DECOMPOSITIONS AND DIMENSION OF CLOSED SETS IN R^n*

BY

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1. **Introduction.** It is our purpose to give a characterization of the dimension of closed sets immersed in R^n (euclidean space of n dimensions). This is done in terms of certain properties of the decompositions of these sets into a countable infinity of closed sets. The results are well known for finite decompositions of compact sets, but have never been shown to be so intrinsic a property of dimension as to remain valid under countable decompositions. This may be due to the fact that the proof seems to require much of the technique and many of the results of Alexandroff,† which are of very recent development. We may state our principal result as follows: *A closed subset F of R^n is of dimension r if and only if there exists an $\epsilon>0$, such that F may be decomposed into the sum of a countable infinity of closed sets $F_1, F_2, \cdots, F_s, \cdots$ of diameter less than ϵ, for which* $\dim F_i\cdot F_j\leqq r-1$, $i\neq j$, *but for any such decomposition there exists a pair of integers m and n such that* $\dim F_m\cdot F_n=r-1$.

This result follows quite readily when we have proved the following: If F is a closed subset of R^n, p a point of F, $F_1, F_2, \cdots, F_s, \cdots$ a decomposition of F into closed sets, z^{n-r-1} a cycle in $S(p, \epsilon)-F$, which does not bound in $S(p, \epsilon)-F$ but does bound in $S(p, \epsilon)-F_i$, $i=1, 2, \cdots$, then there exists a pair of integers m and n such that $\dim F_m\cdot F_n\cdot S(p, \epsilon)\geqq r-1$. From this we obtain an interesting result which may be considered a generalization of a theorem due to Miss Mullikin.‡ We show that the sum of a countable number of closed sets, no one of which separates R^n, and the dimension of whose intersections taken pairwise does not exceed $n-3$, cannot separate R^n.

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2. **Notation.** The notation and definitions used in the sequel are widely employed. For example, $\delta(M)$ refers to the diameter of a point set M, $\rho(M_1, M_2)$ to the distance between the sets M_1 and M_2, $S(M, \epsilon)$ to the set of points x such that $\rho(M, x)<\epsilon$. We shall denote the boundary of a point set M by $B(M)$.

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† Especially his article *Dimensionstheorie*, Mathematische Annalen, vol. 106 (1932), pp. 161–238.

‡ A. M. Mullikin, *Certain theorems relating to plane connected point sets*, these Transactions, vol. 24 (1922), pp. 144–162.

The superscript attached to a symbol representing a chain will denote the dimension of the chain. The complex composed of the simplices of a chain C^i will be denoted by $|C^i|$. Cycles and chains occurring in this article will be assumed to have integral coefficients, although most of the results are just as valid if chains modulo m ($m \geqq 0$) or rational chains are used. The relation expressing the fact that the cycle z^r is (is not) the boundary of a chain in the domain D will be written $z^r \cong 0$ in D (z^r non-$\cong 0$ in D).

3. **Dimension of simplices.** Some years ago Sierpiński* proved that no continuum M can be expressed as the sum of a countable number of closed and proper subsets of M whose intersections are mutually vacuous. Remembering that the vacuous set is of dimension -1 and applying this theorem to the one-simplex, we have as an immediate result:

COROLLARY. *No one-simplex can be decomposed into the sum of a countable number of closed sets of diameter less than $\epsilon > 0$, where ϵ is less than the diameter of the simplex, and the dimension of the intersection of any pair of these closed sets is minus one.*

When the theorem is stated in this form, one is led to anticipate the more general statement:

THEOREM 1. *No r-simplex T^r can be decomposed into the sum of a countable number of closed sets of diameter less than $\epsilon > 0$, where ϵ is less than the diameter of T^r and the dimension of the intersection of any pair of these closed sets is at most $r-2$.*

Proof. Suppose the theorem false. Letting $F_1, F_2, \cdots, F_s, \cdots$ denote the closed sets referred to in the statement of the theorem, we have

$$T^r = F_1 + F_2 + \cdots + F_s + \cdots$$

and

$$\dim F_i \cdot F_j \leqq r - 2, \quad \delta(F_i) < \epsilon, \quad i \neq j \qquad (i, j = 1, 2, \cdots).$$

Denote the totality of intersections $F_i \cdot F_j$ by $P_1, P_2, \cdots$ and their sum by $P = \sum_{i=1}^{\infty} P_i$. P_i is closed.

Since the sum of a countable infinity of closed sets of dimension at most $r-2$ is of dimension at most $r-2$, it follows that P cannot fill any domain† in T^r.‡ But in the complement of each F_i, since F_i is of diameter less than

* W. Sierpiński, *Un théorème sur les continus*, Tôhoku Mathematical Journal, vol. 13 (1918), pp. 300–304.

† A domain is a connected open set. See P. Alexandroff and H. Hopf, *Topologie*, vol. 1, p. 51.

‡ P. Urysohn, *Mémoire sur les multiplicités Cantoriennes*, Fundamenta Mathematicae, vol. 8 (1927), pp. 337–341.

$\delta(T^r)$, there exists a domain. From this it follows readily that at least two of the sets $F_i - P$ $(i=1, 2, \cdots)$ are non-vacuous.

Let p be a point of $F_{i_1} - P$, and q a point of $F_{i_2} - P$, $i_1 \neq i_2$. We may assume p and q to be interior points of T^r. Now P_1 is a closed set, and dim $P_1 \leqq r-2$. Consequently p can be joined to q by a polygonal line t_1 in the interior of $T^r - P_1$.* We enclose t_1 in a domain D_1 whose closure does not meet P_1.

Suppose we have constructed the domains $D_1, D_2, \cdots, D_{i-1}$, where

1. $\bar{D}_k \subset \bar{D}_{k-1}$,
2. $\bar{D}_k \cdot P_k = 0$, and
3. $D_k \supset p+q$ $(k=1, 2, \cdots, i-1)$.

In the construction of D_i we observe that dim $P_i \cdot D_{i-1} \leqq r-2$. Hence p and q can be joined by a polygonal line t_i lying in $D_{i-1} - P_i \cdot D_{i-1}$, which we then enclose in a domain D_i whose closure is contained in D_{i-1} and does not meet P_i.

We thus obtain the sequence of continua

$$\bar{D}_1, \bar{D}_2, \cdots, \bar{D}_i, \cdots, \tag{a}$$

where (a) satisfy relations 1, 2, and 3 above. $\Pi = \bar{D}_1 \cdot \bar{D}_2 \cdot \cdots \cdot \bar{D}_i \cdot \cdots$ is a continuum containing p and q, and, as is easily seen, containing no point of P. But the decomposition of Π into the closed sets $\Pi \cdot F_i$ $(i=1, 2, \cdots)$,

$$\Pi = \sum_{i=1}^{\infty} \Pi \cdot F_i,$$

affords a contradiction to Sierpiński's theorem, since at least two of these sets ($\Pi \cdot F_{i_1}$ and $\Pi \cdot F_{i_2}$) are non-vacuous, whereas the intersection of any pair is vacuous. This contradiction establishes the theorem.

