# ON DIFFERENTIAL GEOMETRY IN THE LARGE, I (MINKOWSKI'S PROBLEM)*

BY

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**Introduction.** Hermann Minkowski,† in a fundamental paper on convex bodies, proposed the following Problem (M): to determine a convex, three-dimensional body $B$ whose surface admits of a given Gaussian curvature $K(n)>0$, assigned as a continuous function of the direction of the interior normal $n$ to the surface. After having stated three obviously necessary conditions for the function $K(n)$, Minkowski proceeds to solve an analogous Problem (M′) for polyhedra. He then considers a passage to the limit among the solutions of problems (M′) approximating (M), and establishes their convergence to a convex body $B_0$. This construction leaves open the question as to whether $B_0$ is a solution of (M). Minkowski remarks that $B$, if it exists, is, to within a translation, the uniquely determined solution of a certain third Problem (M″) of the calculus of variations and that $B_0$, too, is a solution of (M″).

Now if we assume (H): the surface of $B_0$ is differentiable to a sufficiently high order, then $B_0$ solves (M). However, Minkowski does not discuss (H), but proves instead that the mixed volume $V(B_0, B_0, C)$ of $B_0$ with an arbitrary convex body $C$ may be computed as though $B_0$ were a solution of (M) and that, furthermore, a convex body is, except for a translation, uniquely determined by its mixed volume with the totality of convex bodies. While later authors have modified Minkowski's methods, there has been no improvement of his results as far as the hypothesis (H) is concerned.

Thus Minkowski's results are open to the same criticism that could be raised against the early solutions of Plateau's problem: namely, that instead of solving the proposed problem, a more general problem is treated whose solution coincides with that of the former only if the latter solution satisfies certain highly restrictive conditions, and no indication is presented that these conditions are actually satisfied.

The present paper contains a solution of (M) for the case of analytic $K(n)$. It does not involve the Brunn-Minkowski inequalities, nor, indeed, the idea of mixed volume. It uses instead the author's results on elliptic and

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† Minkowski, *Werke*, pp. 231–276, entitled *Volumen und Oberfläche*. See especially §10. Also Bonnesen-Fenchel, *Theorie der konvexen Körper*, Ergebnisse, vol. 3, no. 1 (1934), which contains a bibliography of related papers.

analytic equations of the Monge-Ampère type and establishes the existence of an analytic $B$ by a continuity method.

In order to make this treatment of (M) complete, a new proof of the uniqueness of $B$ is included. This proof is based on a modification of a beautiful idea of Cohn-Vossen* who, in a similar problem, reduced the uniqueness problem to the determination of a certain topological index of a vector field. Our modification is such as to allow an application of this idea to a wider class of related uniqueness problems in the large.

1. Let $u(x, y)$ be a homogeneous polynomial solution of Laplace's equation of degree $n>2$. A suitable rotation of the $(x, y)$-system transforms $u(x, y)$ into a constant multiple of $\mathcal{R}[(x+iy)^n]$, and the asymptotic directions of the surface $u=u(x, y)$, determined by

$$u_{xx}dx^2 + 2u_{xy}dxdy + u_{yy}dy^2 = 0, \tag{1}$$

undergo the same rotation as the coordinates. Assuming this rotation effected, we obtain for the asymptotic directions

$$\mathcal{R}[(x + iy)^{n-2}(d(x + iy))^2] = 0,$$

or, in polar coordinates $r$, $\theta$,

$$\mathcal{R}[r^{n-2}e^{i(n-2)\theta}(dx + idy)^2] = 0.$$

Thus the vector $dx+idy$ of the asymptotic direction forms the angle $-(n-2)\theta/2$ or $-(n-2)\theta/2+\pi/2$ with the $x$-axis; the two asymptotic directions are perpendicular, and either direction turns through an angle $-(n-2)\pi$ as we follow it along a Jordan curve containing the origin in its interior. In other words: the asymptotic directions form two distinct fields of directions with the origin as a singular point of index $-(n-2)/2$. We notice furthermore that the discriminant of (1) is a constant multiple of $r^{2n-4}$ and vanishes only for $r=0$.

