COMPARISON OF PRODUCTS OF METHODS OF SUMMABILITY*

BY

RALPH PALMER AGNEW

1. **Introduction.** A sequence s_n of complex numbers (or complex-valued functions) is called summable to L by the method of summability

$$(A) \qquad S_n = \sum_{k=1}^{\infty} a_{nk} s_k,$$

determined by the matrix $A \equiv (a_{nk})$ of real or complex constants, if the transform S_n exists and $\lim_{n\to\infty} S_n = L$. The matrix A (and method of summability A) is called row-finite if for each n, $a_{nk}=0$ for all sufficiently great k; and is called triangular if $a_{nk}=0$ for $k>n$. The method A is regular if $s_n \to L$ implies $S_n \to L$. Necessary and sufficient conditions that A be regular are, by the Silverman-Toeplitz theorem,

$$(1.1) \qquad \sum_{k=1}^{\infty} |a_{nk}| < M, \qquad M = \text{constant},$$

$$(1.2) \qquad \text{for each } k, \qquad \lim_{n\to\infty} a_{nk} = 0,$$

$$(1.3) \qquad \lim_{n\to\infty} \sum_{k=1}^{\infty} a_{nk} = 1.$$

The set of sequences summable A is called the convergence field of A.

Let

$$(B) \qquad T_n = \sum_{k=1}^{\infty} b_{nk} s_k$$

denote a second method of summability.

In case each sequence summable B is summable A to the same value, A is said to include B and we write $A \supset B$. In case $A \supset B$ and $B \supset A$, A and B are called equivalent and we write $A \sim B$. In case the equality $L_A = L_B$ holds for each sequence summable A to L_A and summable B to L_B, the methods A and B are called mutually consistent (or consistent).

In terms of A and B it is possible to define two "products," each of which is a new method of summability. The *iteration product*, ordinarily denoted by

* Presented to the Society, February 20, 1937; received by the editors March 27, 1937.

AB, is the method which associates with a given sequence the A transform of its B transform, that is,

$$(AB) \qquad U_n = \sum_{p=1}^{\infty} a_{np}T_p = \sum_{p=1}^{\infty}\sum_{k=1}^{\infty} a_{np}b_{pk}s_k.$$

Thus s_n is summable AB to L if $\lim U_n = L$. The *composition product* is also at times denoted by AB; it is the method whose matrix is the product AB (which we denote by $A \cdot B$) of the matrices A and B. Thus we write

$$(A \cdot B) \qquad V_n = \sum_{k=1}^{\infty}\sum_{p=1}^{\infty} a_{np}b_{pk}s_k,$$

and s_n is summable $A \cdot B$ to L if $V_n \to L$.

We observe that U_n and V_n are, if they exist, respectively the "sum by rows" and the "sum by columns" of the double series

$$(1.4) \qquad \begin{aligned} & a_{n1}b_{11}s_1 + a_{n1}b_{12}s_2 + a_{n1}b_{13}s_3 + \cdots \\ & + a_{n2}b_{21}s_1 + a_{n2}b_{22}s_2 + a_{n2}b_{23}s_3 + \cdots \\ & + a_{n3}b_{31}s_1 + a_{n3}b_{32}s_2 + a_{n3}b_{33}s_3 + \cdots \\ & + \cdots \cdots \cdots \cdots \cdots \cdots \cdots . \end{aligned}$$

If A and B are regular and s_n is bounded, the series (1.4) converges absolutely and $U_n = V_n$; but without these restrictions it is not so obvious that $U_n = V_n$. There is in fact the possibility that AB and $A \cdot B$ may fail to be equivalent or even consistent.

It is the main object of this paper to compare pairs selected from the four transformations A, B, AB, and $A \cdot B$, considering in each case questions of inclusion, equivalence, and consistency. It appears that unless either or both of the matrices (a_{nk}) and (b_{nk}) are assumed to belong to restricted types, the results obtained are largely negative. These negative results are established by examples. Several examples are explicitly given, each for two reasons. In the first place each example, consisting of two regular methods of summability satisfying prescribed conditions and a sequence, can be manufactured only after considerable experimentation. In the second place the examples are largely of such obviously pathological character that they leave hope of obtaining positive theorems involving matrices of restricted types. Some such theorems are given in this paper, particularly in §11. It is doubtless true that more (and better) theorems of this kind will appear in the future.

In §12 we compare AB with $A'B'$ and $A \cdot B$ with $A' \cdot B'$ where the pair A, A' and the pair B, B' represent closely related methods of summability.

In §13 we deal briefly with multiple products and in §14 with kernel transformations.

2. **Comparison of methods A and B.** It is well known that two regular row-finite methods A and B with $a_{nk} \geqq 0$, $b_{nk} \geqq 0$ may be such that all of the relations $A \supset B$, $B \supset A$, $A \sim B$ are false and in fact such that A and B are inconsistent.*

3. **Comparison of methods A and AB.** It follows from §2 that we can choose regular row-finite methods A and B, with $a_{nk} \geqq 0$, $b_{nk} \geqq 0$, and a sequence s_n summable A to L_A and summable B to $L_B \neq L_A$. It is easy to see that regularity of A implies that s_n is summable AB to L_B. Thus A and AB may be inconsistent, and there is no hope of showing that $A \supset AB$, $AB \supset A$ or $A \sim AB$.

4. **Comparison of B and AB.** Elementary examples show that $B \supset AB$ may be false, and hence that B and AB need not be equivalent, even when A and B are assumed to be regular, row-finite and $a_{nk} \geqq 0$, $b_{nk} \geqq 0$.

However if A is regular and s_n is summable B to L, then $T_n \to L$ and regularity of A imply $U_n \to L$ so that s_n is summable AB to L. It follows that if A is regular, then $AB \supset B$. This implies that B and AB must be consistent.