By a slight modification in the method of constructing the domains D_i (that is, by constructing D_i as a chain of regions whose diameters are less than $1/i$), we could have been taken Π to be an arc.† We should thus have obtained an incidental proof of the following theorem:

THEOREM 2. *The complement in an n-dimensional simplex (or in R^n) of the sum of a countable infinity of closed sets of dimension at most $n-2$ is arcwise connected.*‡

Although, as observed in Theorem 1, an r-simplex cannot be decomposed into small closed sets with mutual intersections of dimension at most $r-2$,

* Cf. P. Urysohn, loc. cit., p. 307.

† For a discussion of this method of characterizing an arc, see R. L. Moore, *On the foundations of plane analysis situs*, these Transactions, vol. 17 (1916), pp. 133–139.

‡ This was first proved for the case $n=2$, i.e., for the plane, by J. R. Kline, *Concerning the complement of a countable infinity of point sets of a certain type*, Bulletin of the American Mathematical Society, vol. 23 (1916–1917), pp. 290–292.

it can always be decomposed into a countable number of closed sets, of arbitrarily small diameter, whose intersections taken pairwise are of dimension at most $r-1$ (for example, by a simplicial subdivision). We may therefore characterize the dimension of a simplex in the following way:

THEOREM. *A simplex T is of dimension r if and only if for any ϵ, where $\delta(T)>\epsilon>0$, T may be decomposed into the sum of a countable infinity of closed sets $F_1, F_2, \cdots, F_s, \cdots$ of diameter less than ϵ, for which* $\dim F_i \cdot F_j \leqq r-1$, *$i \neq j$, but for any such decomposition there exists a pair of integers m and n such that* $\dim F_m \cdot F_n = r-1$.

This, as well as Theorem 1, is a special case of more general considerations to be developed independently in a following section. Its chief interest lies in the simple proof based entirely on set-theoretic considerations. The same, or similar, methods do not seem adequate for a treatment of closed sets in R^n.

4. **Some preliminary lemmas and considerations.** After this simple characterization of the dimension of a simplex, it is quite natural to define the dimension of a closed set F in an analogous fashion. The definition is an inductive one, where the vacuous set is defined to be of dimension minus one.

DEFINITION. A closed set F is said to be of *dimension r* if there exists an $\epsilon>0$, such that F may be decomposed into the sum of a countable infinity of closed sets $F_1, F_2, \cdots, F_s, \cdots$ of diameter less than ϵ for which $\dim F_i \cdot F_j \leqq r-1$, $i \neq j$, but for any such decomposition there exists a pair of integers m and n such that $\dim F_m \cdot F_n = r-1$.

It is our aim in the sequel to show the complete equivalence between this definition of dimension and the Menger-Urysohn definition applied to closed sets. To accomplish this we prove several lemmas and theorems. The first of these, Lemma A_r, is based on the notion of ϵ-modification and simple r-dimensional obstruction introduced by Alexandroff in his article *Dimensionstheorie*, previously referred to, and certain methods and theorems proved there. For the sake of completeness we shall define ϵ-modification and simple r-dimensional obstruction and state the results used in this paper.

DEFINITION. Given a chain K and a positive number ϵ, the chain K' will be called an *ϵ-modification of K* if to each simplex x of K there corresponds a chain y in K' satisfying the following conditions:

a. If $x_i^h \rightarrow \sum c^j x_j^{h-1}$ then $y_i^h \rightarrow \sum c^j y_j^{h-1}$.

b. If $h=0$ (that is, if x_i^h is a vertex), then $x_i^0 = y_i^0$.

c. The sum $|x_i^h| + |y_i^h|$ is contained in a sphere of radius ϵ.

d. $K=\sum a_i x_i$ implies $K'=\sum a_i y_i$, where the y_i's are the chains corresponding to the simplices x_i.

Remark. The ϵ-modification will be called simple if the bounding relations are taken modulo m, $m \geqq 0$, and only integral coefficients appear. But if the y_i^h are chains with rational coefficients, then this is called an ϵ-modification modulo 0.

From the definition of ϵ-modification it is a short step to prove that if K' is an ϵ-modification of K, then for every simplicial transformation f, of K' into K, where each vertex of y_i^h goes into some vertex of x_i^h,

$$f(K') = K. \tag{1}$$

DEFINITION. $F \subset R^n$, and x is a point of F. F is said to be a *simple r-dimensional obstruction in the neighborhood of x* if there exists an $\epsilon > 0$ so that for every δ sufficiently small $S(x, \delta) - F$ contains an $n-r-1$ dimensional cycle modulo 0, which does not bound in $S(x, \epsilon) - F$.

We can now state Alexandroff's theorem:

THEOREM. *The set F is r-dimensional in the sense of Menger-Urysohn if and only if F is a simple r-dimensional obstruction in the neighborhood of at least one point, but forms no simple k-dimensional obstruction in the neighborhood of any point, if $k > r$.*

We turn now to the proof of several preliminary lemmas.

LEMMA A_r. *Suppose*

1. *D a domain in R^n,*
2. *F a set closed relative to D,*
3. *the dimension* of F at most equal to r,*
4. *z^{n-r-2} a cycle in $D-F$,*
5. *K^{n-r-1} a chain bounded by z^{n-r-2} in D,*

are given. Under these conditions, if ϵ_1 is positive and less than $\frac{1}{2}\rho(K^{n-r-1}, B(D))$, there exists in $S(K^{n-r-1}, \epsilon_1) - F$ a chain C^{n-r-1} bounded by z^{n-r-2}, such that $C^{n-r-1} - K^{n-r-1} \simeq 0$ in D.

Proof. Denote the set $F \cdot \overline{S(K^{n-r-1}, \epsilon_1)}$ by F'. F' is a compact subset of R^n of dimension at most r (conditions 2 and 3). From the theorem of Alexandroff just quoted, F' cannot form an $(r+i)$-dimensional obstruction in the neighborhood of any point. Hence given an $\epsilon_i > 0$, we can find an $\epsilon_{i+1} > 0$, $\epsilon_{i+1} < \epsilon_i$, such that if $z^{n-r-i-1}$ is a cycle in $R^n - F'$, and $\delta(z^{n-r-i-1}) < \epsilon_{i+1}$, then there exists a chain C_1^{n-r-i} satisfying the conditions:

$$\text{(a)} \qquad C_1^{n-r-i} \rightarrow z^{n-r-i-1} \qquad (\text{in } R^n - F'),$$

$$\text{(b)} \qquad \delta(C_1^{n-r-i}) < \tfrac{1}{2}\epsilon_i, \qquad (i = 1, 2, \cdots, n-r-1).$$

* Here, and in the sequel, whenever the term dimension is used it is to be understood in the sense of Menger-Urysohn.