2. We prove now the following lemma:

LEMMA. *Let $F(x, y, u, p, q, r, s, t)$ be analytic in the neighborhood of $(x_0, y_0, u_0, p_0, q_0, r_0, s_0, t)$ and $4(\partial F/\partial r)(\partial F/\partial t)-(\partial F/\partial s)^2>0$. Let $u(x, y)$ and its first derivatives $p$, $q$ and second derivatives $r$, $s$, $t$ be a solution of $F=0$, analytic in a neighborhood of $(x_0, y_0)$ and such that $u(x_0, y_0)=u_0$, $p(x_0, y_0)=p_0, \cdots, t(x_0, y_0)=t_0$. Assume $u'(x, y)$ and its derivatives $p'$, $q'$, $r'$, $s'$, $t'$ to be a second analytic solution of $F=0$, coinciding together with its derivatives $p'$, $q'$, $r'$, $s'$, $t'$, with $u$ and its derivatives at the point $(x_0, y_0)$. Then the difference $U=u-u'$ represents a surface $U(x, y)$ whose Gaussian curvature is negative in a sufficiently small*

* Cohn-Vossen, *Zwei Sätze über die Starrheit der Eiflächen*, Göttinger Nachrichten, 1927, pp. 125–134.

*neighborhood of* $(x_0, y_0)$, *with the exception of* $(x_0, y_0)$ *itself, and the index of either of its distinct asymptotic directions is negative at* $(x_0, y_0)$, *unless* $u(x, y)$ *is identically equal to* $u'(x, y)$.

In terms of the second derivatives $R$, $S$, $T$ of $U(x, y)$ the asympototic directions of $U=U(x, y)$ are given by

$$Q \equiv Rdx^2 + 2Sdxdy + Tdy^2 = 0; \tag{2}$$

and (2) admits of real solutions $dx$, $dy$ if $RT-S^2<0$. To show that $RT-S^2<0$ express $F(x, y, u(x, y), p(x, y), \cdots)-F(x, y, u'(x, y), p'(x, y), \cdots)$ as a power series in $(x-x_0, y-y_0)$ and observe that the terms of lowest degree are given by

$$E \equiv a\overline{R} + b\overline{S} + c\overline{T},$$

where $\overline{R}$, $\overline{S}$, $\overline{T}$ are the second derivatives of the non-vanishing terms $\overline{U}$ of lowest degree $n>2$ in the development of $U$, and $a$, $b$, $c$ are the values of $\partial F/\partial r$, $\partial F/\partial s$, $\partial F/\partial t$ at $(x_0, y_0)$, for which by hypothesis $4ab-c^2>0$. Since $F(\cdots, u(x, y), \cdots)-F(\cdots, u'(x, y), \cdots)=0$, we have $E=0$, and this is possible only for $\overline{R}\overline{T}-\overline{S}^2<0$ or $\overline{R}=\overline{S}=\overline{T}=0$. Now $\overline{R}$, $\overline{S}$, $\overline{T}$ vanish simultaneously only at $(x_0, y_0)$. For there exists a suitable linear transformation of the $(x, y)$-plane with determinant 1 which leaves $(x_0, y_0)$ invariant and transforms $E=0$ into Laplace's equation. $\overline{U}$ is thereby transformed into a harmonic homogeneous polynomial of degree $n>2$ in $(x-x_0, y-y_0)$, and for such polynomials we proved in §1 that the discriminant of the second derivatives vanishes only at $(0, 0)$. Hence the development of $RT-S^2$ starts with the negative term $\overline{R}\overline{T}-\overline{S}^2$ and we conclude that $RT-S^2$ is negative for sufficiently small $|x-x_0|$, $|y-y_0|$ and vanishes only at $x=x_0$, $y=y_0$.

Since the linear transformation does not change the index of a field of directions, we conclude from §1 that the index of the field

$$\overline{Q} \equiv \overline{R}dx^2 + 2\overline{S}dxdy + \overline{T}dy^2 = 0 \tag{3}$$

is negative. Since, for sufficiently small values of $|x-x_0|$, $|y-y_0|$, the directions of the field (2) differ arbitrarily little from those of (3), the index of (3) is negative at $(x_0, y_0)$. This completes the proof of the Lemma.

