DIFFERENTIABLE FUNCTIONS DEFINED IN
ARBITRARY SUBSETS OF EUCLIDEAN
SPACE*

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1. Introduction. In a former paper† we studied the differentiability of a
function defined in closed subsets of Euclidean n-space E. We consider here
the differentiability "about" an arbitrary point of a function defined in an
arbitrary subset of E. We show in Theorem 1 that any function defined in a
subset A of E which is differentiable about a subset B of E may be extended
over E so that it remains differentiable about B. This theorem is a generaliza-
tion of AE Lemma 2. We show further that any function of class C^m about a
set B is of class C^{m-1} about an open set B' containing B. In the second part of
the paper we consider some elementary properties of differentiable functions,
such as: the sum or product of two such functions is such a function.‡ We
end with the theorem that differentiability is a local property.§

2. Definitions and elementary properties. We use a one-dimensional nota-
tion as in AE. Thus f_k(x) = f_{k_1} \ldots f_{k_n}(x_1, \ldots, x_n), x^l = x_1^{i_1} \cdots x_n^{i_n}, l! = l_1! \cdots l_n!,
D_kf(x) = \frac{\partial^{i_1+\cdots+i_n}f(x)}{\partial x_1^{i_1} \cdots \partial x_n^{i_n}}, \text{etc.}; we set \sigma_k = k_1 + \cdots + k_n, r_{xy} = distance from x to y. We always set f(x) = f_0(x). Suppose the functions f_k(x) for
\sigma_k \leq m are defined in the subset A of Euclidean n-space E. Define R_k(x'; x)
for x, x' in A by

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* Presented to the Society, January 2, 1936; received by the editors October 26, 1935.
† Analytic extensions of differentiable functions defined in closed sets, these Transactions, vol. 36
(1934), pp. 63–89. We refer to this paper as AE. See also Functions differentiable on the boundaries of
regions, Annals of Mathematics, vol. 35 (1934), pp. 482–485, and Differentiable functions defined in
closed sets, I, these Transactions, vol. 36 (1934), pp. 369–387, which we refer to as F and D respec-
tively.

P. Franklin in Theorem 1 of a paper Derivatives of higher order as single limits, Bulletin of the
American Mathematical Society, vol. 41 (1935), pp. 573–582, has given a necessary and sufficient
condition for the existence of a continuous mth derivative. We remark that this theorem is exactly
the special case of Theorem I of D obtained by letting f(x) be defined in an interval. It is also a special
case of Theorem 2 of the author's Derivatives, difference quotients, and Taylor's formula, Bulletin of
the American Mathematical Society, vol. 40 (1934), pp. 89–94 (see also Errata, p. 894). For his as-
sumption is easily seen to imply the needed uniformity condition; it also implies at once that f(x) is
continuous, so that no considerations of measurability are necessary. His Theorem 2 should be com-
pared with Theorems II and III of D.

‡ If the set is closed, these theorems may be proved by first extending the functions through-
out E.

§ For the case of one variable this follows from D, Theorem I.
Let \( x^0 \) be an arbitrary point of \( E \). If for each \( k \) \((\sigma_k \leq m)\) and every \( \epsilon > 0 \) there is a \( \delta > 0 \) such that

\[
| R_k(x'; x) | \leq r_{x^0}^{m-\epsilon k} \epsilon \quad \text{if } x, x' \text{ in } A, \ r_{xx} < \delta, \ r_{x'x} < \delta,
\]

we shall say that \( f(x) \) is of class \( C^m \) in \( A \) about \( x^0 \) in terms of the \( f_k(x) \), or, \( f(x) \) is \((C^m, A, x^0, f_k(x))\). If this is true for each \( x^0 \) in \( B \), we say \( f(x) \) is \((C^m, A, B, f_k(x))\), and replace "about \( x^0 \)" by "about \( B \."\) We say \( f(x) \) (defined in \( A \)) is of class \( C^m \) in \( A \) about \( B \), or, \( f(x) \) is \((C^m, A, B)\), if there exist functions \( f_k(x) \) \((\sigma_k \leq m)\) defined in \( A \) such that \( f(x) \) is \((C^m, A, B, f_k(x))\). If \( B = A \) in the last two definitions, we leave out the words "about \( B \);" this is in agreement with the previous definitions. We say \( f(x) \) is \((C^\infty, A, B, f_k(x))\) if \( f(x) \) is \((C^m, A, B, f_k(x))\) for each \( m \). Any function defined in \( A \) is \((C^{-1}, A, E)\).