Now given ϵ_1, we find successively the numbers $\epsilon_2, \epsilon_3, \cdots, \epsilon_{n-r}$, subject to these conditions.

Let $|K^{n-r-1}|$ be subdivided into simplices of diameter less than ϵ_{n-r}. We may assume that none of the vertices of the subdivided complex lie on F, for by an arbitrarily small displacement, leaving z^{n-r-2} intact, the complex can be made to satisfy this condition. We denote the chain obtained from the subdivision of $|K^{n-r-1}|$ by $(K^{n-r-1})'$, wherein the orientation of the simplices will be that induced by their carriers in K^{n-r-1}, and their coefficients will be the same as those of their carriers.

Consider any one-simplex of $(K^{n-r-1})'$, say x_k^1. Its boundary $\dot{x}_k^1$ is a zero-cycle of $R_n - F'$ whose diameter is less than ϵ_{n-r}. We can find a chain y_k^1 in $R^n - F'$ bounded by $\dot{x}_k^1$ and such that $\delta(y_k^1) < \frac{1}{2}\epsilon_{n-r-1}$, from the restrictions on $\epsilon_2, \epsilon_3, \cdots, \epsilon_{n-r}$. The vertices of x_k^1 are a subset of the vertices of y_k^1.

Suppose now that the chains y_k^i have been constructed and ordered to the simplices x_k^i in such a way as to preserve incidence relations. Assume moreover that

1°. $\delta(y_k^i) < \frac{1}{2}\epsilon_{n-r-i}$,

2°. $y_k^i \subset R^n - F'$,

3°. every vertex of x_k^i is a vertex belonging to y_k^i.

If x_k^{i+1} is a simplex of $(K^{n-r-1})'$ and $\sum c^j x_j^i$ is its boundary, it follows from 1° and 3° that

$$\delta\left(\sum c^j y_j^i\right) < \epsilon_{n-r-i}.$$

Also, from 2° and the preservation of incidence under the ordering, $\sum c^j y_j^i$ is a cycle in $R^n - F'$. Hence from the conditions on the ϵ_s, $(s = 1, 2, \cdots, n-r)$, and from (a) and (b), there exists a chain

$$y_k^{i+1} \rightarrow \sum c^j y_j^i \quad \text{in} \quad R^n - F',$$

such that

$$\delta(y_k^{i+1}) < \tfrac{1}{2}\epsilon_{n-r-i-1}.$$

Each vertex of x_k^{i+1} is a vertex of y_k^{i+1} (from 3°). We are careful throughout the above process to choose the y_j^i's corresponding to simplices of $(z^{n-r-2})'$ as the simplices themselves. This may be done since $(z^{n-r-2})'$ is contained in $R^n - F'$. Continuing this process we arrive at an ϵ_1-modification of the chain $(K^{n-r-1})'$ which we may denote by C_*^{n-r-1}.

If

$$(K^{n-r-1})' = \sum a^j x_j^{n-r-1},$$

then

$$C_*^{n-r-1} = \sum a^j y_j^{n-r-1},$$

and from the construction, we have

$$C_*^{n-r-1} \rightarrow (z^{n-r-2})' \quad \text{in} \quad R^n - F.$$

Since $C_*^{n-r-1} \subset S(K^{n-r-1}, \epsilon_1)$, we can replace F' by F in the preceding relation. There exists a chain C_{**}^{n-r-1} such that

$$C_{**}^{n-r-1} \rightarrow z^{n-r-2} - (z^{n-r-2})' \quad \text{in} \quad |z^{n-r-2}|.$$

(C_{**}^{n-r-1} may be obtained by the so-called cylinder construction,† on z^{n-r-2}, in which the base is subdivided into isomorphism with $(z^{n-r-2})'$ and the vertical lines are collapsed into points.)

The chain

$$C^{n-r-1} = C_*^{n-r-1} + C_{**}^{n-r-1}$$

satisfies the statement of the lemma. C^{n-r-1} is bounded by z^{n-r-2}, and we must show that $C^{n-r-1} - K^{n-r-1} \cong 0$. We do this in two steps:

(c) $$C_*^{n-r-1} - (K^{n-r-1})' \cong 0 \quad \text{in} \quad S(K^{n-r-1}, \epsilon_1).$$

(d) $$C_{**}^{n-r-1} + (K^{n-r-1})' - K_{\cdot}^{n-r-1} \cong 0 \quad \text{in} \quad |K^{n-r-1}|.$$

Perform a simplicial transformation of C_*^{n-r-1} into $(K^{n-r-1})'$ in such a way that the vertices of y_k^i are transformed into the vertices of x_k^i $(i=0, 1, \cdots, n-r-1)$. Then

$$f(C_*^{n-r-1}) = (K^{n-r-1})'$$

follows from the discussion about relation (1). If x is a point of C_*^{n-r-1}, then $\rho(x, f(x)) < \epsilon_1$. Therefore the straight line $xf(x)$‡ lies in $S(K^{n-r-1}, \epsilon_1)$. Let G^{n-r-1} be a chain isomorphic to C_*^{n-r-1}. Denote the cylinder formed by the product of the interval $I(0 \leq t \leq 1)$ and $|G_*^{n-r-1}|$ by $|G^{n-r}|$, and subdivide and orient $|G^{n-r}|$ so that

$$\dot{G}^{n-r} = G^{n-r-1} \times 0 - G^{n-r-1} \times 1 - Z^{n-r-2} \times I,$$

where z^{n-r-2} is isomorphic with $(z^{n-r-2})'$. We now cnstruct a continuous transformation g such that every point x of $G^{n-r-1} \times 0$ corresponds to its image y under the isomorphism between $G^{n-r-1} \times 0$ and C_*^{n-r-1}, while $x \times 1$ goes into $f(y)$, and $x \times t$, where $0 < t < 1$, goes into the point dividing the line $yf(y)$ in the ratio $t:(1-t)$. Then

$$g(G^{n-r}) \subset S(K^{n-r-1}, \epsilon_1),$$

† For a discussion of this method see P. Alexandroff and H. Hopf, *Topologie*, vol. 1, pp. 196–198.