5. **Comparison of A and $A \cdot B$.** When A and B are determined as in §3, the series (1.4) from which U_n and V_n are computed reduces to a finite sum, and obviously $U_n = V_n$; hence in this case $A \cdot B \sim AB$. It follows from §3 that A and $A \cdot B$ may be inconsistent, and there is no hope of showing that $A \supset A \cdot B$, $A \cdot B \supset A$, or $A \sim A \cdot B$.

6. **Comparison of B and $A \cdot B$.** Elementary examples of row-finite transformations show that $B \supset A \cdot B$ may be false and hence that B and $A \cdot B$ need not be equivalent.

One might expect to be able to show that if A and B are regular, then $A \cdot B \supset B$. But this is impossible. The author† has given an example of transformations A and B (having some significant properties in addition to regularity) and a sequence s_n which is summable B but non-summable $A \cdot B$.

In this paper we go further and prove in §9 the following theorem:

THEOREM 6.1. *There exist regular transformations A and B with $a_{nk} \geqq 0$, $b_{nk} \geqq 0$ for which B and $A \cdot B$ are inconsistent.*

7. **Comparison of AB and $A \cdot B$.** It is not true that regularity of A and B

---

* Questions involving inconsistency of transformations are discussed in detail in Agnew, *On ranges of inconsistency of regular transformations and allied topics*, Annals of Mathematics, vol. 32 (1931), pp. 715–722.

† Agnew, *Products of methods of summability*, Bulletin of the American Mathematical Society, vol. 42 (1936), pp. 547–549.

implies either $AB \supset A \cdot B$ or $A \cdot B \supset AB$. In fact we prove in §9 the following theorem:

THEOREM 7.1. *There exist regular transformations A and B with $a_{nk} \geqq 0$, $b_{nk} \geqq 0$, for which AB and $A \cdot B$ are inconsistent.*

This theorem evidences the necessity of noticing a distinction between the iteration product AB and the composition product $A \cdot B$.

The transformation A of Theorem 7.1 cannot be row-finite. For if A is row-finite, then all of the terms lying below some row of the double series (1.4) from which U_n and V_n are computed vanish, and existence of U_n implies existence of V_n and the equality $V_n = U_n$. Thus we have the theorem:

THEOREM 7.2. *If A is row-finite, then $A \cdot B \supset AB$.*

This implies that if A is row-finite, then AB and $A \cdot B$ must be consistent. It is however impossible to go further and prove that AB and $A \cdot B$ must be equivalent. We prove the following theorem:

THEOREM 7.3. *There exist regular transformations A and B with A row-finite, $a_{nk} \geqq 0$, $b_{nk} \geqq 0$, and a sequence s_k such that s_k is summable $A \cdot B$ but non-summable AB.*

Let $p_1, p_2, p_3, \cdots$ denote in order the primes $2, 3, 5, \cdots$. For each $n = 1, 2, 3, \cdots$, let $a_{n,p_n} = a_{n,p_n^2} = 1/2$ and let $a_{nk} = 0$ otherwise. If n is neither a prime nor a square of a prime, let $b_{nn} = 1$, and $b_{nk} = 0$ otherwise. For each $n = 1, 2, \cdots$ let $b_{p_n,k} = 0$ when $k \neq p_n, p_n^3, p_n^5, \cdots$, and let $b_{p_n,k} = 2^{-\alpha}$ when k is of the form $p_n^{2\alpha-1}$. Let $b_{p_n^2,k} = 0$ when $k \neq p_n^2, p_n^4, p_n^6, \cdots$, and let $b_{p_n^2,k} = 2^{-\alpha}$ when k is of the form $p_n^{2\alpha}$. These matrices a_{nk} and b_{nk} define methods A and B of summability satisfying the hypotheses of the theorem. Observe that $a_{nk} = b_{nk} = 0$ when $n > k$. Let the sequence s_k be defined by the formulas: $s_k = 2^{\alpha+1}/\alpha$ when k is of the form $p_n^{2\alpha-1}$; $s_k = -2^{\alpha+1}/\alpha$ when k is of the form $p_n^{2\alpha}$; and $s_k = 0$ otherwise.

It can be shown that for this example the double series (1.4) from which U_n and V_n are computed becomes (after omission of rows and columns of zeros)

$$\begin{aligned} &1 + 0 + \tfrac{1}{2} + 0 + \tfrac{1}{3} + 0 + \tfrac{1}{4} + \cdots \\ &+ 0 - 1 + 0 - \tfrac{1}{2} + 0 - \tfrac{1}{3} + 0 - \cdots . \end{aligned} \tag{7.31}$$

It is apparent that, for each n, the sum by columns of this series is 0, that is, $V_n = 0$; and that the sum by rows does not exist, that is, U_n does not exist. Thus the sequence of s_n of the example is summable $A \cdot B$ to 0 and is non-summable AB. This proves Theorem 7.3.

The transformation B of Theorem 7.3 cannot be row-finite. For if both

A and B are row-finite, then $U_n = V_n$ for every n and equivalence of AB and $A \cdot B$ follows.

In spite of the fact that the transformations AB and $A \cdot B$ of Theorem 7.1 need not be consistent, there is a large class of sequences (including all bounded sequences and all unilaterally bounded real sequences) over which they must be equivalent.

We shall say that a sequence s_n lies in an angle less than π in the complex plane if there exist a point z_0, an angle θ_0, and a positive angle $\phi < \pi/2$ such that for each n

$$s_k = z_0 + \rho_k e^{i(\theta_0 + \theta_k)} \tag{7.32}$$

where $\rho_k \geqq 0$ and $|\theta_k| \leqq \phi$.

THEOREM 7.4. *If A and B are regular transformations with*

$$a_{nk} \geqq 0, \qquad b_{nk} \geqq 0, \qquad n, k = 1, 2, \cdots, \tag{7.41}$$

then each sequence s_n which lies in an angle less than π in the complex plane and which is summable to L by one of the methods AB and $A \cdot B$ is also summable to L by the other one.