Remark. We might define in an obvious manner such relations as \((C^m, A, x^0)\), \((C^\infty, A, A, B)\). To study them would require a study of the different possible definitions of the \( f_k(x) \) if \( f(x) \) is \((C^m, A, B)\). The \( f_k(x) \) are not in general determined by \( f(x) \). Thus if \( A = B \) is the \( x_1 \)-axis, only the \( f_k(x) \) with \( k_2 = \cdots = k_n = 0 \) are determined by \( f(x) \). It is not obvious for what point sets \( A \) the \( f_k(x) \) are all determined by \( f(x) \).

If \( f(x) \) is \((C^m, A, B, f_k(x)) \((m \geq 0)\), then the \( f_k(x) \) are continuous at each point of \( B \);* that is, the \( f_k(x) \) may be defined in \( B - B \cdot A \) so that this will be true. To show this, take \( x^0 \) in \( B \), set \( \epsilon = 1 \), and choose \( \delta \) so that (2) holds for any \( k \) \((\sigma_k \leq m)\). Take \( x \) in \( A \) within \( \delta \) of \( x^0 \) (if there is such a point); then (1) and (2) show that \( f_k(x') \) is bounded for \( x' \) in \( A \) within \( \delta \) of \( x^0 \) \((\sigma_k \leq m)\). Now let \( \{x^i\} \) be any sequence of points of \( A, x^i \rightarrow x^0 \); (1) and (2) show that \( \{f_k(x^i)\} \) is a regular sequence.

If \( A \) is open and \( f(x) \) is \((C^m, A, A, f_k(x))\), then \( D_k f(x) \) exists and equals \( f_k(x) \) in \( A \) \((\sigma_k \leq m)\). (See AE.) If \( x^0 \) is an isolated point of \( A \) or is at a positive distance from \( A \), then \( f(x) \) is \((C^m, A, x^0, f_k(x))\) for any \( f_k(x) \). If \( f(x) \) is \((C^m, A, B, f_k(x)) \) or \((C^m, A, B)\], and \( A' \) is in \( A \), \( B' \) is in \( B \), then \( f(x) \) is \((C^m, A', B', f_k(x)) \) or \((C^m, A', B')\]. Also \( f(x) \) is \((C^0, A, B)\) if and only if it is continuous at each point of \( B \). If \( f(x) \) is \((C^m, A, B, f_k(x))\), then it is \((C^m', A, B, f_k(x))\) for all \( m' < m \); a stronger theorem is proved in Theorem 2. If \( f(x) \) is \((C^m, A, B, f_k(x))\), then \( f_k(x) \) is \((C^{m-k}, A, B, f_i(x))\).

3. Extension theorems. We prove here a theorem which gives the maximum range of differentiability of a function, and a theorem about the still larger range of differentiability of a function to an order one less.

* Or better, "continuous in \( A \) about \( B \)."
Theorem 1. If \(f(x)\) is \((C^m, A, B, f_k(x))^*\) \((m \text{ finite or infinite})\), then the \(f_k(x)\) may be extended throughout \(E\) so that \(f(x)\) is \((C^m, E, B, f_k(x))\).†

We note, conversely, that if \(f(x)\) is not \((C^m, A, x_0, f_k(x))\), then no extension of \(f(x)\) will be so. We remark also that \(f(x)\) may be made analytic in \(E - \overline{A}\) \((\overline{A} = A \text{ plus limit points})\).