3. We may now prove the following:

THEOREM 1. *Two closed convex analytic surfaces* $S$ *and* $S'$ *are congruent if they possess the same positive Gaussian curvature* $K$ *at points for which their inner normals are parallel and similarly directed.*

By parallel normals we map $S$ and $S'$ on the unit sphere $\sigma$. An arbitrary equator divides $\sigma$ and, since the map is one-to-one, $S$ and $S'$, into two regions

in each of which the sphere as well as $S$ and $S'$ assume the form $Z=Z(X, Y)$ in suitably chosen rectangular coordinates. Upon introducing

$$x = \frac{\partial Z}{\partial X}, \qquad y = \frac{\partial Z}{\partial Y} \tag{4}$$

as independent variables instead of $(X, Y)$ and setting

$$H(x, y) = -Z(x, y) + xX(x, y) + yY(x, y), \tag{5}$$

we obtain

$$X = H_x, \qquad Y = H_y, \qquad Z = -H + xX + yY. \tag{6}$$

How $S$ and $S'$ satisfy the condition that their curvatures for corresponding parallel normals, i.e., for the same value of $(x, y)$, are the same positive function $K(x, y)$, whence $S$ and $S'$ are solutions of

$$\left(1 + \left(\frac{\partial Z}{\partial X}\right)^2 + \left(\frac{\partial Z}{\partial Y}\right)^2\right)^2 K(x, y) = \frac{\partial^2 Z}{\partial X^2}\,\frac{\partial^2 Z}{\partial Y^2} - \left(\frac{\partial^2 Z}{\partial X \partial Y}\right)^2$$

or

$$(1 + x^2 + y^2)^2 K(x, y) = \frac{\partial(x, y)}{\partial(X, Y)},$$

or, finally, of

$$H_{xx}H_{yy} - H_{xy}^2 = K^{-1}(x, y)(1 + x^2 + y^2)^{-2}. \tag{7}$$

It is readily shown that the second fundamental form of the surface (6) is

$$(H_{xx}dx^2 + 2H_{xy}dxdy + H_{yy}dy^2)(1 + x^2 + y^2)^{-1/2}. \tag{8}$$

Suppose that the formulas (4) and (5) lead to a function $H(x, y)$ if applied to $S$, and to $H'(x, y)$ if applied to $S'$. Then the statement of Theorem 1 is equivalent to the equation

$$H'(x, y) = H(x, y) + l(x, y),$$

where $l(x, y)$ is a linear function of $(x, y)$.

Consider, with Cohn-Vossen, the congruence points of $S$ and $S'$, i.e., those points for which their normals and their second fundamental forms coincide. If all of their points were congruence points, we should have

$$H_{xx} = H'_{xx}, \qquad H_{xy} = H'_{xy}, \qquad H_{yy} = H'_{yy},$$

and the theorem is proved. In the alternative case we shall show the existence of at least one congruence point. The two second differential forms of $S$ and $S'$ may both be assumed to be positive definite. The equation

$$(H_{xx} - H'_{xx})dx^2 + 2(H_{xy} - H'_{xy})dxdy + (H_{yy} - H'_{yy})dy^2 = 0 \tag{9}$$

obtained by setting the two forms equal to one another, determines two distinct directions tangential to $\sigma$ at each point of $\sigma$ which is not a congruence point. For the ellipses of the $(dx, dy)$-plane

$$H_{xx}dx^2 + 2H_{xy}dxdy + H_{yy}dy^2 = 1$$

and

$$H'_{xx}dx^2 + 2H'_{xy}dxdy + H'_{yy}dy^2 = 1$$

have, by (7), the same area $\pi(1+x^2+y^2)K^{1/2}(x, y)$. As they are concentric but not identical, they intersect in four distinct points; the ratios of their coordinates are the two distinct solutions $dx$: $dy$ of (9).

It is impossible to construct a field of tangential directions on the sphere without singularities, and the sum of the indices of these singularities equals 2 if there are only finitely many singularities. Now choose, at an arbitrary point of $\sigma$, one of the two directions (9) and extend, by continuity, this choice over the whole of $\sigma$. If there were no congruence points, we should obtain a field of tangential directions on $\sigma$ without singularities. Hence there is at least one congruence point $(x_0, y_0)$.

Subtracting if necessary a linear function from $H'(x, y)$ we may assume that, at $(x_0, y_0)$, $H(x, y)$ and $H'(x, y)$ coincide with their derivatives up to the second order, without affecting the truth of (7) nor that of (9). But now our lemma implies that unless $H$ and $H'$ are identical, the congruence point $(x_0, y_0)$ is isolated and has a negative index. Summing over all indices of all singularities we still obtain a negative number in contradiction to the general fact mentioned above. Hence $S$ and $S'$ are congruent.