‡ Here $xf(x)$ means the straight line from x to $f(x)$.

and

$$g(G^{n-r})^{\cdot} = g(\dot{G}^{n-r}) = C_{*}^{n-r-1} - (K^{n-r-1})' + g(Z^{n-r-2} \times I).$$

But $g(Z^{n-r-2}\times I)^{\cdot}=0$, since g sends the $(n-r-1)$-chain $Z^{n-r-2}\times I$ into the $(n-r-2)$-chain $(z^{n-r-2})'$. This establishes relation (c).

To establish (d), it is sufficient to note that the left-hand member is the boundary of the cylinder on K^{n-r-1} with base subdivided into $(K^{n-r-1})'$ and vertical lines degenerated into points. Adding (c) and (d), and using the definition of C^{n-r-1}, we have

$$C^{n-r-1} - K^{n-r-1} \cong 0 \quad \text{in} \quad S(K^{n-r-1}, \epsilon_1).$$

This completes the proof of the lemma.

Lemma B. *Assume*

1. *K a complex,*
2. *F a closed subset of K,*
3. *z^r an r-cycle on K which does not bound in $K-F$,*
4. *K_i^{r+1} $(i=1, 2)$ two chains of K bounded by z^r,*
5. *K^{r+2} a chain bounded by the cycle $K_1^{r+1}-K_2^{r+1}$.*

Then the intersection I of F and K^{r+2} contains a continuum M joining K_1^{r+1} and K_2^{r+1}.†

Proof. Suppose I contains no continuum M joining K_1^{r+1} and K_2^{r+1}. Then there exists an $\epsilon<0$, such that whenever K^{r+2} is subdivided into simplices of diameter less than ϵ, and H^{r+2} is the collection of those simplices having a non-vacuous intersection with I, no component of H^{r+2} joins K_1^{r+1} and K_2^{r+1}. Assume the contrary. There exists a sequence of simplicial subdivisions of K^{r+2}, say $K_1^{r+2}, K_2^{r+2}, \cdots, K_i^{r+2}, \cdots$ where the simplices of K_i^{r+2} are of diameter less than $\epsilon_i>0$, $\lim_{i\to\infty}\epsilon_i=0$, and the collection H_i^{r+2}, of simplices of K_i^{r+2} meeting I, contains a component joining K_1^{r+1} and K_2^{r+1}. Denote these components by $L_1, L_2, \cdots, L_i, \cdots$. We can choose a subsequence of the L_i for which the limit inferior‡ is non-vacuous. Since $L_i \subset H_i^{r+2} \subset S(I, \epsilon_i)$, the limit superior of our subsequence is a subset of I. From a well known theorem of L. Zoretti,§ the limit superior is a continuum M. Moreover, since the L_i

† This lemma was stated and proved first by R. L. Wilder, *Generalized closed manifolds*, Annals of Mathematics, vol. 35 (1934), p. 879, for the case $K=R^n$ and $F=\Gamma^{n-r-1}$, a cycle linking z^r. In a footnote on the same page Wilder mentions the truth of the lemma when $K=R^n$ and F is any closed set. This latter statement would have been sufficient for our needs. However the formulation of Lemma C has intrinsic interest, and we offer it even though its full generality is not needed here.

‡ For a definition of the terms "limit inferior" and "limit superior" see Kuratowski, *Topologie* I, p. 152.

§ See, e.g., Hausdorff, *Mengenlehre*, 1927 edition, p. 163.

join K_1^{r+1} and K_2^{r+1}, M does likewise, contrary to our assumption of the falsity of the lemma.

Now let ϵ and H^{r+2} be defined as in the preceding paragraph. Let $'C^{r+2}$ denote the sum of the components of H^{r+2} which meet K_1^{r+1}. We form the following chain from the simplices of C^{r+2}: If x^{r+2} is a simplex of $'C^{r+2}$, then x^{r+2} is assumed to have the same orientation and coefficient as its carrier in K^{r+2}. We denote this chain by $(C^{r+2})'$. Similarly, form the chains $(K^{r+2})'$, $(K_1^{r+1})'$, $(K_2^{r+1})'$, the subdivisions of the chains K^{r+2}, K_1^{r+1}, K_2^{r+1}. If L^{r+1} is the boundary of $(C^{r+2})'$, we have

$$(C^{r+2})' \rightarrow L^{r+1}$$

and

$$(K^{r+2})' \rightarrow (K_1^{r+1})' - (K_2^{r+1})'.$$

L^{r+1} contains points of I only in those simplices which are contained in $(K_1^{r+1})'$. For $(C^{r+2})'$ does not meet $(K_2^{r+1})'$, and if x^{r+1} is a simplex of L^{r+1} containing a point of I, then all simplices having x^{r+1} on their boundary belong to $(C^{r+2})'$. Hence x^{r+1} can only occur with coefficient different from zero when it belongs to either $(K_1^{r+1})'$ or $(K_2^{r+1})'$. Those simplices of L^{r+1} which meet I belong to $(K_1^{r+1})'$ and have the same coefficient as they have in $(K_1^{r+1})'$. It follows that $(K_1^{r+1})' - L^{r+1}$ is a chain in $|K^{r+2}| - I$. But

$$L^{r+1} \rightarrow 0$$

and

$$(K_1^{r+1})' \rightarrow (z^r)'.$$

Hence

$$(K_1^{r+1})' - L^{r+1} \rightarrow (z^r)'$$

in $|K^{r+2}| - I$, and therefore in $K - F$. Since $z^r \cong (z^r)'$ in $|z^r|$, we have z^r does bound in $K - F$ contrary to the hypothesis of the lemma. This final contradiction completes the proof.

Let F be a subset of the domain $D \subset R^n$. F is closed relative to D and decomposed into the sets $F_1, F_2, \cdots, F_i, \cdots$, each closed relative to D. Denote the totality of products $F_i \cdot F_j$, $i \neq j$, by $P_1, P_2, \cdots, P_i, \cdots$, and their sum by $P = \sum_{i=1}^{\infty} P_i$. Let $H_i = F_i - P$ $(i = 1, 2, \cdots)$.

Lemma C. *If infinitely many of the sets H_i are non-vacuous, then infinitely many of them contain points in the complement of S, the limit superior of $H_1, H_2, \cdots, H_i, \cdots$.*†

† It would be sufficient for our purposes to show that at least two of the sets H' contain points in the complement of S, but the proof is the same in either case.

Proof. Suppose there were a number s such that $S \supset \sum_{i=s}^{\infty} H_i$. Let p_1 be a point of the first non-vacuous set, say H_{s_1}.

$$F_i \cdot p_1 = 0 \qquad (i = 1, 2, \cdots, s_1 - 1).$$

For if $F_{i'} \supset p_1$, $i' < s_1$, then $F_{i'} \cdot F_{s_1} = P_k \supset p_1$, and $H_{s_1} \cdot p_1 = p_1 \cdot (F_{s_1} - P) \subset p_1 \cdot (F_{s_1} - P_k) = 0$. Consequently $F - (F_1 + F_2 + \cdots + F_{s_1 - 1}) \supset p_1$. As $F_1 + F_2 + \cdots + F_{s_1 - 1}$ is closed relative to D, there is a neighborhood N_1 of p_1 such that $N_1 \subset D - (F_1 + F_2 + \cdots + F_{s_1 - 1})$.