To prove the theorem, we first extend the \(f_k(x)\) through \(\overline{A} - A\) as follows: Take any \(x^0\) in \(\overline{A} - A\). Let \(f_k(x^0)\) be the upper limit of \(f_k(x^i)\) for sequences \(\{x^i\}, x^i \to x^0, x^i \in A\), if this is finite; otherwise, set \(f_k(x^0) = 0\). Next we extend the \(f_k(x)\) throughout \(E - \overline{A}\) by the method of AE Lemma 2. We shall assume in the proof that \(m\) is finite. If \(m = \infty\), we prove \(C^{m'}\) for every integer \(m'\).

The only alteration needed is that AE §12 should be used; but this makes no essential change.

As \(E - \overline{A}\) is open, \(f(x)\) is \((C^m, E, E - \overline{A}, f_k(x))\); we must show that \(f(x)\) is \((C^m, E, B - A, f_k(x))\). Take a fixed point \(x^0\) in \(B - \overline{A}\). Let us say \((k, e, A, A)\) holds if there is a \(\delta > 0\) such that (2) holds whenever \(x\) is in \(A_1\), \(x'\) is in \(A_2\), and \(r_{x^0} < \delta, r_{x'x^0} < \delta\). We must prove \((k, e, E, E)\) for each \(k\) \((\sigma_k \leq m)\) and each \(e > 0\).

First we prove \((k, e, \overline{A}, \overline{A})\). Set \(e' = e/[2(m + 1)^{n}]\), and let \(\delta\) be the smallest of the \(\delta\)'s given by \((l, e', A, A)\) for \(\sigma_l \leq m\). Let \(U\) be the spherical neighborhood of \(x^0\) of radius \(\delta\); then \(f_l(x)\) is bounded in \(U \cdot \overline{A}\) \((\sigma_l \leq m)\). Given \(x\), \(x'\) in \(U \cdot \overline{A}\), choose sequences \(\{x^i\}, \{x'^i\}\) of points of \(U \cdot A\), with \(x^i \to x, x'^i \to x'\). Suppose first \(\sigma_k = m\). Then we may take these sequences so that \(f_k(x^i) \to f_k(x)\), \(f_k(x'^i) \to f_k(x')\), and the desired inequality for \(R_k(x'; x)\) follows from that for \(R_k(x^i; x^i)\). Suppose now that \(\sigma_k < m\). Relations (1) and (2) with \(k\), \(x^i\), \(x^i\) replaced by \(k\), \(x\), \(x\) show that for any such \(\{x^i\}\), \(\{f_i(x^i)\}\) is a regular sequence \((\sigma_l < m)\); hence \(f_i(x^i) \to f_i(x)\), and similarly \(f_i(x'^i) \to f_i(x')\) \((\sigma_l < m)\). Relation (1) now shows that for \(i\) large enough, \(\Delta = R_k(x'; x) - R_k(x^i; x^i)\) differs as little as we please from

\[ - \sum_{\sigma_l = m - \sigma_k} \frac{f_{k+l}(x) - f_{k+l}(x^i)}{l!} \cdot \frac{(x' - x)^l}{l!}. \]

As \(|f_i(x) - f_i(x^i)| \leq e'\) \((\sigma_i = m)\) and \(|(x' - x)^l| \leq r_{x^0}^l\), \(|\Delta| \leq (m + 1)^n e' r_{x^0}^{m - \sigma_k}\) for \(i\) large enough; the inequality again follows.

Next we prove \((k, e, \overline{A}, E - \overline{A})\). Set \(e' = e/[2 \cdot 4^m (m + 1)^n]\), and define \(\eta\) in terms of \(e'\) and then \(\delta\) as in AE §11, using \((k, \eta, \overline{A}, \overline{A})\). Take \(x\) in \(\overline{A}\) and \(x'\) in \(E - \overline{A}\), each within \(\delta/4\) of \(x^0\). By AE (6.3) and the equation following (11.6),

