{meas} \left[ U(\zeta, \gamma) = 0 \right] \leq (2\pi)^{-1} e^{D} \operatorname{meas} \left[ U(0, \gamma) = 0 \right], \]
which proves Lemma 3 in the case where \( R \) has no vertices on \( |z| = 1 \).
In the general case by cutting off the vertices lying on $|z| = 1$ we can always divide $R_0$ into two parts,
\begin{equation}
R_0 = R_0^* + R_0^{**},
\end{equation}
such that
\begin{equation}
\sigma(R_0^{**}) < \delta
\end{equation}
and that $R_0^*$ has a finite NE-diameter, say $D = D(\delta)$. We may always suppose $R_0^*$ to contain the origin. The set
\begin{equation}
B = \sum_{s} S(R_0^{**})
\end{equation}
is invariant under $G$. The set $R_0^*$ and all sets $R_*^*$ congruent to it cover the complement of $B$ in $|z| < 1$. We note that all $R_*^*$ have the same NE-diameter $D$. Consider all
\begin{equation}
R_*^* \subset K_1.
\end{equation}
Every point of $K_{l-D}$ belongs either to one of these $R_*^*$ or to $B$. For, if it belongs to any $R_*^*$ at all, this set must be contained in $K_1$ since its diameter equals $D$. Hence
\begin{equation}
K_{l-D} \subset \sum R_*^* + BK_{l-D}, \quad R_*^* \subset K_1.
\end{equation}
Since $h_i \leq 1$ we obtain from (25)
\begin{equation}
M_\epsilon(l - D) \leq \left[ \frac{\sigma(BK_{l-D})}{\sigma(K_{l-D})} \right]^2 + \frac{q(l)}{\sigma^2(K_1)} \sum \int \int \int \int R_*^* + BK_{l-D}, \quad R_*^* \subset K_1.
\end{equation}
the summation being confined to all
\begin{equation}
R_*^* \subset K_1, \quad R_*^* \subset K_1.
\end{equation}
Here the second term satisfies the same inequalities as before. The first term is less than
\begin{equation}
a^2\sigma^2(BR_0) = a^2\sigma^2(R_0^{**}) < a^2\delta^2,
\end{equation}
by Lemma 1 and (41). On proceeding as before to the limit as $l \to \infty$, $\epsilon \to 0$, we obtain inequality (39) with the additional term $a^2\delta^2$ in the right-hand side. A suitable choice of $\delta$ evidently leads to the general proof of Lemma 3. Theorem B, and therefore the proposed theorem on surfaces $\Sigma$ of the first kind, is herewith completely proved.

6. Surfaces of the second kind. For a group $G$ of the second kind, the fundamental region has on its boundary one or several arcs of the unit circle. We shall consider only the case where $R$ has no zero angle vertices on $|z| = 1$,
i.e., where the surface Σ has no cusps. These arcs and their images under \( G \) are known to lie everywhere dense on \(|z| = 1\). We shall prove that this set \( \omega \) has the measure of the whole of \(|z| = 1\). Let \( \alpha \) be one of the complete arcs of \(|z| = 1\) belonging to the boundary of \( R \). There will be two images of \( R \) adjacent to \( R \) along the two sides of \( \alpha \) which end at the two end points of \( \alpha \), respectively. In particular, there are two arcs of the set \( \omega \) immediately adjacent on both sides of \( \alpha \). This shows that the end points of any arc \( \alpha \) are interior points of the set \( \omega \) introduced above. Since \( \omega \) is invariant under \( G \), the Poisson integral \( u(z) \) whose boundary values on \(|z| = 1\) are zero on \( \omega \) and one elsewhere must also be invariant under \( G \),

\[
u(S(z)) = u(z)
\]

for all \( S \) of \( G \). All we have to prove is that \( u \equiv 0 \). Indeed, \( u(z) \) has, in the sense of ordinary convergence, the boundary value zero on every closed arc \( \alpha \). On account of its invariance, \( u \) takes all its values in \( R \). Since a harmonic function always attains its extrema on the boundary, \( u(z) \) must attain them on the (closed) part of \(|z| = 1\) belonging to the boundary of \( R \), whence \( u \equiv 0 \).

If \( R \) has vertices on \(|z| = 1\) an elementary proof could still be given as well as for Lemma 2, for instance by applying Green's formula

\[
\iint \text{grad}^2 u \, dx \, dy = \iint u \frac{\partial u}{\partial n} \, ds
\]

to a region obtained by diminishing \( R \) suitably (cutting off the vertices on \(|z| = 1\) in a suitable way).

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