Since $S \supset p_1$, N_1 contains a point $p_2 \subset H_{s_2}$, where $s_2 > s_1$. In the same way we find a neighborhood $N_2 \subset N_1$ such that $\overline{N}_2 \cdot (F_1 + F_2 + \cdots + F_{s_2 - 1}) = 0$. Continuing this process, we obtain a sequence of integers $s_1 < s_2 < \cdots, s_i \geqq i$, and neighborhoods $N_1, N_2, \cdots, N_i, \cdots$ such that

$$\overline{N}_{i+1} \subset N_i,$$

and

$$\overline{N}_i \cdot (F_1 + F_2 + \cdots + F_{s_i - 1}) = 0.$$

Since $\overline{N}_i$ is compact, $\prod_{i=1}^{\infty} \overline{N}_i \neq 0$. If x is a point of $\prod_{i=1}^{\infty} \overline{N}_i$, there is an integer t such that $F_{t-1} \supset x$. But x is also contained in $\overline{N}_t$, while $\overline{N}_t \cdot (F_1 + F_2 + \cdots + F_{s_t - 1}) = 0$. This contradiction establishes the lemma.

5. **Principal theorems.** Let F be a closed subset of R^n, p a point of F, and ϵ a positive number. $F_1, F_2, \cdots, F_s, \cdots$ is a decomposition of F into closed sets.

THEOREM 3. *If there exists in $S(p, \epsilon) - F$ a cycle z^{n-r-1}, which does not bound in $S(p, \epsilon) - F$ but does bound in $S(p, \epsilon) - F_i$, $i = 1, 2, \cdots$, then there is a pair of integers m and n, $m \neq n$, such that,*

$$\dim F_m \cdot F_n \cdot S(p, \epsilon) \geqq r - 1.$$

Proof. If we assume the existence of the cycle z^{n-r-1}, F contains a closed subset A which is irreducible with respect to the property

$$z^{n-r-1} \text{ non-} \simeq 0 \quad \text{in} \quad S(p, \epsilon) - A. \tag{2}$$

This may be seen by first showing that when $F \supset F' \supset F'' \supset \cdots \supset F^{(k)} \supset \cdots$ are closed sets and z^{n-r-1}non-$\simeq 0$ in $S(p, \epsilon) - F^{(k)}$, then $F_\omega = \prod_{k=1}^{\infty} F^{(k)}$ is likewise closed and z^{n-r-1}non-$\simeq 0$ in $S(p, \epsilon) - F_\omega$. Then, from a well known theorem due to Brouwer,† the existence of A follows. To prove the first point it is sufficient to remark that if K^{n-r} is any chain in $S(p, \epsilon)$ bounded by z^{n-r-1}, then K^{n-r} has a compact and non-vacuous intersection with each of the sets $F^{(k)}$. The product of these intersections is non-vacuous and belongs to F_ω.

† See K. Menger, *Dimensionstheorie*, Leipzig and Berlin, 1928, p. 69.

Let $A_i = F_i \cdot A$. A_i is closed, and

$$A = \sum_{i=1}^{\infty} A_i. \tag{3}$$

From $F_i \supset A_i$ follows $S(p, \epsilon) - A_i \supset S(p, \epsilon) - F_i$. By hypothesis, z^{n-r-1} bounds in $S(p, \epsilon) - F_i$. Hence

$$z^{n-r-1} \cong 0 \quad \text{in} \quad S(p, \epsilon) - A_i \qquad (i = 1, 2, \cdots). \tag{4}$$

Also $A_i \cdot A_j \cdot S(p, \epsilon) \subset F_i \cdot F_j \cdot S(p, \epsilon)$, which gives

$$\dim A_i \cdot A_j \cdot S(p, \epsilon) \leqq \dim F_i \cdot F_j \cdot S(p, \epsilon).$$

From this point to the completion of the proof, we shall suppose the theorem false, that is, $\dim F_i \cdot F_j \cdot S(p, \epsilon) \leqq r-2$, for all i, j, $(i \neq j)$. Consequently

$$\dim A_i \cdot A_j \cdot S(p, \epsilon) \leqq r - 2, \quad \text{for} \quad i \neq j.$$

Arranging the countable collection of products $A_i \cdot A_j \cdot S(p, \epsilon)$ in a sequence, and renaming them $P_1, P_2, \cdots$, we have $P = \sum_{i=1}^{\infty} P_i$ is of dimension at most $r-2$. This is a consequence of the well known result that the sum of a countable number of sets, closed relative to a domain and each of dimension at most $r-2$, is itself of dimension at most $r-2$. We have thus constructed a set A and a subdivision $A_1, A_2, \cdots$ which bear the same relation to z^{n-r-1} and $S(p, \epsilon)$ as do F and its subdivision and in addition is irreducible with respect to the non-bounding of z^{n-r-1} in $S(p, \epsilon) - A$.

We now prove that if

$$A(t) = \sum_{i=1}^{t} A_i,$$

then

$$z^{n-r-1} \cong 0 \quad \text{in} \quad S(p, \epsilon) - A(t), \qquad t = 1, 2, \cdots. \tag{5}$$

This is demonstrated for the case $t=2$. A simple induction then carries the demonstration to any finite t. For, if by assumption, z^{n-r-1} bounds in $S(p, \epsilon) - A(t-1)$, then $A(t-1)$ and A_t satisfy the same conditions as do A_1 and A_2, and $A(t-1) + A_t = A(t)$. ($A(t-1)$ is closed since it is the sum of a finite number of closed sets. Moreover, $A(t-1) \cdot A_t \cdot S(p, \epsilon) = A_1 \cdot A_t \cdot S(p, \epsilon) + A_2 \cdot A_t \cdot S(p, \epsilon) + \cdots + A_{t-1} \cdot A_t \cdot S(p, \epsilon)$, being the sum of a finite number of sets closed relative to $S(p, \epsilon)$, and each of dimension at most $r-2$, satisfies the relation $\dim A(t-1) \cdot A_t \cdot S(p, \epsilon) \leqq r-2$.) We turn to the proof for $t=2$.