* Or merely locally \((C^m, A, B)\); see Theorem 6.
† If \(A = B\) is closed, then \(B\) may be replaced by \(E\); the present proof then gives a proof of AE Lemma 2 which makes no use of AE Lemma 1.
\[ R_k(x'; x) = D_k f(x') - \psi_k(x'; x) = \sum_l \frac{R_{k+l}(x^*, x)}{l!} (x' - x^*)^l + \sum_{s=1}^{t} \sum_l \left( \begin{array}{l} k \\ l \end{array} \right) D_l \phi_{k,s}(x') \xi_{s;k-l}(x'), \]

where \( x^* \) is a point of \( \overline{A} \) distant \( \delta_*/4 \) from \( x' \), \( \delta_*/4 \) being the distance from \( x' \) to \( \overline{A} \). As \( r_{x^*x} \leq 2r_{xz} \), \( r_{x'z} \leq 2r_{xz} \), and \( \delta_* \leq 4r_{xz} \), we find with the help of AE (11.8)

\[ | R_k(x'; x) | < (m + 1)^n (2r_{xz})^{m-\sigma_k} \eta + (4r_{xz})^{m-\sigma_k} \epsilon'/2 < r_{xz} \epsilon. \]

Next we prove \((k, \epsilon, E - \overline{A}, \overline{A})\). As is easily seen from AE (6.3) or by F (6) with \( x^{i-1}, x^i \) replaced by \( x, x' \),

\[ R_k(x'; x) = \sum_l \frac{R_{k+l}(x'; x^*)}{l!} (x' - x^*)^l. \]

Set \( \epsilon' = \epsilon/(m+1)^n \), and take the smallest \( \delta \) given by \((k+l, \epsilon', \overline{A}, E - \overline{A})\) for \( \sigma_i \leq m - \sigma_k \). The required inequality now follows at once.

Finally we must show \((k, \epsilon, E - \overline{A}, E - \overline{A})\). Set \( \epsilon' = \epsilon/[2(n(m+1)^n] \), and take \( \delta \) smaller than the \( \delta/4 \) given by AE §11 with \( \epsilon \) replaced by \( \epsilon' \) and smaller than the \( \delta \)'s given by \((k+l, \epsilon', \overline{A}, E - \overline{A})\) and \((k+l, \epsilon', E - \overline{A}, \overline{A})\) for \( \sigma_i \leq m - \sigma_k \). Now take \( x \) and \( x' \) in \( E - \overline{A} \) within \( \delta \) of \( x^0 \); we must consider two cases. Case I: The line segment \( S = xx' \) lies wholly in \( E - \overline{A} \). By AE (11.2), \( \left| f_s(y) - f_s(x') \right| < 2\epsilon' \) for \( y \) on \( S (\sigma_i \leq m) \); the desired inequality now follows from F, Lemma 3. Case II: There is a point \( x^* \) of \( \overline{A} \) on \( S \). From AE (6.3), or F (6) with \( x^{i-1}, x^i \) replaced by \( x, x^* \), we find

\[ R_k(x'; x) = R_k(x'; x^*) + \sum_l \frac{R_{k+l}(x^*, x)}{l!} (x' - x^*)^l, \]

and the inequality again follows.

**Theorem 2.** If \( f(x) \) is \((C^m, A, B, f_k(x)) \) \((m \text{ finite})\), then there is an open set \( B' \) containing \( B \) such that \( f(x) \) is \((C^{m-1}, A, B', f_k(x)) \).

For each \( x \) in \( B \), let \( \delta(x) \) be the largest of the numbers \( \delta \) for which (2) holds for all \( k (\sigma_k \leq m) \) with \( \epsilon \) replaced by 1. Let \( U(x) \) be the set of all points \( x' \) within \( \delta(x) \) of \( x \); then \( B' \) is the sum of all \( U(x) \). The set \( B' \) is open. To prove \((C^{m-1}, A, B', f_k(x)) \), take any \( x^0 \) in \( B' \) and any \( \epsilon > 0 \). For some \( x^* \) in \( B \), \( r_{x^*x^0} < \delta(x^*) \). There is an \( M \) such that \( |f_k(y)| < M \) for \( y \) in \( A \cdot U(x^*) \) \((\sigma_k \leq m).\) Let \( \delta \) be the smaller of \( \delta(x^*) - r_{x^*x^0} \) and \( \epsilon/[2(m+1)^n M + 2] \). Now take any \( x \) and \( x' \) in \( A \) within \( \delta \) of \( x^0 \). We are interested in the remainders

