ON EFFECTIVE SETS OF POINTS IN RELATION TO INTEGRAL FUNCTIONS*

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1. Introduction. Let \( f(z) \) be an integral function and let \( M(r, f) = \max_{|z| \leq r} |f(z)| \). The order \( \rho \) and the type \( \kappa(f) \) of \( f(z) \) are defined by the relations

\[
\rho = \limsup_{r \to \infty} \frac{\log \log M(r, f)}{\log r}; \quad \kappa(f) = \limsup_{r \to \infty} \frac{\log M(r, f)}{r^\rho}.
\]

Let \( \{z_n\}_n, z_n = r_n e^{i\theta_n} \) be a distinct sequence of complex numbers such that

\[
0 < r_1 \leq r_2 \leq \cdots \leq r_n \to \infty
\]
as \( n \to \infty \). Let \( \rho_1 > 0 \) be any number. The type \( \kappa(f, \rho_1, \{z_n\}) \) of \( f(z) \) over the set \( \{z_n\} \) is defined by the relation

\[
\kappa(f, \rho_1, \{z_n\}) = \limsup_{n \to \infty} \frac{\log |f(z_n)|}{|z_n|^{\rho_1}}.
\]

If \( f(z) \) is of order \( \rho \), it is evident that \( \kappa(f, \rho_1, \{z_n\}) \leq 0 \) when \( \rho_1 > \rho \). If \( \rho_1 \leq \rho \), the value of \( \kappa(f, \rho_1, \{z_n\}) \) can vary from \(-\infty\) to \( \infty \).

1.1. Definition. Let \( f_0 \) be a function of order \( \rho \); we shall say that \( \{z_n\} \) is an effective set, or briefly an \( E \)-set, for \( f_0 \) when \( \kappa(f_0, \rho_1, \{z_n\}) = \kappa(f) \).

1.2. It is easy to see that any given function \( f(z) \) always possesses an \( E \)-set; for, on \( |z| = r \), there is at least one point \( z(r) \) such that \( M(r, f) = |f(z(r))| \); also, a sequence \( \{r_n\}, r_1 < r_2 < \cdots < r_n \to \infty \) as \( n \to \infty \), exists for which

\[
\kappa(f) = \lim_{n \to \infty} \frac{\log M(r_n, f)}{r_n^\rho};
\]

hence \( \{z(r_n)\} \) is an effective set for \( f(z) \). A more interesting question is to ascertain whether all functions of a given class specified by some simple property possess an \( E \)-set in common. In this paper an attempt is made to answer this question.

1.3. We denote \( C(\rho, d) \) the class of all functions of order \( \rho \) and type less than \( d \) where \( \rho \) and \( d \) are any two given positive numbers. We regard all functions of order less than \( \rho \) as of order \( \rho \) and minimal type, that is \( \kappa(f) = 0 \), un-

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less \( f(z) = 0 \) in which case \( \kappa(f) = \kappa(f, \rho, [z_n]) = -\infty \) for all \( \rho \) and \( [z_n] \). These are included in \( C(\rho, d) \) for the purposes of this paper.

2. We shall, first, discuss a few general properties of an \( E \)-set for a given class \( C(\rho, d) \).

**Theorem 1.** In order that a set \([z_n] \) may be an \( E \)-set for a class \( C(\rho, d) \) it is necessary that

(i) the exponent of convergence (which we shall speak of as the order) of \([z_n] \) cannot be less than \( \rho \);

(ii) if the order of \([z_n] \) be \( \rho \), any function with zeros at \( z = z_n \) must be of order \( \rho \) and type not less than \( d \) unless such a function is identically zero;

(iii) the set \([\theta_n] \) of amplitudes of \([z_n] \) must be everywhere dense in \( 0 \leq \theta \leq 2\pi \).

**Proof.** If \([z_n] \) were of order \( \rho' < \rho \), the canonical product \( \sigma(z) \) with simple zeros at \([z_n] \) is of order \( \rho' \) and therefore is of order \( \rho \) and minimal type so that, by the definition of \( C(\rho, d) \)

\[
\kappa(\sigma) = \kappa(\sigma, \rho, [z_n]) = 0.
\]

But \( \sigma(z_n) = 0 \) so that

\[
\kappa(\sigma, \rho, [z_n]) = -\infty.
\]