The gist of this theorem is that for regular transformations A, B satisfying (7.41), the two transformations AB and $A \cdot B$ are equivalent in so far as application to sequences lying in an angle less than π is concerned.

To prove the theorem, suppose first that s_k is a sequence given by (7.32) for which V_n exists. Then

$$V_n = \sum_{k=1}^{\infty} \sum_{p=1}^{\infty} a_{np} b_{pk} [z_0 + \rho_k e^{i\theta_0} e^{i\theta_k}], \tag{7.42}$$

and, where ξ_k and η_k are the real and imaginary parts of $\rho_k e^{i\theta_k}$,

$$V_n = z_0 \sum_{k=1}^{\infty} \sum_{p=1}^{\infty} a_{np} b_{pk} + e^{i\theta_0} \sum_{k=1}^{\infty} \sum_{p=1}^{\infty} a_{np} b_{pk} [\xi_k + i\eta_k]. \tag{7.43}$$

Since $a_{nk} \geqq 0$, $b_{nk} \geqq 0$, $\xi_k \geqq 0$, and η_k is real, this implies convergence of

$$\sum_{k=1}^{\infty} \sum_{p=1}^{\infty} a_{np} b_{pk} |\xi_k|. \tag{7.44}$$

But $|\eta_k| \leqq \xi_k \tan \phi$ so that (7.44) converges when $|\xi_k|$ is replaced by $|\eta_k|$. It now follows easily that both series in (7.43) converge absolutely and hence that the series in (7.42) converges absolutely. Therefore

$$U_n = \sum_{p=1}^{\infty} \sum_{k=1}^{\infty} a_{np} b_{pk} [z_0 + \rho_k e^{i\theta_0} e^{i\theta_k}] \tag{7.45}$$

exists and $U_n = V_n$; hence $V_n \to L$ implies also $U_n \to L$. We can show similarly that $U_n \to L$ implies also $V_n \to L$, and Theorem 7.4 is proved.

8. **A double series.** In the next section we use the double series

$$
\begin{array}{l}
1 - \tfrac{1}{2} + 0 - \tfrac{1}{4} + 0 - \tfrac{1}{6} + 0 - 0 + 0 - 0 \ + \cdots \\
+ 0 - 0 + \tfrac{1}{3} - 0 + 0 - 0 + 0 - \tfrac{1}{8} + 0 - \tfrac{1}{10} + \cdots \\
+ 0 - 0 + 0 - 0 + \tfrac{1}{5} - 0 + 0 - 0 + 0 - 0 \ + \cdots \\
+ 0 - 0 + 0 - 0 + 0 - 0 + \tfrac{1}{7} - 0 + 0 - 0 \ + \cdots \\
+ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{array}
\tag{8.1}
$$

whose terms u_{nk} may be defined as follows: for each odd k

$$
\begin{aligned}
u_{nk} &= 1/k \text{ if } n = (k+1)/2, \\
&= 0 \text{ otherwise;}
\end{aligned}
\tag{8.2}
$$

and for each even k

$$
\begin{aligned}
u_{nk} &= -1/k \text{ if } n = n_k, \\
&= 0 \text{ otherwise,}
\end{aligned}
\tag{8.3}
$$

where n_k is the smallest n for which

$$
u_{n1} + u_{n2} + u_{n3} + \cdots + u_{n,k-1} - 1/k \geqq 0. \tag{8.4}
$$

The harmonic series $\sum 1/k$ being divergent, it is easy to see that for each n, the infinite series $u_{n1} + u_{n2} + \cdots$ converges to 0. Hence the double series (8.1) converges by rows to 0 and by columns to $\log 2 = 1 - 1/2 + 1/3 - 1/4 + \cdots$. Moreover it can be shown by arithmetic methods that for each n, $u_{nk} = 0$ for all sufficiently great k; that is, each row of (8.1) contains only a finite number of non-vanishing terms.

9. **Proof of Theorems** 6.1 **and** 7.1. We can now prove the following theorem of which Theorems 6.1 and 7.1 are obvious corollaries:

THEOREM 9.1. *There exists a pair of regular transformations A and B with B row-finite,*

$$
a_{nk} \geqq 0, \qquad b_{nk} \geqq 0, \qquad n, k = 1, 2, \cdots, \tag{9.11}
$$

$$
\sum_{k=1}^{\infty} a_{nk} = 1, \qquad \sum_{k=1}^{\infty} b_{nk} = 1, \qquad n = 1, 2, \cdots, \tag{9.12}
$$

and a sequence s_k such that

$$
T_n = \sum_{k=1}^{\infty} b_{nk} s_k = 0, \qquad n = 1, 2, \cdots, \tag{9.13}
$$

$$U_n = \sum_{p=1}^{\infty} a_{np}T_p = \sum_{p=1}^{\infty}\sum_{k=1}^{\infty} a_{np}b_{pk}s_k = 0, \qquad n = 1, 2, \cdots, \tag{9.14}$$

and

$$V_n = \sum_{k=1}^{\infty}\sum_{p=1}^{\infty} a_{np}b_{pk}s_k = 1, \qquad n = 1, 2, \cdots. \tag{9.15}$$

Let the positive integers $1, 2, 3, \cdots$ be displayed as a double sequence h_{nk} so that $h_{11}=1$, $h_{21}=2$, $h_{12}=3$, $h_{31}=4$, $h_{22}=5$, $h_{13}=6$, $h_{14}=7$, etc. For each $n=1, 2, 3, \cdots$, let a_{nk} be defined for $k=1, 2, 3, \cdots$ by the formula

$$\begin{aligned} a_{nk} &= 0, \qquad k \neq h_{n1}, h_{n2}, h_{n3}, \cdots, \\ &= 2^{-r}, \qquad k = h_{nr}, r = 1, 2, \cdots. \end{aligned} \tag{9.16}$$

This matrix a_{nk} determines a regular method A of summability with $a_{nk} \geqq 0$ and $\sum_{k=1}^{\infty} a_{nk} = 1$ for each n.