We can find chains C_j^{n-r} in $S(p, \epsilon) - A_j$, $j = 1, 2$, (relation (4)) such that

$$C_j^{n-r} \to z^{n-r-1} \quad \text{in} \quad S(p, \epsilon) - A_j, \qquad j = 1, 2.$$

Denote the cycle $C_1^{n-r}-C_2^{n-r}$ by Z^{n-r}. Let K^{n-r+1} be a chain bounded by Z^{n-r} in $S(p, \epsilon)$. $A_1 \cdot A_2 \cdot S(p, \epsilon)$, Z^{n-r}, K^{n-r+1}, $S(p, \epsilon)$ satisfy the same conditions as do F, Z^{n-r}, K^{n-r+1}, D, respectively, in Lemma A_{r-2}. Hence there exists a chain C^{n-r+1} such that

$$C^{n-r+1} \to z^{n-r} \quad \text{in} \quad S(p, \epsilon) - A_1 \cdot A_2.$$

Now if z^{n-r-1} did not bound in $S(p, \epsilon)-A(2)$, then C^{n-r+1}, $A(2)$, z^{n-r-1}, C_i^{n-r}, C^{n-r+1} would satisfy the same conditions as K, F, z^r, K_i^{r+1}, K^{r+2}, respectively, in Lemma B. Hence $A(2) \cdot |C^{n-r+1}|$ would contain a continuum M joining C_1^{n-r} and C_2^{n-r}. This is impossible.

$$M = M \cdot A_1 + M \cdot A_2.$$

Neither of these sets is vacuous, since $A_2 \supset M \cdot |C_1^{n-r}| \neq 0$ and $A_1 \supset M \cdot |C_2^{n-r}| \neq 0$. The intersection $(M \cdot A_1) \cdot (M \cdot A_2) = M \cdot (A_1 \cdot A_2) \subset |C^{n-r+1}| \cdot (A_1 \cdot A_2) = 0$. This negates the connectedness of M and therefore establishes relation (5).

Infinitely many of the sets $S(p, \epsilon) \cdot A_i - P$ are non-vacuous. Suppose to the contrary that only

$$S(p, \epsilon) \cdot A_1 - P, \; S(p, \epsilon) \cdot A_2 - P, \; \cdots, \; S(p, \epsilon) \cdot A_t - P$$

are non-vacuous. Then

$$S(p, \epsilon) \cdot A = S(p, \epsilon) \cdot A(t) + P.$$

But there exists a chain C^{n-r} (relation (5)) such that

$$C^{n-r} \to z^{n-r-1} \quad \text{in} \quad S(p, \epsilon) - A(t).$$

We then find a neighborhood D of C^{n-r} which is small enough to lie in $S(p, \epsilon)$ and to exclude $A(t)$. Since $A \cdot D = P \cdot D$, it would follow that $\dim A \cdot D \leqq r-2$. Again applying Lemma A_{i-2}, z^{n-r-1} would bound in $D - A \cdot D$ and consequently in $S(p, \epsilon)-A$, contradicting relation (2).

Now $S(p, \epsilon) \cdot A = \sum_{i=1}^{\infty} S(p, \epsilon) \cdot A_i$, and infinitely many of the sets $S(p, \epsilon) \cdot A_i - P$ are non-vacuous. Hence, by Lemma C, there exist two points p_1 and p_2 belonging to different sets, say, $A_{i_1}-P$ and $A_{i_2}-P$, and such that neither p_1 nor p_2 belongs to the limit superior of $S(p, \epsilon) \cdot A_1 - P$, $S(p, \epsilon) \cdot A_2 - P, \cdots$. We can find a number α sufficiently small so that $\overline{S(p_k, \alpha)}$ will meet no set of $S(p, \epsilon) \cdot A_1 - P$, $S(p, \epsilon) \cdot A_2 - P, \cdots$ other than $A_{i_k} - P$, $(k=1, 2)$. The sets $A - S(p_k, \alpha)$ $(k=1, 2)$ are closed and proper subsets of A. It follows from the irreducibility of A that there exist chains C_k^{n-r} such that

$$C_k^{n-r} \to z^{n-r-1} \quad \text{in} \quad S(p, \epsilon) - \{A - S(p_k, \alpha)\}.$$

Choose a number ϵ_1 such that ϵ_1 is smaller than either of the numbers

$\frac{1}{2}\rho(C_k{}^{n-r}, A-S(p_k, \alpha))$. We now replace the chains $C_k{}^{n-r}$ by chains C_{1k}^{n-r}, lying in $S(C_k{}^{n-r}, \epsilon_1)-P_1$, which are bounded by z^{n-r-1}. This is of course possible by Lemma A_{i-1}. (P_1 is closed relative to $S(C_k{}^{n-r}, \epsilon_1)$ and of dimension less than $r-1$.) The cycle $C_{11}^{n-r}-C_{12}^{n-r}$ lies in the complement of P_1 in $S(p, \epsilon)$ and hence bounds in $S(p, \epsilon)-P_1$. This may be shown by allowing $C_{11}^{n-r}-C_{12}^{n-r}$ to bound some chain in $S(p, \epsilon)$, and then displacing this chain to another chain $C_1{}^{n-r+1}$ bounded by $C_{11}^{n-r}-C_{12}^{n-r}$ in $S(p, \epsilon)-P_1$; a permissible operation as shown by Lemma A_{r-2}.

Let us suppose that we have constructed the following chains:

$$C_1^{n-r+1},\ C_2^{n-r+1},\ \cdots,C_t^{n-r+1}. \tag{6}$$

Assume further that these chains satisfy the following conditions:

(a) $\rho(C_i^{n-r+1}, P_i) > \alpha_{ii} > 0$.

(b) $\alpha_{si} = \rho(C_s^{n-r+1}, P_i) > \frac{1}{2}\alpha_{ii}$ for $s > i$.

(c) $C_i^{n-r+1} \to C_{i1}^{n-r} - C_{i2}^{n-r}$ in $S(p, \epsilon)$.

(d) $C_{ik}^{n-r} \to z^{n-r-1}$ in $S(p, \epsilon) - \{A - S(p_k, \alpha)\}$, $k = 1, 2$.

(e) $d_i = \rho(C_i^{n-r+1}, B(S(p, \epsilon))) > d > 0$.

We proceed with the construction of C_{t+1}^{n-r+1}.

Choose a number δ smaller than the minimum of the numbers

a^0. $\alpha_{ti} - \frac{1}{2}\alpha_{ii}$, $i = 1, 2, \cdots, t$,

b^0. $\rho(C_{tk}^{n-r}, A - S(p_k, \alpha))$, $k = 1, 2$,

c^0. $d_t - d$.