This contradiction shows that the order of \([z_n] \) cannot be less than \( \rho \). A similar argument proves (ii). To prove (iii), suppose that \([\theta_n] \) is not everywhere dense in \((0, 2\pi)\). Then there is a \( \theta_0 \) such that \( \theta_0 - \delta \leq \theta \leq \theta_0 + \delta \) contains no \( \theta_n, \delta > 0 \) being sufficiently small. We can suppose without loss of generality that \( \theta_0 = 0 \) so that the angle \( |\theta| \leq \delta \) does not contain any point of \([z_n] \). Now, let \( H_\rho(z) \) be defined by

\[
H_\rho(z) = \begin{cases} \prod_{n=1}^{\infty} \left( 1 + \frac{z}{n^{1/\rho}} \right), & 0 < \rho \leq \frac{1}{2}, \\ \sum_{n=0}^{\infty} \frac{z^n}{\Gamma\left( \frac{n}{\rho} + 1 \right)}, & \rho > \frac{1}{2}. \end{cases}
\]

It is known that, if \( z \) is outside \( |\theta| \leq \delta \),

\[
\limsup_{|z| \to \infty} \frac{\log |H_\rho(z)|}{|z|^\rho} < \kappa(H_\rho).
\]

By considering a function of the form \( H_\rho(\eta z) \) where \( \eta \) is such that \( \eta^\rho \kappa(H_\rho) < d \), we conclude that \([z_n] \) cannot be an \( E \)-set for \( H_\rho(\eta z) \) which obviously belongs to \( C(\rho, d) \). Hence \([\theta_n] \) is everywhere dense in \((0, 2\pi)\). It may be noted that

* For \( 0 < \alpha \leq 1 \), see Paley and Wiener, *Fourier Transforms in the Complex Domain*, p. 79; for \( \alpha > \frac{1}{2} \), \( H_\rho(z) \) are Mittag-Leffler’s functions, *Acta Mathematica*, vol. 29 (1905), pp. 101–181.
since an $E$-set remains an $E$-set when any other set is added to it, we cannot expect to improve upon the result (i) of Theorem 1.

2.1. We shall now give a general criterion for a set $[z_n]$ to form an $E$-set for a class $C(p, d)$. Let $A_n(h)$ denote the circle with center $z_n$ and radius $|z_n|^{-h}$; and let $A(h)$ denote the system of circles $A_n(h)$, $n = 1, 2, \ldots$. We prove

**Theorem 2.** The set $[z_n]$ of order $p$ will form an $E$-set for $C(p, d)$ provided there exists a function $g(z)$ with simple zeros at $z = z_n$ and $h > p$ such that the following relations hold:

$$
\lim_{n \to \infty} \frac{\log |g'(z_n)|}{|z_n|^h} = d; \quad \lim_{|z| \to \infty} \frac{\log |g(z)|}{|z|^h} = d
$$

as $|z| \to \infty$ outside the circles $A(h)$.

2.2. We shall establish two lemmas in the first place.

**Lemma 1.** There exists a sequence $[R_n]$, $R_1 < R_2 < \cdots < R_n \to \infty$ as $n \to \infty$, $R_{n+1} \leq aR_n$, $a > 1$ being given, such that no circle $|z| = R_n$ cuts any circle of $A(h)$.

**Proof.** Let $b > 1$. Consider the ring $r \leq |z| \leq br$. The sum of the diameters of those circles of $A(h)$ whose centers lie in the ring cannot exceed

$$
\sum_{r \leq |z| \leq br} |z_n|^{-h}
$$

which is less than a fixed positive constant since $\sum |z_n|^{-h}$ converges when $h > p$. Therefore if $r$ is sufficiently large, there is at least one circle $|z| = R$ in the ring $r \leq |z| \leq br$ which does not cut any circle of $A(h)$. Hence, there is an $n_0$ such that for all $n \geq n_0$, the ring $b^n \leq |z| \leq b^{n+1}$ contains a circle of the type required. Taking $b = a^{1/t}$ we get the required result.

**Lemma 2.** Any function $g(z)$ satisfying the condition (ii) of Theorem 2 is of order $p$ and type $d$.

**Proof.** Let $a > 1$ be given and let $[R_n]$ be the sequence of Lemma 1. On $|z| = R_n$, we have, by (ii)

$$
M(R_n, g) \leq \exp [(d + e)R_n^p]
$$

for $n \geq n_0 = n_0(e)$. Since $M(r, g)$ is an increasing function of $r$ and $R_{n+1} \leq aR_n$, we get for all $r \geq r_0 = r_0(e)$

$$
M(r, g) \leq \exp [a^p(d + e)r^p],
$$

so that $\kappa(g) \leq \alpha d$ and since $a$ is any number greater than one, we get $\kappa(g) \leq d$. But obviously $d \leq \kappa(g)$. Hence $\kappa(g) = d$.