The double series (1.4) from which U_n and V_n are computed takes (after removal of rows of zeros) the form

$$\begin{aligned} &\tfrac{1}{2}b_{h_{n1},1}s_1 + \tfrac{1}{2}b_{h_{n1},2}s_2 + \tfrac{1}{2}b_{h_{n1},3}s_3 + \cdots \\ &+ \tfrac{1}{4}b_{h_{n2},1}s_1 + \tfrac{1}{4}b_{h_{n2},2}s_2 + \tfrac{1}{4}b_{h_{n2},3}s_3 + \cdots \\ &+ \tfrac{1}{8}b_{h_{n3},1}s_1 + \tfrac{1}{8}b_{h_{n3},2}s_2 + \tfrac{1}{8}b_{h_{n3},3}s_3 + \cdots \\ &+ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots. \end{aligned} \tag{9.17}$$

Let, for each n and r,

$$b_{h_{n,r},k} = 0, \qquad k \neq h_{n,r}, h_{n,r+1}, \cdots. \tag{9.18}$$

Then the double series (9.17) takes (after removal of columns of zeros) the form

$$\begin{aligned} &\tfrac{1}{2}b_{h_{n1},h_{n1}}s_{h_{n1}} + \tfrac{1}{2}b_{h_{n1},h_{n2}}s_{h_{n2}} + \tfrac{1}{2}b_{h_{n1},h_{n3}}s_{h_{n3}} + \cdots \\ &+ \quad 0 \quad + \tfrac{1}{4}b_{h_{n2},h_{n2}}s_{h_{n2}} + \tfrac{1}{4}b_{h_{n2},h_{n3}}s_{h_{n3}} + \cdots \\ &+ \quad 0 \quad + \quad 0 \quad + \tfrac{1}{8}b_{h_{n3},h_{n3}}s_{h_{n3}} + \cdots \\ &+ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots. \end{aligned} \tag{9.19}$$

We observe that if $n' \neq n''$, then the "variables" $b_{\alpha\beta}$ and s_γ appearing in (9.19), when $n=n'$, are distinct from those appearing when $n=n''$. For each $n=1, 2, 3, \cdots$ let the elements $b_{\alpha\beta}$ and s_γ appearing in (9.19) be determined so that the terms of the two series (9.19) and (8.1) in corresponding positions will be equal, and the non-vanishing elements of the sequence

$$b_{h_{n,r}h_{n,r}}, b_{h_{n,r}h_{n,r+1}}, b_{h_{n,r}h_{n,r+2}}, \cdots$$

will be equal in order to

$$1/2d,\ 1/2^2d,\ \cdots,\ 1/2^\delta d$$

where $d=1/2+1/2^2+\cdots+1/2^\delta$.

This gives a complete and unique determination of the elements of the matrix $B\equiv(b_{nk})$. It is clear that B satisfies the hypotheses of the theorem, regularity being implied by the conditions

$$b_{nk}\geqq 0,\qquad \sum_{k=1}^{\infty} b_{nk}=1,$$

and the fact that $b_{nk}=0$ when $n>k$.

It follows from identity of (9.19) and (8.1) and the fact that each row of (8.1) converges to 0, that $T_p=0$ for each $p=1, 2, \cdots$; and since (8.1) and hence (9.19) converge by rows to 0 and by columns to log 2 we have $U_n=0$ and $V_n=\log 2$. If finally we divide each s_n determined above by log 2, then T_p, U_n, and V_n will be divided by log 2 and we obtain (9.13), (9.14), and (9.15).

10. **Remarks on Theorem** 9.1. The author has been unable to find an example less recondite than the one just given to prove Theorem 9.1. In case the requirements $a_{nk}\geqq 0$, $b_{nk}\geqq 0$ are removed, we can give simpler examples. For definiteness, and convenience of reference we state the following theorem:

THEOREM 10.1. *If r is a complex number with $0<|r|<1$, then the methods*

$$(A)\qquad S_n=s_n+r^ns_{n+1}+r^{n+1}s_{n+2}+\cdots,$$

$$(B)\qquad T_n=[1/(1-r)]s_{n-1}+[-r/(1-r)]s_n$$

are regular while AB and $A\cdot B$ are inconsistent. The sequence $s_k=(1-r)r^{-k}$ is summable AB to 0 *and $A\cdot B$ to* 1.

Verification is straightforward and left to the reader. We note that if $0<r<1$, the elements a_{nk} are all $\geqq 0$ but some elements b_{nk} are <0; while if $-1<r<0$, all elements b_{nk} are $\geqq 0$ but some elements a_{nk} are <0. If r is not real, the conditions $a_{nk}\geqq 0$, $b_{nk}\geqq 0$ both fail.

The method A of Theorem 10.1 is, for each admissible r, equivalent to convergence. For on one hand A is regular. On the other hand if s_k is summable A to L so that

$$(10.11)\qquad S_n=s_n+r^ns_{n+1}+r^{n+1}s_{n+2}+\cdots$$

exists and $S_n\to L$, then convergence of the series in (10.11) implies that

$$(10.12)\qquad \lim_{\alpha\to\infty}(r^\alpha s_{\alpha+1}+r^{\alpha+1}s_{\alpha+2}+\cdots)=0,$$

and hence that

$$(10.13)\qquad \lim s_n=\lim S_n=L.$$

The method B is not only regular but also has several other features at times desirable in methods of summability. The permissibility of removal or adjunction of elements at the beginning of a sequence is such a feature.

These remarks make it appear likely that significant theorems giving conditions sufficient for consistency of AB and $A\cdot B$ (or for $AB\supset A\cdot B$, or for $A\cdot B\supset AB$, or for $AB\sim A\cdot B$) will involve classes of methods defined by matrices of more or less restricted types rather than involve classes of methods having various ones of the numerous "desirable" properties of methods of summability.