THEOREM 1′. *Consider a sequence of closed convex analytic surfaces $S(\tau)$ of positive curvature such that their corresponding functions $H(x, y, \tau)$ depend analytically on $(x, y, \tau)$ for small values of $\tau$. Suppose that all surfaces $S(\tau)$ have a common point of contact corresponding to the same point $(x, y)$ of $\sigma$. Assume that for each point of $S(0)$ the derivative $(\partial K/\partial\tau)_{\tau=0}$ vanishes. Then we have also $(\partial H/\partial\tau)_{\tau=0}=0$.*

Abbreviate the operator $(\partial/\partial\tau)_{\tau=0}$ by the use of the symbol $\delta$ and consider the field of tangential directions $dx$, $dy$ on $\sigma$, determined by

$$\delta H_{xx}dx^2 + 2\delta H_{xy}dxdy + \delta H_{yy}dy^2 = 0. \tag{9'}$$

Unless $\delta H_{xx}=\delta H_{xy}=\delta H_{yy}=0$, there are, at each point $(x, y)$, two real and distinct directions satisfying (9′). For we deduce from (7) by differentiation

$$H_{yy}\delta H_{xx} + H_{xx}\delta H_{yy} - 2H_{xy}\delta H_{xy} = 0, \tag{7'}$$

which yields $\delta H_{xx}\delta H_{yy}-(\delta H_{xy})^2<0$ in view of $H_{xx}H_{yy}-H_{xy}^2>0$.

As in the preceding proof we conclude first the existence of a point $P$ on $\sigma$ for which $\delta H_{xx} = \delta H_{yy} = \delta H_{xy} = 0$. Assume this not to be true identically.

Since an addition of a linear function of $(x, y)$ to $\delta H$ does not affect the truth of (7′) and (9′), we may assume that the field of directions determined by (9′) in the neighborhood of $P$ corresponds to a function $\delta H$ that vanishes at $P$ together with its derivatives up to the second order. This function $\delta H$ may be considered as the difference between the following two solutions of (7′): $\delta H$ itself and the identically vanishing solution. From the lemma we see that $P$ is an isolated singularity of the field defined by (9′) and that the index of $P$ is negative. The proof of Theorem 1′ can now easily be completed by recalling the conclusion at the end of the proof of Theorem 1.

4. We state now the following:

DEFINITION. *Let $d\omega$ be the surface element of the unit sphere $\sigma$ and let $\xi, \eta, \zeta$ be the cosines of the angle of its normal with the axes of the $(X, Y, Z)$-system. A function $F$ of a point of $\sigma$ is called admissible if it depends analytically on the point of $\sigma$ and if*

$$\int F\xi d\omega = \int F\eta d\omega = \int F\zeta d\omega = 0. \tag{10}$$

THEOREM 2. *For every admissible positive function $K$ on $\sigma$ there exists a closed convex analytic surface $S$ whose curvature, considered as function of the interior normal vector (of length 1), equals $K$.*

First we shall demonstrate the following statement:

II. *Assume that for small values of a parameter $\tau$ an admissible positive function $K$ depends analytically on the point of $\sigma$ and $\tau$, and that for $\tau = 0$ there exists a surface $S(0)$ with $K(0)$ as corresponding curvature function. Then there exists an analytic closed convex surface $S(\tau)$ with $K(\tau)$ as corresponding curvature provided $|\tau|$ is sufficiently small.*

Let $M$ be an arbitrary but analytic function on the unit sphere $\sigma$. Continue its definition from the sphere into the three-dimensional space containing it by assuming $M$ to be zero at the center and linear on every ray issuing from the center. $M$ thereby becomes a homogeneous function of degree 1 in any system of rectangular coordinates $(\xi, \eta, \zeta)$ with the origin as center and $M$ is analytic everywhere except at the origin. We have from the homogeneity

$$M = M_\xi \xi + M_\eta \eta + M_\zeta \zeta. \tag{11}$$

Introduce

$$x = -\xi/\zeta, \qquad y = -\eta/\zeta, \qquad \zeta H(x, y) = -M. \tag{12}$$

We obtain

$$H_x = M_\xi, \qquad H_y = M_\eta. \tag{13}$$

On $\sigma$ the coordinates $(x, y)$ may be used for either hemisphere $\zeta<0$ or $\zeta>0$. From (13) we conclude that $H_x$, $H_y$ are analytic on the whole of $\sigma$ if we define them on the equator $\zeta=0$ by continuity.