If $C_{t+1,k}^{n-r}$ is any chain bounded by z^{n-r-1} in $S(C_{tk}^{n-r}, \delta)$, then $C_{t+1,k}^{n-r}$ satisfies condition (d). To prove this we show that if x is a point of $A-S(p_k, \alpha)$ and y a point of $C_{t+1,k}^{n-r}$, then $\rho(x, y)>0$. $C_{t+1,k}^{n-r}$ is assumed to lie in $S(C_{t,k}^{n-r}, \delta)$, so that there is a point z of $C_{t,k}^{n-r}$ such that $\rho(y, z)<\delta$. Condition b^0 applied to

$$\rho(x, y) \geqq \rho(x, z) - \rho(y, z)$$

yields

$$\rho(x, y) \geqq \rho(C_{t,k}^{n-r}, A - S(p_k,\alpha)) - \delta > 0.$$

If C_{t+1}^{n-r+1} is a chain in $S(C_t{}^{n-r+1}, \delta)$, C_{t+1}^{n-r+1} will satisfy (b), that is,

$$\rho(C_{t+1}^{n-r+1}, P_i) > \tfrac{1}{2}\alpha_{ii}, \qquad i < t + 1.$$

For if x is a point of C_{t+1}^{n-r+1}, y a point of P_i, there is a point z of $C_t{}^{n-r+1}$ such that $\rho(x, z)<\delta$. From $\rho(x, y)\geqq\rho(z, y)-\rho(x, z)$ and condition a^0 on the number δ, we have

$$\rho(x, y) \geqq \rho(C_t^{n-r+1}, P_i) - \delta > \alpha_{ti} - (\alpha_{ti} - \tfrac{1}{2}\alpha_{ii}) = \tfrac{1}{2}\alpha_{ii}.$$

Similar considerations show that C_{t+1}^{n-r+1} would satisfy (e).

In $S(C_{t,k}^{n-r}, \delta) - P_{t+1}$, $(k=1,2)$, we can find a chain $C_{t+1,k}^{n-r}$, and in $S(C_{t,k}^{n-r}, \delta)$ a chain ${}^kC^{n-r-1}$ such that

$$\begin{aligned} C_{t+1,k}^{n-r} &\to z^{n-r-1} \quad \text{in} \quad S(C_{t,k}^{n-r}, \delta) - P_{t+1}, \\ {}^kC^{n-r+1} &\to C_{t+1,k}^{n-r} - C_{t,k}^{n-r} \quad \text{in} \quad S(C_{t,k}^{n-r}, \delta). \end{aligned} \tag{7}$$

(This is justified by Lemma A_{r-2}.) The chain

$$^*C_{t+1}^{n-r+1} = C_t^{n-r+1} + {}^1C^{n-r+1} - {}^2C^{n-r+1} \tag{8}$$

is such that

$$^*C_{t+1}^{n-r+1} \to C_{t+1,1}^{n-r} - C_{t+1,2}^{n-r} \quad \text{in} \quad S(C_t^{n-r+1}, \delta)$$

by (7) and (c) applied to (8). Since the boundary of $^*C_{t+1}^{n-r+1}$ lies in $S(C_t^{n-r+1}, \delta) - P_{t+1}$, by Lemma A_{r-2} we can replace $^*C_{t+1}^{n-r+1}$ by C_{t+1}^{n-r+1}, a chain in $S(C_t^{n-r+1}, \delta) - P_{t+1}$ bounded by $C_{t+1,1}^{n-r} - C_{t+1,2}^{n-r}$.

The set P_{t+1} is closed relative to $S(p, \epsilon)$, and C_{t+1}^{n-r+1} is contained in $S(p, \epsilon)$ and does not meet P_{t+1}. Consequently we can find a number $\alpha_{t+1,t+1}$ for which the relation

$$\rho(C_{t+1}^{n-r+1}, P_{t+1}) > \alpha_{t+1,t+1} > 0$$

holds. We have thus obtained an extension of the system (6) by the addition of C_{t+1}^{n-r+1}. Since we had previously shown that (6) existed for $t=1$, this latter shows that (6) can be extended indefinitely. We may therefore suppose that we have constructed a countable infinity

$$C_1^{n-r+1}, C_2^{n-r+1}, \cdots, C_t^{n-r+1}, \cdots \tag{6'}$$

of chains satisfying relations (a)–(e) inclusive.

Consider any chain of the above sequence. $A \cdot |C_t^{n-r+1}|$ is a closed set, and z^{n-r-1} does not, of course, bound on $|C_t^{n-r+1}| - A \cdot |C^{n-r+1}|$. Applying Lemma B, we see that $A \cdot |C_t^{n-r+1}|$ contains a continuum M_t joining $C_{t,1}^{n-r}$ and $C_{t,2}^{n+r}$. Since these latter chains meet A only in $S(p_2, \alpha)$ and $S(p_1, \alpha)$ respectively, we have

$$M_t \cdot S(p_k, \alpha) \neq 0, \qquad k = 1, 2.$$

From $|C_t^{n-r+1}| \subset S(p, \epsilon - d)$ (condition (e)), it follows that

$$M_t \subset S(p, \epsilon - d),$$

that is, the sequence

$$M_1, M_2, \cdots, M_t, \cdots \tag{9}$$

is uniformly bounded, and the limit superior of (9) is contained in the interior of $S(p, \epsilon)$. From (9) we choose a subsequence

$$M_{t_1}, M_{t_2}, \cdots, M_{t_s}, \cdots \tag{10}$$

of which the limit inferior is non-vacuous. From the theorem of L. Zoretti previously referred to, the limit superior of (10) is a continuum M. Since each M_i is contained in $\overline{S(p, \epsilon-d)}$, M is likewise. From $A \supset M_i$, and A closed, we have

$$A \supset M. \tag{11}$$

We affirm

$$M \cdot P = 0. \tag{12}$$

If we assumed the contrary, there would be some P_i for which $M \cdot P_i \neq 0$. But $M_s \subset |C_s^{n-r+1}|$, and from condition (b) on (6′) we should have

$$\rho(M_s, P_i) > \tfrac{1}{2}\alpha_{ii} > 0, \quad \text{for all} \quad s > i.$$

This leads to the conclusions

$$\rho(M, P_i) \geqq \tfrac{1}{2}\alpha_{ii},$$

and therefore $M \cdot P_i = 0$, which proves (12). Combining (3) and (11), we have

$$M = \sum_{i=1}^{\infty} M \cdot A_i.$$

Moreover

$$(M \cdot A_i) \cdot (M \cdot A_j) \subset M \cdot P = 0, \qquad i \neq j.$$

$M \cdot A_s$ is closed.

We have thus obtained a decomposition of the continuum M into a countable infinity of closed sets whose intersections taken pairwise are vacuous. This is impossible according to Sierpiński's theorem unless all but one of these sets are vacuous. But M must contain a point of $A \cdot \overline{S(p_k, \alpha)}$ $(k=1, 2)$, since each of the sets in (9) does. From the choice of the number α, we have $A_{i_k} - P \supset A \cdot \overline{S(p_k, \alpha)} - P$. M must therefore have a non-vacuous intersection with both A_{i_1} and A_{i_2}. This final contradiction completes the proof of the theorem.