The following theorem indicates the possibility of obtaining constructive theorems involving AB and $A\cdot B$, and is of interest in connection with Theorem 10.1:

THEOREM 10.2. *If A and B are regular transformations with $a_{nk}\geqq 0$, $b_{nk}\geqq 0$ and if B is of the form*

$$T_n = \sum_{k=n-\alpha}^{n+\beta} b_{nk}s_k,$$

where α and β are non-negative integers, then $AB\supset A\cdot B$.

In interpreting B, we agree that $s_k=0$ when $k<1$, and that $b_{pk}=0$ when $k<p-\alpha$ and when $k>p+\beta$. Assuming s_k to be a sequence for which

$$V_n = \sum_{k=1}^{\infty}\sum_{p=1}^{\infty} a_{np}b_{pk}s_k \equiv \sum_{k=1}^{\infty}\sum_{p=k-\beta}^{k+\alpha} a_{np}b_{pk}s_k \tag{10.21}$$

exists, we have for fixed n

$$\begin{aligned} V_n &= \lim_{Q\to\infty}\sum_{k=1}^{Q}\sum_{p=1}^{\infty} a_{np}b_{pk}s_k = \lim_{Q\to\infty}\sum_{p=1}^{\infty} a_{np}\sum_{k=1}^{Q} b_{pk}s_k \\ &= \lim_{Q\to\infty}\left\{\sum_{p=1}^{Q-\beta} a_{np}\sum_{k=1}^{Q} b_{pk}s_k + \sum_{p=Q-\beta+1}^{Q+\alpha} a_{np}\sum_{k=1}^{Q} b_{pk}s_k\right\}, \end{aligned} \tag{10.22}$$

and hence

$$V_n = \lim_{Q\to\infty}\left\{\sum_{p=1}^{Q-\beta} a_{np}\sum_{k=1}^{\infty} b_{pk}s_k + \sum_{p=Q-\beta+1}^{Q+\alpha}\ \sum_{k=Q-\alpha-\beta+1}^{Q} a_{np}b_{pk}s_k\right\}. \tag{10.23}$$

The operations under the limit sign are justified by vanishing of elements b_{nk}. Now convergence of the first series in (10.21) implies that

$$\Delta_k \equiv \sum_{p=1}^{\infty} a_{np}b_{pk}\,|\,s_k\,| \to 0$$

as $k \to \infty$. But since $a_{nk} \geqq 0$ and $b_{nk} \geqq 0$, $0 \leqq a_{np} b_{pk} |s_k| \leqq \Delta_k$ for each fixed p.

Hence

$$\left| \sum_{p=Q-\beta+1}^{Q+\alpha} \sum_{k=Q-\alpha-\beta+1}^{Q} a_{np} b_{pk} s_k \right| \leqq \sum_{p=Q-\beta+1}^{Q+\alpha} \sum_{k=Q-\alpha-\beta+1}^{Q} \Delta_k = (\alpha + \beta) \sum_{k=Q-\alpha-\beta+1}^{Q} \Delta_k \to 0$$

as $Q \to \infty$. This fact and (10.23) imply that

$$U_n = \sum_{p=1}^{\infty} a_{np} \sum_{k=1}^{\infty} b_{pk} s_k \equiv \sum_{p=1}^{\infty} a_{np} \sum_{k=p-\alpha}^{p+\beta} b_{pk} s_k$$

exists and $U_n = V_n$. This argument shows that $AB \supset A \cdot B$, and Theorem 10.2 is proved.

Examples show it is impossible to modify the argument to prove $A \cdot B \supset AB$. For one such example, it suffices to put, for each $n = 1, 2, \cdots$,

$$\begin{aligned} a_{nk} &= 0, & k &\neq n^2, n^4, n^8, \cdots, \\ &= 2^{-\sigma}, & k &= n^{2^\sigma}, & \sigma &= 1, 2, \cdots, \end{aligned} \tag{10.4}$$

and for each $n = 2, 3, \cdots$,

$$\begin{aligned} b_{nk} &= 0, & k &\neq n-1, n, \\ &= 1/2, & k &= n-1, n, \end{aligned} \tag{10.5}$$

while $b_{11} = 1$ and $b_{1k} = 0$ for $k > 1$. The sequence defined by $s_n = (-1)^n \log n$ is summable AB to 0 and is non-summable $A \cdot B$. We note in passing that (10.5) defines a regular Nörlund method of summability which is included by the arithmetic mean method.

11. **The arithmetic mean and generalizations.** Let $p_1, p_2, p_3, \cdots$ be a sequence of constants with

$$P_n = p_1 + p_2 + \cdots + p_n \neq 0, \qquad n = 1, 2, \cdots. \tag{11.01}$$

Let P denote the method of summability associated with the transformation

$$T_n = (p_1 s_1 + p_2 s_2 + \cdots + p_n s_n)/P_n. \tag{P}$$

The transformations P differ from the more familiar Nörlund methods in order of distribution of the "weights" p_k; but share with Nörlund methods the property of reducing to the important arithmetic mean method C_1 (or M) when $p_k = 1$ for each k. The theorems of this section therefore give facts involving C_1.

THEOREM 11.1. *If A and P (regular or not) are methods of summability with $a_{nk} \geqq 0$, $P_n > 0$, then $AP \supset A \cdot P$.*

The AP and $A\cdot P$ transforms U_n and V_n of a sequence s_k are (if they exist) determined as the sum by rows and the sum by columns of the double series obtained by removing parentheses from the series

$$
\begin{aligned}
&P_1^{-1}a_{n1}(p_1s_1 + 0 + 0 + 0 + \cdots)\\
&+ P_2^{-1}a_{n2}(p_1s_1 + p_2s_2 + 0 + 0 + \cdots)\\
&+ P_3^{-1}a_{n3}(p_1s_1 + p_2s_2 + p_3s_3 + 0 + \cdots)\\
&+ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots.
\end{aligned}
\tag{11.10}
$$

We show that existence of V_n implies existence of U_n and the equality $U_n = V_n$. This follows, on introducing obvious notation and interchanging rows and columns, from the following lemma:

LEMMA 11.2. *If $\theta_n \geqq 0$ for $n = 1, 2, \cdots$, and the series obtained by removing parentheses from the series*

$$
\begin{aligned}
&\sigma_1(\theta_1 + \theta_2 + \theta_3 + \cdots)\\
&+ \sigma_2(\qquad \theta_2 + \theta_3 + \cdots)\\
&+ \sigma_3(\qquad\qquad \theta_3 + \cdots)\\
&+ \cdots\cdots\cdots\cdots
\end{aligned}
\tag{11.20}
$$

converges by rows to Λ, then it also converges by columns to Λ.