We find

$$\begin{aligned} \pm(\xi, \eta, \zeta) &= \left(\frac{x, y, -1}{(1+x^2+y^2)^{1/2}}\right), \\ d\omega &= \frac{dxdy}{(1+x^2+y^2)^{3/2}}. \end{aligned} \tag{14}$$

Whenever $M$ is such that $\partial(H_x, H_y)/\partial(x, y)\neq 0$, the reasoning of p. 260 shows that

$$\frac{\partial(H_x, H_y)}{\partial(x, y)}\cdot(1+x^2+y^2)^2 = \phi(H, H) \tag{15}$$

is the reciprocal Gaussian curvature of the surface

$$X = M_\xi, \qquad Y = M_\eta, \qquad Z = M_\zeta, \tag{16}$$

whence we conclude that $\phi$ remains invariant under the change which $x$, $y$, and $H$ undergo as $\xi$, $\eta$, $\zeta$ are subjected to an orthogonal transformation that is sufficiently near the identity. This fact is obviously independent of the condition $\partial(H_x, H_y)/\partial(x, y)\neq 0$.

The integral over an arbitrary region

$$\mathfrak{J} = \iint \mp (H_{xx}H_{yy} - H_{xy}^2)dxdy = \iint \phi(H, H)\zeta d\omega$$

can be transformed into the integral over its boundary

$$\frac{1}{2}\int (H_x dH_y + H_y dH_x).$$

As the expressions $H_x$, $H_y$ are analytic on the unit sphere we can extend the integration in $\mathfrak{J}$ over the whole of $\sigma$ and obtain $\mathfrak{J}=0$ since the boundary integrals over the equator, generated by the integration over the upper and lower hemispheres, annul each other.

Using the invariance of $\phi(H, H)$, we obtain similarly

$$\iint \phi(H, H)\xi d\omega = \iint \phi(H, H)\eta d\omega = \iint \phi(H, H)\zeta d\omega = 0. \tag{17}$$

We also find for the associated bilinear form

$$\phi(H_1, H_2) = (1 + x^2 + y^2)^2(H_{1xx}H_{2yy} + H_{1yy}H_{2xx} - 2H_{1xy}H_{2xy})$$

that

$$\iint \phi(H_1, H_2)\xi d\omega = \iint \phi(H_1, H_2)\eta d\omega = \iint \phi(H_1, H_2)\zeta d\omega = 0. \tag{18}$$

In other words, $\phi(H_1, H_2)$ is admissible if $M_1$ and $M_2$ are analytic on the sphere.

After these preliminary remarks we return to the given function $K^{-1}(\tau)$ and develop it into a power series in $\tau$,

$$K^{-1}(\tau) = \kappa(\tau) = \kappa_0 + \tau\kappa_1 + \tau^2\kappa_2 + \cdots . \tag{19}$$

We try to determine a function $M(\xi, \eta, \zeta; \tau)$, homogeneous of first degree in $\xi, \eta, \zeta$,

$$M(\tau) = M_0 + M_1\tau + M_2\tau^2 + \cdots , \tag{20}$$

depending analytically on $\xi, \eta, \zeta, \tau$ (excepting the point $\xi=\eta=\zeta=0$) and being in the following relation to $K^{-1}(\tau)$: If we introduce by (12) the quantities $x, y, H(x, y; \tau)$, then (7) holds for all sufficiently small values of $|\tau|$. Let us develop the equation (7)

$$\phi(H, H) = \kappa$$

into a power series in $\tau$ and set the coefficients of both members equal. Denoting again by the operator $\delta$ the differentiation with respect to $\tau$ at $\tau=0$, we find the equations:

$$\delta^n M = \delta^n H(x, y)/(1 + x^2 + y^2)^{1/2}, \quad M_0 = H(x, y)/(1 + x^2 + y^2)^{1/2}, \tag{21}$$

$$H_{xx}\delta H_{yy} + H_{yy}\delta H_{xx} - 2H_{xy}\delta H_{xy} = \kappa_1(x, y)(1 + x^2 + y^2)^{-2},$$

$$\begin{aligned} H_{xx}\delta^2 H_{yy} + H_{yy}\delta^2 H_{xx} - 2H_{xy}\delta^2 H_{xy} = {} & 2!\kappa_2(x, y)(1 + x^2 + y^2)^{-2} \\ & - 2(\delta H_{xx}\delta H_{yy} - \delta H_{xy}\delta H_{xy}), \end{aligned} \tag{22}$$

. . . . . .