Corollary. *In R^n let F' be a closed set and z_1^{n-r-1} a cycle in $R^n - F'$ which does not bound in $R^n - F'$. Moreover let $F' = F_1' + F_2' + \cdots + F_s' + \cdots$ where F_s' is closed $(s=1, 2, \cdots)$ and z_1^{n-r-1} bounds in $R^n - F_s'$. Then there exists a pair of integers m and n such that* $\dim F_m' \cdot F_n' \geqq r-1$.

To show this we let p be any point of F' and ϵ some positive number. R^n is homeomorphic to $S(p, \epsilon)$, and the homeomorphism f may be so taken as to leave p invariant. If we take $F=\overline{f(F')}$, $F_s=\overline{f(F_s')}$ and $z^{n-r-1}=f(z_1{}^{n-r-1})$, then, since

$$F_i \cdot F_j \cdot S(p, \epsilon) = f(F_i') \cdot f(F_j'),$$

we have from Theorem 3,

$$\dim f(F_m') \cdot f(F_n') \geqq r - 1$$

for some pair of integers m and n. But because of the invariance of dimension under homeomorphism, this implies

$$\dim F_m' \cdot F_n' \geqq r - 1.$$

Another very interesting application of Theorem 3, or rather the corollary to Theorem 3, is the generalization of a theorem first proved by Miss Mullikin.† Miss Mullikin showed that the sum of a countable number of closed sets, no one of which separates the plane, and whose intersections taken pairwise are vacuous, cannot separate the plane. Recalling that the vacuous set is of dimension minus one, and the plane is R^2, we see that this is a particular case of the following theorem:

THEOREM 4. *In R^n let $A_1', A_2', \cdots$ be a countable collection of closed sets no one of which separates R^n, and such that*

$$\dim A_i' \cdot A_j' \leqq n - 3, \qquad \textit{for} \qquad i \neq j.$$

Under these conditions the sum $S=\sum_{i=1}^{\infty} A_i'$ cannot separate R^n.

Proof. Let us suppose, to the contrary, that S does separate R^n. Then

$$R^n - S = M_1 + M_2,$$

where M_1 and M_2 are non-vacuous and mutually separated. From the well known result that if a set X separates a connected set M then some closed subset Y of X also separates M,‡ it follows that S contains a closed subset A which separates M_1 from M_2 in R^n. Setting $A_i = A \cdot A_i'$, we have $A=\sum_{i=1}^{\infty} A_i$ (A_i closed). Taking a_1 a point of M_1 and a_2 a point of M_2, we have $a_1 - a_2 \cong 0$ in $R^n - A_i$ but $a_1 - a_2$ non-$\cong 0$ in $R^n - A$. It follows from the corollary to Theorem 3 that there exists a pair of integers m and n such that $\dim A_m \cdot A_n \geqq n-2$. Since $A_i' \supset A_i$, $\dim A_m' \cdot A_n' \geqq n-2$ contrary to the hypothesis of the theorem.

We return now to our discussion of the dimension of closed sets. If the

† A. M. Mullikin, *Certain theorems relating to plane connected point sets*, these Transactions, vol. 24 (1922), pp. 144–162.

‡ See Knaster and Kuratowski, Fundamenta Mathematicae, vol. 2 (1921), pp. 234–235.

dimension of the closed set F is r, then F is a simple r-dimensional obstruction at a point p of F. We can find an $\epsilon>0$ and a cycle z^{n-r-1} in $S(p, \epsilon)-F$ such that z^{n-r-1} does not bound in $S(p, \epsilon)-F$. Denote by d the distance between F and z^{n-r-1}. Now let F be decomposed into the sum of a countable number of closed sets $F_1, F_2, \cdots, F_s, \cdots$ each of which is of diameter smaller than d. These sets will satisfy the hypotheses of Theorem 3, if $z^{n-r-1}\simeq 0$ in $S(p, \epsilon)-F_s$.

If $F_s\cdot S(p, \epsilon)=0$, the above is certainly true. Let us assume therefore that q_s is a point of $F_s\cdot S(p, \epsilon)$. $\delta_s=\delta(F_s)<d$. Choose two numbers α and β such that $\delta_s<\alpha<\beta<d$. $S(q_s, \alpha)\supset F_s$ and $S(p, \epsilon)-\overline{S(q_s, \alpha)}\supset z^{n-r-1}$. We may assume that $r\neq 0$, since the theorem we are about to prove is trivial for the case $r=0$. We now show that z^{n-r-1} can be deformed into a point in $S(p, \epsilon)-\overline{S(q_s, \alpha)}$, from which $z^{n-r-1}\simeq 0$ in $S(p, \epsilon)-F_s$ follows. This may be done in two steps. First project z^{n-r-1} on $S(p, \epsilon)\cdot B(S(q_s, \beta))$, with center of projection q_s. Denote the projection of z^{n-r-1} by $z_{1/2}^{n-r-1}$. Since $S(p, \epsilon)\cdot B(S(q_s, \beta))$ is either equal to $B(S(q_s, \beta))$ or homeomorphic to a hemisphere of $B(S(q_s, \beta))$, our second step, deforming $z_{1/2}^{n-r-1}$ into a point is possible (note $n-r-1\neq n-1$). The projection can be considered a deformation with the parameter varying from 0 to $\frac{1}{2}$ as z^{n-r-1} moves to $z_{1/2}^{n-r-1}$ and from $\frac{1}{2}$ to 1 in the second step.

We can therefore say that if a set F is of dimension r, there exists a number d such that any decomposition of F into a countable infinity of closed sets of diameter less than d has the property that the intersection of at least one pair is of dimension at least $r-1$. Conversely, if a closed set F in R^n can be ϵ-decomposed into a countable infinity of closed sets with the intersection of any pair of dimension at most $r-1$, and if ϵ is arbitrary, then the dimension of F is at most r. From the fact that any closed set of dimension r can be decomposed into a countable infinity of closed sets of diameter less than any preassigned ϵ, whose intersections, taken pairwise are of dimension at most $r-1$ (if the set is compact, these may even be taken to be finite in number) we can state the following theorem:

THEOREM 5. *A closed subset F of R^n is of dimension r if and only if there exists an $\epsilon>0$, such that F may be decomposed into the sum of a countable infinity of closed sets $F_1, F_2, \cdots, F_s, \cdots$, of diameter less than ϵ with* $\dim F_i\cdot F_j\leqq r-1$, *$i\neq j$, but for any such decomposition there exists a pair of integers m and n such that* $\dim F_m\cdot F_n=r-1$.

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