To prove the lemma, let

$$R_n = \theta_n + \theta_{n+1} + \theta_{n+2} + \cdots. \tag{11.21}$$

If $R_n = 0$ for some n, then $\theta_n = 0$ for all sufficiently great n, and the conclusion of the lemma obviously holds. Hence we may assume $R_n \neq 0$ for all n. Let

$$\omega_n = \sigma_n(\theta_n + \theta_{n+1} + \theta_{n+2} + \cdots) = \sigma_n R_n. \tag{11.22}$$

Then

$$\sum \omega_n = \Lambda, \qquad \sigma_n = \omega_n/R_n. \tag{11.23}$$

The sum of n columns of the series (11.20) is

$$K_n = \sigma_1(R_1 - R_{n+1}) + \sigma_2(R_2 - R_{n+1}) + \cdots + \sigma_n(R_n - R_{n+1}). \tag{11.24}$$

This can be written

$$K_n = \sum_{k=1}^{n} \beta_{nk}\omega_k, \tag{11.25}$$

where

$$\beta_{nk} = 1 - R_{n+1}/R_k. \tag{11.26}$$

The fact that $R_n \to 0$ monotonely as $n \to \infty$ enables us to show that (11.25) defines a regular method of evaluating series,* that is, $\sum \omega_n = \Lambda$ implies $K_n \to \Lambda$. This completes the proof of Lemma 11.2 and hence the proof of Theorem 11.1.

It is impossible to strengthen Theorem 11.1 by proving that $A \cdot P \supset AP$, even when P is the arithmetic mean transformation C_1. In fact, we prove the following theorem:

THEOREM 11.3. *Corresponding to each transformation of the form*

$$(P) \qquad T_n = (p_1 s_1 + p_2 s_2 + \cdots + p_n s_n)/P_n,$$

where $p_n \neq 0$, $P_n = p_1 + \cdots + p_n > 0$ *for each* $n = 1, 2, \cdots$, *there is a regular transformation*

$$(A) \qquad S_n = \sum_{k=1}^{\infty} a_{nk} s_k,$$

with $a_{nk} \geqq 0$, *such that* $A \cdot P$ *does not include* AP.

Let $\rho_1, \rho_2, \cdots$ denote in order the odd primes. For each $n = 1, 2, 3, \cdots$ let

$$(11.30) \qquad \begin{aligned} a_{nk} &= 0, & k &\neq 2\rho_n, 2\rho_n^2, 2\rho_n^3, \cdots \\ &= 2^{-\sigma}, & k &= 2\rho_n^\sigma, & \sigma &= 1, 2, \cdots. \end{aligned}$$

The double series, of which U_n is the sum by rows and V_n the sum by columns, becomes, after removing rows of zeros and factoring the remaining rows

$$(11.31) \qquad \begin{aligned} &(2P_\tau)^{-1}(p_1 s_1 + \cdots + p_{2\tau-1} s_{2\tau-1} + p_{2\tau} s_{2\tau} + 0 + \cdots \\ &+ (4P_{\tau^2})^{-1}(p_1 s_1 + \cdots + p_{2\tau-1} s_{2\tau-1} + p_{2\tau} s_{2\tau} + \cdots + p_{2\tau^2} s_{2\tau^2} + \cdots \\ &+ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots, \end{aligned}$$

where $\tau = \rho_n$. For each $n = 1, 2, \cdots$, let

$$(11.32) \qquad s_{2\tau^\sigma} = 1/2^\sigma P_{\tau^\sigma} p_{2\tau^\sigma}, \qquad \sigma = 1, 2, \cdots,$$

and let $s_{2k} = 0$ when k is not of the form p_n^σ. For each k, let

$$(11.33) \qquad s_{2k-1} = -p_{2k} s_{2k}/p_{2k-1}.$$

We now have a complete and unique determination of a regular matrix a_{nk} and a sequence s_k. For each n, the series (11.31) converges by rows to 0 and fails to converge by columns. Therefore the sequence s_k is summable AP to 0 and is non-summable $A \cdot P$.

* Carmichael, Bulletin of the American Mathematical Society, vol. 25 (1918), pp. 97–131.

12. **Comparison of AB with $A'B'$ and $A\cdot B$ with $A'\cdot B'$.** It is trivially easy to see that if A, A', B, B' are regular transformations and there is an index n_0 such that $a_{nk}=a'_{nk}$ and $b_{nk}=b'_{nk}$ when $n\geqq n_0$, then $A\sim A'$ and $B\sim B'$. A comparison of the two methods AB and $A'B'$, or the two methods $A\cdot B$ and $A'\cdot B'$ is not so simple. We can however prove the following theorem:

THEOREM 12.1. *Let*

$$(A),\ (A') \qquad S_n = \sum_{k=1}^{\infty} a_{nk}s_k, \qquad S'_n = \sum_{k=1}^{\infty} a'_{nk}s_k,$$

$$(B),\ (B') \qquad T_n = \sum_{k=1}^{\infty} b_{nk}s_k, \qquad T'_n = \sum_{k=1}^{\infty} b'_{nk}s_k,$$

be four regular methods of summability, and let an index N exist such that $a_{nk}=a'_{nk}$, and $b_{nk}=b'_{nk}$, for $n\geqq N$.