Since by hypothesis $K^{-1}(\tau)$ is admissible for all $\tau$ in question, the same is true for the coefficients:

$$\iint \kappa_\nu\xi d\omega = \iint \kappa_\nu\eta d\omega = \iint \kappa_\nu\zeta d\omega = 0, \qquad (\nu = 0, 1, 2, \cdots).$$

Suppose that we have found $\delta^n M(\xi, \eta, \zeta)$ for $n<m$, in accordance with (22), and analytic on the sphere. Then the preliminary remarks show that the right-hand member in the $m$th equation yields zero if we multiply it in turn by

$$(1 + x^2 + y^2)^2\xi d\omega, \qquad (1 + x^2 + y^2)^2\eta d\omega, \qquad (1 + x^2 + y^2)^2\zeta d\omega$$

and integrate over the whole sphere. We therefore are entitled to apply a theorem of Hilbert* in which he states the now well known alternative for elliptic differential equations on the sphere for the special case of the equation with admissible right-hand member $r$,

$$(1 + x^2 + y^2)^2[H_{xx}(\delta H)_{yy} + H_{yy}(\delta H)_{xx} - 2H_{xy}(\delta H)_{xy}] = r,$$
$$\delta H \cdot (1 + x^2 + y^2)^{-1/2} = \delta M,$$

and establishes the existence of a solution $\delta M$, which is differentiable infinitely many times as the coefficients of the differential equation are. Applying Hilbert's result to the $m$th equation (22) we find the function $\delta^m M$. Since this equation is elliptic because $S(0)$ is convex and accordingly $H_{xx}H_{yy} - H_{xy}^2 > 0$, we conclude from the analyticity of the equation the analyticity of $\delta^m M$ on the sphere.

It remains to be seen that for sufficiently small $|\tau|$ the series

$$M = M_0 + \sum_1^\infty \frac{\delta^n M}{n!}\tau^n$$

converges. This can be true only if we eliminate the arbitrariness that affects the determination of $\delta^m M$, since we may add to $\delta^m M$ an arbitrary linear combination of $\xi$, $\eta$, $\zeta$ with constant coefficients and still retain a solution of the differential equation for $\delta^m M$. Denote by $L[v]$ the linear elliptic differential expression in $v$ on the sphere which in coordinates $(x, y)$ reduces to

$$L(v) \equiv (1 + x^2 + y^2)^2(H_{xx}u_{yy} + H_{yy}u_{xx} - 2u_{xy}H_{xy}), \quad u = v(1 + x^2 + y^2)^{1/2}.$$

Since $L[v] = 0$ has precisely the three linearly independent solutions $v = \xi, \eta, \zeta$, $L[v]$ admits of a Green's function of the second kind $G(A; B)$, where $A$ and $B$ are two points of the sphere. With the aid of one such $G(A; B)$ we solve the equations (22), written in the abbreviated form

$$L[\delta^\nu M] = f^\nu, \qquad (\nu = 1, 2, \cdots),$$

by setting

$$\delta^\nu M = \iint G(A; B) f^\nu(B) d\omega_B, \tag{23}$$

and thereby disposing of the arbitrariness in the determination of $\delta^\nu M$.

We shall say "a function $f$ on the sphere satisfies a Hölder condition of ex-

* D. Hilbert, *Grundzüge einer Allgemeinen Theorie der Linearen Integralgleichungen*, Leipzig and Berlin, 1912, p. 250.

ponent $\alpha$ and coefficient $C$" if for arbitrary distinct points $Q_1$ and $Q_2$ of spherical distance $\overline{Q_1Q_2}>0$ we have

$$|f(Q_1) - f(Q_2)| \leqq C\overline{Q_1Q_2}^{\alpha},$$

where $\alpha$ is a positive constant less than 1. Similarly "the derivatives of $f$ satisfy a Hölder condition of exponent $\alpha$ and coefficient $C$" if $0<\alpha<1$ and for arbitrary $Q_1 \neq Q_2$

$$|f_s(Q_1) - f_s(Q_2)| \leqq C\overline{Q_1Q_2}^{\alpha}, \qquad |f_n(Q_1) - f_n(Q_2)| \leqq C\overline{Q_1Q_2}^{\alpha},$$

where $f_s$ is the derivative in the direction of the great circle joining $Q_1$ to $Q_2$ and $f_n$ is the derivative in the direction normal to this circle. Similarly for higher derivatives.