*† For the proof, see the paragraph following the remark.
\[ R_k'(x'; x) = \sum_{i=1}^{m-\sigma_k} \frac{f_{k+1}(x)}{i!} (x' - x)^i + R_k(x'; x) \]

with \( \sigma_k < m \). As \( r_{xx'} < 2\delta \),

\[ |R_k'(x'; x)| \leq (m + 1)^n M r_{xx'} \frac{m-\sigma_k}{n} + r_{xx'} \frac{m-1-\sigma_k}{n} \epsilon. \]

**Corollary.** If \( f(x) \) is of class \( C^m \) in any given point set about \( B \), then it may be extended through an open set \( B' \) containing \( B \) so that it is of class \( C^{m-1} \) in \( B' \) and of class \( C^m \) in \( B' \) about \( B \).

4. Composite functions, etc. We prove here three theorems.

**Theorem 3.** If \( f \) and \( g \) are of class \( C^m \) in \( A \) about \( B \), then so are \( f + g \) and \( f - g \), with

\[ (f \pm g)_k = f_k \pm g_k. \]

This is obvious.

**Theorem 4.** If \( f \) and \( g \) are of class \( C^m \) in \( A \) about \( B \), then so is \( fg \), and \( f/g \) if \( g \neq 0 \). The derivatives are given by the ordinary formulas. Thus

\[ (fg)_k = \sum_l \binom{k}{l} f_{l} g_{k-l}. \]

We might prove this theorem directly, but it follows from Theorem 5: \( fg \) and \( f/g \) are functions (of two variables) of class \( C^\infty \) of the functions \( f \) and \( g \). (The condition \( B \) in \( A \) is obtained by using Theorem 1.)

**Theorem 5.** Let \( A \) and \( B \) be subsets of \( n \)-space \( E_n \), and let \( A' \) and \( B' \) be subsets of \( v \)-space \( E_v \). Let \( f^i(x) \) be \( (C^m, A, B, f^i_k(x)) \) \((i = 1, \ldots, v)\), and let \( g(y) \) be \( (C^m, A', B', g_k(y)) \) \((m \text{ finite or infinite})\). Suppose \( B \) is in \( A \), \( x \) in \( A \) implies

\[ y = (y_1, \ldots, y_v) = (f^1(x), \ldots, f^v(x)) = f(x) \]

in \( A' \), and \( x \) in \( B \) implies \( f(x) \) in \( B' \). Then the function

\[ h(x) = g(f^1(x), \ldots, f^v(x)) = g(f(x)) \]

is \( (C^m, A, B, h_k(x)) \); the \( h_k(x) \) are given by the ordinary formulas (9) for derivatives.

As a consequence of this theorem, the definition of being of class \( C^m \) is independent of the coordinate system chosen. If the condition \( x \) in \( A \) \([\text{or} B]\) does not imply \( f(x) \) in \( A' \) \([\text{or} B']\), we may apply the theorem to any subset \( A_1 \) \([\text{or} B_1]\) of \( A \) \([\text{or} B]\) for which it does. We shall suppose \( m \) is finite; if \( m = \infty \), we merely apply the reasoning below for each positive integer.
Suppose first \( u^1(x), \ldots, u^r(x) \) are functions of class \( C^m \) in an open set \( \Gamma \) of \( E_n \), suppose \( v(y) \) is of class \( C^m \) in an open set \( \Gamma' \) of \( E_n \), and suppose \( x \) in \( \Gamma \) implies \( u(x) \) in \( \Gamma' \). Letting \( R'^1, S' \) denote remainders for \( u^1, v \), Taylor's formula gives

\[
(5) \quad u_k^i(x') = D_ku^i(x') = \sum_{\sigma_i \leq m-\sigma_k} \frac{u_{k+1}^i(x)}{l!} (x' - x)^l + R^i_k(x'; x),
\]