2.3. **Proof of Theorem 2.** Let $f(z)$ be any function of $C(p, d)$ and $[R_*]$ the sequence of Lemma 1 for some $a > 1$. Let
\[ I_v = \frac{1}{2\pi i} \int_{|z| = R_v} \frac{x^m f(x)}{g(x)} \frac{dx}{x - z}, \]

where \( m \geq 0 \) is an integer. By (ii) we find that

\[ I_v \to 0 \]

as \( v \to \infty \) uniformly in any fixed circle \( |z| \leq R \). But

\[ I_v = \frac{z^m f(z)}{g(z)} - \sum_{|z_n| < R_v} \frac{z_n^m f(z_n)}{g'(z_n)} \frac{1}{z - z_n}, \]

while by (i) the series

\[ \sum_{n=1}^{\infty} \left| \frac{z_n^m f(z_n)}{g'(z_n)} \frac{1}{z - z_n} \right| \]

converges uniformly except at the points \( z = z_n \). Therefore (1) and (2) give

\[ f(z) = \frac{g(z)}{g'(z)} \frac{z^m f(z)}{g(z)} = \sum_{n=1}^{\infty} \frac{z_n^m f(z_n)}{g'(z_n)} \frac{1}{z - z_n}. \]

Let \( \kappa(f, \rho, [z_n]) = \beta \leq \kappa(f) < d \). Choose \( \eta \) so that \( 0 < \eta < d - \beta \) and \( \lambda \) so that \( d\lambda^\rho = d - \beta - \eta \). Let \( \chi(z) = c_0 + c_1 z + c_2 z^2 + \cdots \) be any integral function of order \( \rho \) and type not exceeding \( d\lambda^\rho \). Then by (i) the double series

\[ \sum_{(m,n)} \left| \frac{c_m z_n^m f(z_n)}{g'(z_n)} \frac{1}{z - z_n} \right| \]

converges uniformly except at \( z = z_n \), so that (3) gives

\[ f(z) \chi(z) \equiv g(z) = \sum_{n=1}^{\infty} \frac{f(z_n) \chi(z_n)}{g'(z_n)} \frac{1}{z - z_n}. \]

In (4) we can take \( \chi(z) = g(\lambda z) \) since in this case \( \kappa(\chi) = d\lambda^\rho \) by Lemma 2. So (4) gives

\[ f(z) = \frac{g(z)}{g(\lambda z)} \sum_{n=1}^{\infty} \frac{f(z_n) g(\lambda z_n)}{g'(z_n)} \frac{1}{z - z_n}. \]

Let \( A(\lambda) \) denote the circles around the zeros of \( g(\lambda z) \) similar to \( A(h) \). Then, given \( a > 1 \), we can, just as in Lemma 1, choose a sequence \( [R_n], R_{n+1} \leq aR_n \), such that the circles \( |z| = R_n \) do not cut any circle of either \( A(h) \) or \( A(\lambda) \). Using (i), (ii) and the choice of \( \lambda \), we get from (5),

\[ M(R_n, f) \leq \exp [(d - d\lambda^\rho + \epsilon) R_n^\rho] \]

for \( n \geq n_0 = n_0(\epsilon) \). Starting from (6), an argument of the type used in Lemma 2
shows that $k(f) \leq d - d\lambda = \beta + \eta$ and since $\eta$ is subject to the sole restriction $0 < \eta < d - \beta$ we get $k(f) \leq \beta$. Since $\beta \leq \kappa(f)$, we get $\kappa(f) = \beta$ which is the result required.

2.4. In some cases it is possible to conclude that the relation (i) of Theorem 2 follows from (ii). The circles of $A(h)$ determine a sequence of non-overlapping domains $D_1, D_2, \ldots, D_n, \ldots$. Let $p_n$ denote the number of points of $[z_n]$ lying in $D_n$. We shall prove

**Lemma 3.** If $p_n \geq P$, a fixed positive number, then (ii) of Theorem 2 involves (i).

**Proof.** Let $z_n$ be contained in $D_{q_n}$. Let

$$P_n(z) = \prod_{z_n \in D_{q_n}} \left(1 - \frac{z}{z_n}\right),$$

and

$$g(z) = P_n(z)Q_n(z).$$

The greatest and the least distances of the boundary of $D_{q_n}$ from the origin lie in the interval $(|z_n| - H, |z_n| + H)$ where $H = \sum_{n=1}^{\infty} |z_n|^{-\eta}$. Since the degree of $P_n(z)$ does not exceed $P$, we have

$$\lim_{|z| \to \infty} \frac{\log |P_n(z)|}{|z|^P} = 0,$$

as $|z| \to \infty$ outside the domains $D_n$, uniformly in $n$. Therefore on the boundary of $D_{q_n}$, we have by (ii)

$$\exp \left[ (d - \epsilon)(|z_n| - H)^P \right] \leq |Q_n(z)| \leq \exp \left[ (d + \epsilon)(|z_n| + H)^P \right].$$