With these notations the familiar estimates of the theory of linear elliptic differential equations are applied to the integral $v(A)=\int\int G(A;B)f(B)d\omega_B$ and lead to the following:

LEMMA. *If $f$ is bounded by $C$ and satisfies a Hölder condition of exponent $\alpha$ and coefficient $C$, then $v(A)=\int\int G(A;B)f(B)d\omega_B$ and its first and second derivatives are bounded by $hC$ and satisfy a Hölder condition of exponent $\alpha$ and coefficient $Ch$ where $h$ does not depend on $f$.*

From the convergence of the series (19), with the use of the Heine-Borel theorem, we derive the existence of a number $\bar{\rho}>0$ such that

$$|\kappa_\nu| < \frac{1}{\bar{\rho}^\nu}, \qquad |\kappa_\nu'| < \frac{1}{\bar{\rho}^\nu}, \qquad (\nu = 1, 2, \cdots),$$

where $\kappa_\nu'$ stands for the directional derivative of $\kappa_\nu$ at an arbitrary point in an arbitrary direction. Hence there also exists a number $\rho>0$ such that $|\kappa_\nu|<1/\rho^\nu$ and $\kappa_\nu$ satisfies a Hölder condition of a certain exponent $\alpha$ and coefficient $1/\rho^\nu$.

The first equation (22) shows that $\delta M$ and its first and second derivatives are bounded by $h/\rho$ and satisfy a Hölder condition of exponent $\alpha$ and coefficient $h/\rho$.

Consider the equation

$$z^2(\tau) = \frac{1}{64h^2} - \sum_1^\infty \frac{\tau^n}{\rho^n}. \tag{24}$$

It admits of two roots for $z(\tau)$ and in particular the root

$$z(\tau) = -\frac{1}{8h} + \sum_1^\infty \frac{c_n\tau^n}{n!\rho^n}.$$

We have the following system of recurrent formulas from (24):

$$2z(0)z'(0) = -\frac{1}{\rho} \qquad \text{or} \qquad c_1 = \frac{4h}{\rho},$$

$$2z(0)z''(0) = -\frac{2!}{\rho^2} - 2z'^2(0) \qquad \text{or} \qquad c_2 = \frac{4h\cdot 2!}{\rho^2} + 8hc_1^2,$$

$$2z(0)z'''(0) = -\frac{3!}{\rho^3} - 6z'(0)z''(0) \qquad \text{or} \qquad c_3 = \frac{4h\cdot 3!}{\rho^3} + 24hc_1c_2,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

On the other hand consider the successive bounds and Hölder coefficients $C_\nu$ for $\delta^\nu M$ and its first and second derivatives as obtained from (22). We have evidently the same law by which to form the successive inequalities*

$$C_1 \leqq \frac{4h}{\rho},$$

$$C_2 \leqq 4h\cdot\frac{2!}{\rho^2} + 8hC_1^2,$$

$$C_3 \leqq 4h\cdot\frac{3!}{\rho^3} + 24hC_1C_2,$$

$$\cdots\cdots\cdots$$

As all terms involved are positive we conclude $C_\nu \leqq c_\nu$, $(\nu = 1, 2, \cdots)$. Hence the series

$$M(\tau) = M(0) + \sum_1^\infty \frac{\delta^n M}{n!}\tau^n$$

converges together with its first and second derivatives with respect to $\xi$, $\eta$, $\zeta$ uniformly for sufficiently small $|\tau|$ since $z(\tau)$ is a majorant series. In order to complete the proof of statement II we have to show that $M(\tau)$ depends analytically on $\xi$, $\eta$, $\zeta$ for all sufficiently small $|\tau|$ and $(\xi, \eta, \zeta) \neq (0, 0, 0)$.

Denoting by $M_n(\tau)$ the $n$th partial sum of $M(\tau)$, by $\kappa_n(\tau)$ that of $\kappa(\tau)$, and by $H_n(x, y; \tau)$ the function $M_n(\tau)(1+x^2+y^2)^{1/2}$, we have for arbitrary $\epsilon > 0$ and uniformly for all points of the sphere

$$|\Delta_n| \equiv |(H_{nxx}H_{nyy} - H_{nxy}^2)(1 + x^2 + y^2)^2 - \kappa_n(\tau)| < \epsilon,$$

provided $n$ is large enough. For $\Delta_n$ is a polynomial whose term of lowest degree in $\tau$ is of degree $n$; if we replace everywhere in $\Delta_n$ the $\delta M$, $\delta^2 M$, $\cdots$ and

* Observe the invariance of $\phi(\delta H, \delta H)$.

their first and second derivatives with respect to $(x, y)$ by their upper bounds $C$, and form the sum of the absolute values of all terms thus obtained, we obtain less than the $n$th remainder in the similarly formed majorant of

$$(H_{xx}(\tau)H_{yy}(\tau) - H_{xy}^2(\tau))(1 + x^2 + y^2)^2, \text{ where } H(\tau) = M(\tau)\cdot(1+ x^2 + y^2)^{1/2};$$

and this majorant converges since it is a polynomial of convergent power series in $\tau$.