Then the two methods of summability

$$(AB) \qquad U_n = \sum_{p=1}^{\infty} a_{np} \sum_{k=1}^{\infty} b_{pk}s_k,$$

$$(A'B') \qquad U'_n = \sum_{p=1}^{\infty} a'_{np} \sum_{k=1}^{\infty} b'_{pk}s_k$$

are consistent, and the two methods

$$(A\cdot B) \qquad V_n = \sum_{k=1}^{\infty} \sum_{p=1}^{\infty} a_{np}b_{pk}s_k$$

$$(A'\cdot B') \qquad V'_n = \sum_{k=1}^{\infty} \sum_{p=1}^{\infty} a'_{np}b'_{pk}s_k$$

are consistent.

We prove first that (AB) and $(A'B')$ are consistent. Suppose s_k is a sequence summable AB to L and summable $A'B'$ to L' so that $U_n\to L$, $U'_n\to L'$. Then

$$T_p = \sum_{k=1}^{\infty} b_{pk}s_k, \qquad T'_p = \sum_{k=1}^{\infty} b'_{pk}s_k$$

must exist for each $p=1, 2, \cdots$. Letting o_α denote generically quantities depending on α which converge to 0 as $\alpha\to\infty$, we find

$$\begin{aligned} U_n &= \sum_{p=1}^{N-1} a_{np}T_p + \sum_{p=N}^{\infty} a_{np}\left[\sum_{k=1}^{N-1} b_{pk}s_k + \sum_{k=N}^{\infty} b_{pk}s_k\right] \\ &= o_n + \sum_{p=N}^{\infty} a_{np}\left[o_p + \sum_{k=N}^{\infty} b_{pk}s_k\right] \\ &= o_n + \sum_{p=N}^{\infty} a_{np} \sum_{k=N}^{\infty} b_{pk}s_k. \end{aligned}$$

Likewise

$$U_n' = o_n + \sum_{p=N}^{\infty} a_{np}' \sum_{k=N}^{\infty} b_{pk}' s_k.$$

But over the ranges of summation in this series, we have, when $n \geqq N$, $a_{np}' = a_{np}$ and $b_{pk}' = b_{pk}$. Hence $U_n = o_n + U_n'$, and it follows that $L = L'$. Thus AB and $A'B'$ are consistent.

To prove that $A \cdot B$ and $A' \cdot B'$ are consistent, let s_k be a sequence summable $A \cdot B$ to Λ and summable $A'B'$ to Λ' so that $V_n \to \Lambda$, $V_n' \to \Lambda'$. Since $a_{np}' = a_{np}$ when $n \geqq N$, we can write for $n > N$

$$V_n = \sum_{k=1}^{\infty} \left(\sum_{p=1}^{N} a_{np} b_{pk} s_k + \sum_{p=N+1}^{\infty} a_{np} b_{pk} s_k \right),$$

$$V_n' = \sum_{k=1}^{\infty} \left(\sum_{p=1}^{N} a_{np} b_{pk}' s_k + \sum_{p=N+1}^{\infty} a_{np} b_{pk}' s_k \right).$$

Since $b_{pk}' = b_{pk}$ when $p > N$, it follows that when $n \geqq N$

$$V_n - V_n' = \sum_{k=1}^{\infty} \sum_{p=1}^{N} a_{np}(b_{pk} - b_{pk}') s_k.$$

An application of the following lemma with $\delta_{pk} = (b_{pk} - b_{pk}') s_k$ shows that $V_n - V_n' \to 0$ as $n \to \infty$ and hence that $\Lambda = \Lambda'$ and that $A \cdot B$ and $A' \cdot B'$ are consistent.

LEMMA 12.2. *If*

$$\lim_{n \to \infty} a_{np} = 0, \qquad p = 1, 2, \cdots, N, \tag{12.21}$$

and

$$\Delta_n = \sum_{k=1}^{\infty} \sum_{p=1}^{N} a_{np} \delta_{pk} \tag{12.22}$$

exists for each sufficiently great n (say $n \geqq n_0$), then

$$\lim_{n \to \infty} \Delta_n = 0. \tag{12.23}$$

We prove this lemma by induction, considering first the case where $N = 1$ and accordingly

$$\Delta_n = \sum_{k=1}^{\infty} a_{n1} \delta_{1k}, \qquad n \geqq n_0.$$

If $a_{n1}=0$ for all sufficiently great n, then obviously $\Delta_n \to 0$. Otherwise we can choose $n \geqq n_0$ such that $a_{n1} \neq 0$ and conclude existence of

$$B_1 = \sum_{k=1}^{\infty} \delta_{1k},$$

so that

$$\Delta_n = a_{n1} B_1.$$

Since $a_{n1} \to 0$, it follows that $\Delta_n \to 0$. Thus the lemma holds when $N=1$.

In case $N=2$, we have

$$\Delta_n = \sum_{k=1}^{\infty} (a_{n1}\delta_{1k} + a_{n2}\delta_{2k}), \qquad n \geqq n_0. \tag{12.24}$$

Suppose two multipliers μ_1 and μ_2, not both zero, exist such that

$$\mu_1 a_{n1} + \mu_2 a_{n2} = 0, \qquad n > n_0.$$

Then supposing $\mu_2 \neq 0$ (the case $\mu_1 \neq 0$ is analogous) and putting $\rho = -\mu_1/\mu_2$ we have

$$a_{n2} = \rho a_{n1}, \qquad n > n_0,$$

and substitution in (12.24) gives

$$\Delta_n = \sum_{k=1}^{\infty} a_{n1}(\delta_{1k} + \rho\delta_{2k}).$$

Now $\Delta_n \to 0$ follows from truth of the lemma for $N=1$.