Since $Q_n(z)$ does not vanish in $D_{q_n}$, (7) holds in the interior of $D_{q_n}$, in particular, at $z = z_n$. Hence

$$\lim_{n \to \infty} \frac{\log |Q_n(z_n)|}{|z_n|^P} = d.$$

Moreover

$$g'(z_n) = P_n'(z_n)Q_n(z_n),$$

and arguing as before we get

$$\lim_{n \to \infty} \frac{\log |g'(z_n)|}{|z_n|^P} = \lim_{n \to \infty} \frac{\log |Q_n(z_n)|}{|z_n|^P} = d.$$

So the lemma is proved.

3. Using Theorem 2, we shall set up an $E$-set for a given class $C(\rho, d)$. We first establish the following
Lemma 4. Let $\rho = 2/\alpha$ and

$$\sigma_p(z) = \prod_{n=1}^{\infty} \left(1 - \frac{z^n}{\rho^{\alpha n}}\right).$$

Then $\sigma_p(z)$ is an integral function of order $\rho$ satisfying (i) and (ii) of Theorem 2, with $d = \alpha/4 = 1/(2p)$.

Proof. If we prove (ii) with some $h > \rho$, (i) will follow from Lemma 3 since the circles of $A(h)$ are obviously non-overlapping after a certain stage. Let $\rho > 0$ be any integer and let $z$ lie in the ring $\rho^a \leq |z| \leq (\rho+1)^a$.

$$\sigma_p(z) = \left(1 - \frac{z^p}{\rho^a}\right) \left(1 - \frac{z^n}{(p+1)^n(p+1)}\right) F_p(z),$$

where

$$F_p(z) = \prod_{n=1}^{p-1} \left(1 - \frac{z^n}{\rho^{an}}\right) \prod_{n=p+2}^{\infty} \left(1 - \frac{z^n}{\rho^{an}}\right) = S_1 \times S_2,$$

say. We have

$$\log |S_1| = \frac{1}{2} \rho(p - 1) \log |z| - \alpha \sum_{n=1}^{p-1} n \log n + \log \prod_{n=1}^{p-1} \left|1 - \frac{\rho^{an}}{z^n}\right|$$

$$= \frac{\alpha}{4} \rho^2 + O(\log \rho) + \log \prod_{n=1}^{p-1} \left|1 - \frac{\rho^{an}}{z^n}\right|$$

$$= \frac{\alpha}{4} |z|^{2/\alpha} + O(|z|^{1/\alpha} \log |z|) + \log \prod_{n=1}^{p-1} \left|1 - \frac{\rho^{an}}{z^n}\right|.$$

Now, since $\rho^a \leq |z| \leq (\rho+1)^a$, we have

$$\prod_{n=1}^{\infty} \left(1 - \frac{\rho^{an}}{\rho^{an}}\right) \leq \prod_{n=1}^{p-1} \left|1 - \frac{\rho^{an}}{z^n}\right| \leq \prod_{n=1}^{p-1} \left(1 + \frac{\rho^{an}}{\rho^{an}}\right).$$

Using the fact that $(1+1/x)^x$ steadily increases and $(1-1/x)^{-x}$ steadily decreases as $x$ varies in $0 < x < \infty$, we get

$$\left(\frac{n}{p}\right)^{\alpha n} = \left\{(1 - \frac{n}{p})^{p/(p-n)}\right\}^{\alpha n(p-n)/p} \leq \exp\left[-\frac{\alpha n(p-n)}{p}\right]$$

$$\leq \exp\left[-\frac{\alpha}{2} \cdot \min(n, p-n)\right].$$

Therefore (12) gives

$$0 < a = \left\{\prod_{n=1}^{\infty} \left(1 - e^{-\alpha n/2}\right)\right\}^2 \leq \prod_{n=1}^{p-1} \left|1 - \frac{\rho^{an}}{z^n}\right| \leq \left\{\prod_{n=1}^{\infty} \left(1 + e^{-\alpha n/2}\right)\right\}^2 = b,$$
so that by (11),

$$\log | S_1 | = \frac{\alpha}{4} | z |^{2/\alpha} + O( | z |^{1/\alpha} \log | z | ).$$