Thus $\lim_{n\to\infty}H_n(x, y; \tau)=H(x, y; \tau)$ is a solution of the analytic elliptic equation

$$(H_{xx}(\tau)H_{yy}(\tau) - H_{xy}^2(\tau))(1 + x^2 + y^2) = K^{-1}(\tau)$$

and its second derivatives satisfy a Hölder condition; hence, by a well known theorem,* $H(x, y; \tau)$ is analytic in $x$ and $y$ for every closed bounded region of the $(x, y)$-plane. This result may, of course, be formulated invariantly by stating that $M(\tau)$ depends analytically on the point of the sphere for small values of $|\tau|$.

It will be observed that the proof of statement II which is thereby completed follows precisely the routine way of solving a functional equation in the neighborhood of a value $\tau_0$ of a parameter $\tau$ entering the equation, if it can be solved for $\tau=\tau_0$. Our theorem on Monge-Ampère equations† is, however, the essential tool which makes it possible to derive our present Theorem 2 from the statement II.

Returning to the hypotheses of Theorem 2 we form, with the given positive admissible distribution of reciprocal curvature on $\sigma$, the family of positive admissible distributions

$$K^{-1}(\tau) = 1 - \tau + \tau K^{-1},$$

which for $\tau=0$ reduces to the reciprocal curvature of $\sigma$ itself and for $\tau=1$ to that of the surface to be determined. Let $\tau'$ be the greatest value of $\tau$, $0\leqq\tau'\leqq 1$, such that for every positive $\epsilon$ there exists an analytic surface $S(\tau)$ of curvature $K(\tau)$ with $\tau'-\epsilon<\tau<\tau'$. We shall show that $\tau'=1$. First of all, by II, $\tau'>0$. Since for all values of $\tau$ in $0\leqq\tau\leqq 1$ the curvature $K(\tau)$ is bounded from below by a fixed positive number, a theorem of Bonnet‡ shows that all existing $S(\tau)$ have a diameter which is bounded from above. Now take an arbitrary normal of $\sigma$ and introduce coordinates $(\xi, \eta, \zeta)$ such that its inter-

* E. Hopf, *Über den funktionalen, insbesondere den analytischen Charakter der Lösungen elliptischer Differentialgleichungen zweiter Ordnung*, Mathematische Zeitschrift, vol. 34 (1931), pp. 194–233.

† Hans Lewy, *A priori limitations for solutions of elliptic Monge-Ampère equations*, II, these Transactions, vol. 41 (1937), pp. 365–374, especially Theorem 2′ on p. 374.

‡ Cf. Blaschke, *Differentialgeometrie*, vol. 1, Berlin, 1924, p. 150.

section with $\sigma$ becomes $(0, 0, 1)$. Our introduction of the $(x, y)$-system will give this point the coordinates $(0, 0)$. Let $H(x, y; \tau)\ (1+x^2+y^2)^{-1/2}$ be the distance of the tangent plane of $S(\tau)$ from a fixed point for which we take the center of gravity of $S(\tau)$. Then we have $|H(x, y, \tau)| < 2\beta$ where $\beta$ is the upper bound of the diameter of $S(\tau)$ and $(x, y)$ is restricted to the circle $x^2+y^2<1$. Apply our theorem on Monge-Ampère equations to a sequence of solutions $H(x, y, \tau)$ of (7) for which the parameter $\tau$ tends to $\tau'$. We obtain a subsequence converging to an analytic solution $H(x, y, \tau')$ of (7) with $\tau=\tau'$. Since the origin of the $(x, y)$-system corresponds to an arbitrary normal of $\sigma$, the Heine-Borel Lemma shows the existence of a closed analytic surface $S(\tau')$ of curvature $K(\tau')$. Now $S(\tau')$ may be made the starting point for the construction of $S(\tau)$ for infinitely many values of $\tau$, greater than and close to $\tau'$, with the aid of II. Thus the assumption that $\tau'$ be less than 1 and at the same time the greatest value in every neighborhood of which there are smaller values of $\tau$ admitting a surface $S(\tau)$, has led to a contradiction. Hence $\tau'=1$ and $S(\tau')=S(1)$ exists.

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