\[
(6) \quad v_k(y') = D_kv(y') = \sum_{\sigma_i \leq m-\sigma_k} \frac{v_{k+1}(y)}{l!} (y' - y)^l + S_k'(y'; y),
\]

certain inequalities on the \( R^i_k \) and \( S_k' \) being satisfied. We have set \( \sigma_k' = k_1 + \cdots + k_r \). Set \( w(x) = v(u(x)) \); then (5) and (6) with \( k=0 \) give

\[
(7) \quad w(x') = \sum_i \frac{v_i(u(x))}{l!} \left( \sum_{\sigma_i \geq 1} \frac{u_i(x)}{j!} (x' - x)^j + R'(x'; x) \right)^l + S'(u(x'); u(x)),
\]

where \( S' = S' \). Also, by Taylor's formula,

\[
(8) \quad w_k(x') = \sum_i \frac{w_{k+1}(x)}{l!} (x' - x)^l + T_k'(x'; x).
\]

Subtract (8) with \( k=0 \) from (7); then as \( R', S', \) and \( T' \) all approach 0 to the \( m \)th order as \( x' \to x \),† we may equate coefficients of \( (x' - x)^k \) for \( \sigma_k \leq m. \)‡ Thus we find polynomials

\[
P_k(u^i_p, v_q) \quad (\sigma_p \leq \sigma_k, \sigma_q' \leq \sigma_k; \sigma_k \leq m)
\]

such that, for any \( x \) in \( \Gamma \),

\[
(9) \quad w_k(x) = P_k(u^i_p(x), v_q(u(x))).
\]

Using (8) gives for \( w_k(x') \)

\[
(10) \quad w_k(x') = \sum_i \frac{P_{k+1}(u^i_p(x), v_q(u(x)))}{l!} (x' - x)^l + T_k'(x'; x).
\]

We may also evaluate it by replacing \( x \) by \( x' \) in (9) and using (5) and (6). (In (6) we replace \( y' \) by \( u(x') \) and use (5) again.) Each variable in the resulting polynomial \( P_k \) consists of a polynomial in quantities \( R', S' \), and other quantities; if we multiply out and collect all terms with an \( R' \) or an \( S' \) as a factor, we obtain

† This is clear for \( S' \) if \( m=0 \); if \( m>0 \), then

\[
S'/r' = \left[ S'/|u(x')-u(x)|^m \right] \left[ |u(x')-u(x)|/r' \right]^m,
\]

where \( |y'-y|=r_{y'} \), and the last factor is bounded in \( U \cdot A \).

‡ This is easily proved in succession for \( \sigma_k=0, 1, \ldots \) on letting \( x' \to x \).
$$\frac{x^i}{i!} \left( \frac{\sum_{j=1}^{m} u_j(x)}{j!} (x' - x)^j \right) \right] + Q_k,$$

where $Q_k$ is a polynomial containing an $R'$ or an $S'$ as a factor in each term. It must be understood that $\sum u_{j+i}(x)(x' - x)^j/s!$ appears as the variable in the position of $u_{j+i}$, etc., in $P_k(u_j^i, v_q)$.

We now prove: If $u_j^i (\sigma_k \leq m; i = 1, \cdots, \nu)$, $v_k (\sigma_k \leq m)$ are any numbers, then

$$P_k^*(x; u_j^i, v_q) = P_k \left[ \sum \frac{u_{j+i}}{i!} x^i, \sum \frac{v_{j+i}}{i!} (x')^j \right] - \sum \frac{P_{k+i}(u_j^i, v_q)}{i!} x^i,$$

considered as a polynomial in $x$, contains no terms of degree $\leq m - \sigma_k$. To prove this, define the polynomials

$$w(x) = \sum_{i=0}^{m} \frac{u_i}{i!} x^i, \quad v(y) = \sum_{i=0}^{m} \frac{v_i}{i!} (y - u_0)^i;$$

then $w'(0) = D_k w'(0) = u_j^i$, $v_k(0) = D_k v_k(0) = v_k$. Set $w(x) = v(u(x))$. Replacing $x', x$ by $x, 0$ in (10) and (11) and putting in (12) gives, as $Q_k = 0$ in this case,

$$P_k^*(x; u_j^i, v_q) = T_k'(x; 0).$$

As $T_k' \to 0$ to the $(m - \sigma_k)$th order as $x \to 0$, $P_k^*$ cannot contain any terms of degree $\leq m - \sigma_k$.