A similar argument shows that

$$\log | S_2 | = O(1),$$

so that (9), (10), and (13) give

$$\log | \sigma(z) | = \frac{\alpha}{4} | z |^{2/\alpha} + O( | z |^{1/\alpha} \log | z | )
+ \log \left| 1 - \frac{z^p}{p^{\alpha p}} \right| 1 - \frac{z^{p+1}}{(p+1)^{\alpha (p+1)}} |,$$

where $\rho^a \leq | z | \leq (p+1)^a$. Taking $\hat{h} > \rho = 2/\alpha$, we find from (14) that when $z$ is outside the circles of $A(\hat{h})$ but inside the ring $\rho^a \leq | z | \leq (p+1)^a$, the lemma is proved.

3.1. It is easy to see that if $[z^\nu]$ is an $E$-set for $C(p, d)$ then $[\eta z^\nu]$ is an $E$-set for $C(p, d/\eta^\nu)$. Hence Lemma 4 enables us, in conjunction with Theorem 2, to state

**Theorem 3.** Let $\rho > 0$, $d > 0$ be given. The set of points

$$[(2pd)^{-1/p}\eta^{2/p} \nu^{2/p} z^{\nu}]$$

forms an $E$-set for the class $C(p, d)$. In other words, if $f(z)$ is any function of order $\rho$ and type less than $d$, then

$$\limsup_{n \to \infty} \log M(r, f) \frac{\rho^\nu}{r^\rho} = \kappa(f) = \limsup_{n \to \infty} \log | f((2pd)^{-1/p}\eta^{2/p} \nu^{2/p} z^{\nu}) |.$$
3.3. If \( \sigma(z) \) is the canonical product with simple zeros at the lattice points \( z = m + in, \ m, n = 0, \pm 1, \pm 2, \ldots \), it is known from the pseudo-periodic properties of \( \sigma(z) \) that (i) and (ii) of Theorem 2 hold for \( \sigma(z) \) with \( \rho = 2 \) and \( d = \pi/2 \). Hence the class \( C(2, \pi/2) \) has the peculiarly simple \( E \)-set \( z = m + in, \ m, n = 0, \pm 1, \pm 2, \ldots \).

3.4. I have shown elsewhere* that a function of \( C(2, \pi/2) \) bounded at the lattice-points must be a constant. The question may be asked whether the same is true of an \( E \)-set for \( C(\rho, d) \) for which the conditions of Theorem 2 hold. That this is in fact the case can be shown by using exactly the same method followed in the case of the lattice points.† So we can state

**Theorem 5.** Let \( \{z_n\} \) be a set of points satisfying the conditions (i) and (ii) of Theorem 2. Then any function of order \( \rho \) and type less than \( d \) bounded at the points \( \{z_n\} \) must reduce to a constant.

3.5. As a particular case of Theorem 5 we get

**Theorem 6.** An integral function of order \( \rho \) and type less than \( 1/(2\rho) \) bounded at the points

\[
\begin{align*}
n^{2i/n} e^{2\pi i/n}, & \quad n = 1, 2, 3, \ldots; \nu = 0, 1, 2, \ldots, n - 1
\end{align*}
\]

reduces to a constant.

3.6. It may be noted that an \( E \)-set for a class \( C(\rho, d) \) is a fixture to that class and is independent of the individual functions of the class. Theorems 2–4 throw a good deal of light on the peculiar behaviour of the functions of \( C(2, \pi/2) \) at the lattice points. These latter were, in fact, the starting point of the investigations of this paper. It is very probable that conditions closely allied to those of Theorem 2 are also necessary for an \( E \)-set although I have not succeeded in discovering exactly what these conditions are. The question whether (ii) of Theorem 2 always involves (i) is also unsolved.

† Since \( f(z_n) = 0 \), \( \kappa(f) \leq 0 \); if \( \kappa(f) < 0 \), \( f(z) = 0 \); if \( \kappa(f) = 0 \), then also \( \kappa(f^n) = 0 \), so that formula (3) holds with \( m = 0 \) and \( f^n/p^\nu \), \( \nu = 0, 1, 2, \ldots \), in place of \( f(z) \). An addition and an argument, as in Lemma 2, will show that \( f(z) \) is of finite order, that is, \( f(z) \) is a polynomial which must be a constant since \( f(z_n) = O(1) \).

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