If multipliers μ_1 and μ_2 do not exist as above, then we can choose n_1 and n_2 such that $n_0 < n_1 < n_2$ and

$$\begin{vmatrix} a_{n_1 1} & a_{n_1 2} \\ a_{n_2 1} & a_{n_2 2} \end{vmatrix} \neq 0. \tag{12.25}$$

From

$$\Delta_{n_1} = \sum_{k=1}^{\infty} (a_{n_1 1}\delta_{1k} + a_{n_1 2}\delta_{2k}),$$

$$\Delta_{n_2} = \sum_{k=1}^{\infty} (a_{n_2 1}\delta_{1k} + a_{n_2 2}\delta_{2k}),$$

we conclude existence of

$$a_{n_2 2}\Delta_{n_1} - a_{n_1 2}\Delta_{n_2} = \sum_{k=1}^{\infty} (a_{n_1 1}a_{n_2 2} - a_{n_1 2}a_{n_2 1})\delta_{1k}.$$

This relation and (12.25) enable us to conclude convergence of the first, and similarly we conclude convergence of the second of the series

$$B_1 = \sum_{k=1}^{\infty} \delta_{1k}, \qquad B_2 = \sum_{k=1}^{\infty} \delta_{2k}.$$

Therefore we can write (12.24) in the form

$$\Delta_n = a_{n1}B_1 + a_{n2}B_2,$$

and $\Delta_n \to 0$ follows from $a_{n1} \to 0$, $a_{n2} \to 0$. Thus the lemma holds when $N=2$.

Assuming that the lemma holds when $N<R$, we can show that

$$\Delta_n = \sum_{k=1}^{\infty} \sum_{p=1}^{R} a_{np}\delta_{pk} \to 0$$

in the case where multipliers $\mu_1, \mu_2, \cdots, \mu_R$ not all zero exist such that

$$\mu_1 a_{n1} + \mu_2 a_{n2} + \cdots + \mu_R a_{nR} = 0, \qquad n > n_0;$$

and also in the alternative case where $n_1 < n_2 < \cdots < n_R$ all exist greater than n_0 and such that the determinant

$$\det \left| a_{n_\alpha \beta} \right|, \qquad \alpha, \beta = 1, \cdots, R,$$

does not vanish. The methods are analogous to those of our proof for the case $N=2$. This completes the proof of Lemma 12.2 and hence of Theorem 12.1.

The reader may naturally be disgruntled by the conclusions of Theorem 12.1; the theorem would be more satisfying if we could replace "consistent" by "equivalent," but we cannot do this. It is clear that under the hypotheses of the theorem AB and $A'B$ are equivalent; and $A \cdot B$ and $A' \cdot B$ are equivalent. But AB and AB' (or $A \cdot B$ and $A \cdot B'$) need not be equivalent.

For an example, put $a_{n1} = 1/n$, $n = 1, 2, \cdots$; $a_{nn} = 1 - 1/n$, $n = 2, 3, \cdots$; and $a_{nk} = 0$ when $k \neq 1, n$. Let $b_{1k} = 1/2^p$ when k has the form $4p-3$, and $b_{1k} = 0$ otherwise. Let $b'_{1k} = 1/2^p$ when k has the form $4p-1$ and $b'_{1k} = 0$ otherwise. Let for each $n > 1$

$$\begin{aligned} b_{nk} = b'_{nk} &= 1, \quad k = n, \\ &= 0, \quad \text{otherwise}. \end{aligned}$$

The sequence $s_k = [(-2)^p$ when $k = 4p-1$ and 0 otherwise$]$ is summable AB and $A \cdot B$ to 0 but is non-summable AB' and $A \cdot B'$. The sequence $s_k = [(-2)^p$ when $k = 4p-3$ and 0 otherwise$]$ is summable AB' and $A \cdot B'$ to 0 but is non-summable AB and $A \cdot B$. Thus AB and AB' have overlapping convergence fields, as do $A \cdot B$ and $A \cdot B'$.

13. **Multiple products.** In terms of three methods A, B, C of summability, we can define five types of products: ABC, $A(B\cdot C)$ $(A\cdot B)C$, $A\cdot(B\cdot C)$ and $(A\cdot B)\cdot C$. It is easy to show that the last two methods are equivalent. It follows easily from Theorem 7.1 that each other pair selected from the five products may be inconsistent, even though A, B, C are regular and $a_{nk}\geqq 0$, $b_{nk}\geqq 0$, and $c_{nk}\geqq 0$.

14. **Kernel transformations.** Just as matrices (a_{nk}) serve to define generalizations of $\lim_{n\to\infty}s_n$, so also kernels $a(x, t)$ serve to define generalizations of $\lim_{t\to\infty}s(t)$. The $\mathcal{A}$ transform of a function $s(t)$ is, if it exists, given by

$$(\mathcal{A}) \qquad S(x) = \int_0^\infty a(x, t)s(t)dt,$$

and $s(t)$ is summable $\mathcal{A}$ to L if $S(x)\to L$ as $x\to\infty$.

Since the transformation $\mathcal{A}$ becomes essentially a matrix transformation of sequences when we put $s(t)=s_k$ when $k-1\leqq t<k$ and $a(x, t)=a_{nk}$ when $n-1\leqq x<n$, $k-1\leqq t<k$, so that $S(x)=S_n$ when $n-1\leqq x<n$, it follows that some of our results have immediate application to kernel transformations. We mention only the fact that the *iteration product*

$$(\mathcal{A}\mathcal{B}) \qquad U(x) = \int_0^\infty a(x, \alpha)d\alpha \int_0^\infty b(\alpha, t)s(t)dt$$

and the *composition product*

$$(\mathcal{A}\cdot\mathcal{B}) \qquad V(x) = \int_0^\infty \left\{\int_0^\infty a(x, \alpha)b(\alpha, t)d\alpha\right\} s(t)dt$$

may represent inconsistent methods of summability of functions, even though $\mathcal{A}$ and $\mathcal{B}$ are regular and the kernels $a(x, t)$ and $b(x, t)$ are everywhere nonnegative. It thus appears that a formal change of order of integration in a right member above may not only produce a meaningless integral but may actually produce a wrong answer.

Cornell University,
Ithaca, N. Y.