We return now to the functions $f(x), g(x), h(x)$. Set $h_k(x) = P_k(f^i(x), g_q(f(x)))$. The formulas (10) and (11) hold equally well for the $f, g, h$. Hence using (10), (11), and (12), we find for the remainder for $h_k(x)$

$$T_k(x'; x) = P_k^*(x' - x; f_p^i(x), g_q(f(x))) + Q_k.$$

To show that $h(x)$ is $(C^m, A, B, h_k(x))$, take any $x^0$ in $B$, and set $y^0 = f(x^0)$. As $f(x)$ is continuous in $A$ about $B$, for each neighborhood $V$ of $y^0$ there is a neighborhood $U(V)$ of $x^0$ such that $x$ in $U(V) \cdot A$ implies $f(x)$ in $V \cdot A'$. As $y^0$ is in $B'$, we may take $V$ so that the $g_k(y)$ are bounded in $V \cdot A'$. We may take $U$ in $U(V)$ so small that the $f_k(x)$ are bounded in $U \cdot A$. Because of the property of $P_k^*$, we may obviously take $\delta$ small enough so that $P_k^*$ satisfies an inequality of the nature of (2). Moreover each term in $Q_k$ contains an $R_p(x'; x)$ or an $S_k(u(x'); u(x))$ with $\sigma_p \leq \sigma_k$ or $\sigma_q \leq \sigma_k$; as each such remainder satisfies
an inequality (2) (see a recent footnote) and all other quantities entering into
$Q_k$ are bounded, we may take $\delta$ small enough so that $Q_k$ also satisfies an
inequality (2). Hence the same is true of $T_k$, and the theorem is proved.

5. Differentiability a local property. Our object is to prove

**Theorem 6.** Let $f(x)$ be locally $(C^m, A, B)$ (m finite or infinite). For each
point $x^0$ of $B$ there is a neighborhood $U$ of $x^0$ and functions $f_k(x^0)(x)$ defined in
$U \cdot A$ such that $f(x)$ is $(C^m, U \cdot A, U \cdot B, f_k(x^0)(x))$.† Then $f(x)$ is $(C^m, A, B)$. If
the $f_k(x^0)(x)$ for $\sigma_k \leq p$ are independent (at any $x$ for which they are defined) of $x^0$,
then these functions may be included among the $f_i(x)$ ($\sigma_k \leq m$).

We may take each neighborhood $U$ as an open $n$-cube, so small that
the $f_k(x^0)(x)$ are bounded in $U$. A finite or denumerable number of them,
$C_1, C_2, \ldots$, cover $B$; we may take them so that any one touches at most a
finite number of the others, and so that any boundary point of any $C_i$ is in-
terior to some $C_j$.† By hypothesis, to each $i$ there correspond functions $f_i(x)$,
$\sigma_i \leq m$, such that $f(x)$ is $(C^m, C_i \cdot A, C_i \cdot B, f_i(x))$. In each $C_i$ we define the
function $\pi_i(x)$ as it was defined in I, in AE §9; set

$$
(16) \quad \phi_i(x) = \pi_i(x)/\sum_i \pi_i(x)
$$

in $C_1 + C_2 + \cdots$. Set $g_i(x) = \phi_i(x)f(x)$ in $C_i \cdot A$. By Theorem 4, $g_i(x)$ is
$(C^m, C_i \cdot A, C_i \cdot B)$, and

$$
(17) \quad g(x) = \sum_i \left(\begin{array}{c}
C_i
D\phi_i(x) f_i - i(x).
\end{array}\right)
$$

As the $f_i(x)$ are bounded in $C_i \cdot A$ and the $D\phi_i(x)\to 0$ to infinite order as $x$
approaches the boundary of $C_i$ (see AE §9), the latter statement is true also
of the $g_i(x)$. Hence, evidently, if we set $g_i(x) = 0$ in $A - C_i \cdot A$, $g_i(x)$ is
$(C^m, A, B, g(x))$. Set

$$
(18) \quad f(x) = g_k^1(x) + g_k^2(x) + \cdots ,
$$

which in any $C_i \cdot A$ is a finite sum; this reduces to $f(x)$ for $k = 0$. Theorem 3
shows at once that $f(x)$ is $(C^m, A, B, f_k(x))$. (Given $x^0$ in $B$, to apply Theorem

† Note that "$f(x)$ is locally $(C^m, \cdots)$" is not the same statement as "$f(x)$ is locally $(C^m, \cdots)$
for each $m$."† Let $C^1, C^2, \cdots$ be a denumerable set of the cubes which cover $B$. Express each $C^i$ as the sum
of a denumerable number of cubes $C_j^i$ with the following properties: Each $C_j^i$ is, with its boundary,
interior to $C^i$; the diameter of $C_j^i$, $\delta(C_j^i)$, is $< 1/i$; $\delta(C_j^i)\to 0$ as $j\to \infty$; the cubes $C_j^i$ approach the
boundary of $C^i$ as $j\to \infty$. Now drop out all cubes $C_j^i$ which are interior to larger cubes $C_j^k$; the remain-
ing cubes $C_1, C_2, \cdots$ still cover $B$. To each cube $C_j^i$ corresponds a number $\eta > 0$ such that any point
set of diameter $< \eta$ having points in common with $C_j^i$ lies interior to some $C_j^k$; using this fact, it is
easily seen that any $C_i$ has points in common with but a finite number of the $C_j$.\[License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use]
3, we choose \( \delta \) so small that the points within \( \delta \) of \( x^0 \) lie in but a finite number of the \( C_i \).

To prove the second statement, let \( f^i(x) \) denote the common value of \( f^i(x) \) for \( \sigma_i \leq \rho \). Differentiating \( \sum \phi_i = 1 \) gives

\[
\sum_i D_i \phi_i(x) = \begin{cases} 1 & \text{if } l = 0, \\ 0 & \text{if } \sigma_i > 0. \end{cases}
\]

Define the \( f_k(x) \) as before. Take any \( k \) with \( \sigma_k \leq \rho \); then (17) and (18) give

\[
f_k(x) = \sum_l \left( \begin{array}{c} k \\ l \end{array} \right) f^l \phi_l(x) \sum_i D_i \phi_i(x) = f_k'(x)
\]

in \( C_1 + C_2 + \cdots \). It does not matter how \( f_k(x) \) is defined outside this set.

The second statement in the theorem does not hold for an arbitrary set of \( f_k(x) \), at least using the above method. To see this, take \( n = m = 2, A = B \) = the interval \((-1, 1)\) of the \( x_1\)-axis, \( C_1 = C_2 \) = the square with corners \((\pm 1, \pm 1)\); set \( f = 0 \),

\[
f^{10} = f^{10} = f^{11} = f^{02} = 0, \quad f^{01} = 1,
\]

and \( f_{ij} = -f_{ij} \) on \( A \). Also set

\[
\phi_1(x, y) = \frac{1}{2} + \frac{3}{4} x - \frac{1}{2} x^3, \quad \phi_2(x, y) = \frac{1}{2} - \frac{3}{4} x + \frac{1}{4} x^3.
\]

(Though \( \phi_1 \) and \( \phi_2 \) are not the functions defined above, they have the necessary properties.) We find on \( A \)

\[
g^{11}(x, y) = g^{21}(x, y) = \frac{3}{4} - \frac{3}{8} x^2, \quad f_{11}(x, y) = \frac{3}{8} - \frac{3}{8} x^2 \neq 0.
\]

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