CONCERNING LIMITING SETS IN ABSTRACT SPACES, II*

BY

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In his first paper on limiting sets† the author considered the distributive property in connection with metric spaces. In this paper we consider the property in connection with more general spaces and show that it and weak additional hypotheses imply that every uncountable point set in the space under consideration (1) is α-compact in itself‡ and (2) is separable. It is well known that in a metric space properties (1), (2), and the following, (3), are equivalent: (3) Every point set has the Lindelöf property. Sierpiński has shown that (2) and (3) are independent in a space S.§ In consideration of Sierpiński's result an equivalence involving these properties as stated in Theorem 7 is of considerable interest and is used in showing that (2) holds in Hausdorff space.

Above we discussed certain properties that hold "im grossen." With the help of the first countability axiom, or a more general hypothesis concerning

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† These Transactions, vol. 30 (1928), pp. 668–685. In a topological space the *limiting set* of an aggregate G of sets is the set of all points P of the space such that every neighborhood of P contains points in common with infinitely many distinct elements of G. The elements of G are understood to be sets $h(\alpha, g)$, where α is a number and g is a point set in the space; for the case $\alpha=0$ let $h(\alpha, g)$ be g, and for other values of α let the elements of $h(\alpha, g)$ be α and the points of g. Thus, we may refer to an element $h(\alpha, g)$ of G as a point set if $\alpha=0$. A topological space is said to have the *distributive property* provided that if in that space K is a closed point set, G is a collection of sets, and if each point of K belongs to some subset of K which is the limiting set of a sub-collection of G, then K itself is the limiting set of a sub-collection of G.

For well known definitions and for general information about topological spaces see: M. Fréchet, *Les Espaces Abstraits et leur Théorie Considérée comme Introduction à l'Analyse Générale*, 1928; F. Hausdorff, (I) *Grundzüge der Mengenlehre*, 1914, and (II) *Mengenlehre*, 1927; K. Menger, (I) *Dimensionstheorie*, 1928, and (II) *Kurventheorie*, 1932; R. L. Moore, *Foundations of Point Set Theory*, American Mathematical Society Colloquium Publications, vol. 13, 1932; C. Kuratowski, *Topologie*, 1933; W. Sierpiński, (I) *Introduction to General Topology*, 1934, translated by C. C. Krieger; and P. Alexandroff and H. Hopf, *Topologie*, 1935.

‡ A space or a point set is *α-compact* (*in itself*) provided that every uncountable point set in it has a limit point (contains a limit point of itself). Cf. W. Gross, *Zur Theorie der Mengen in denen ein Distanzbegriff definiert ist*, Sitzungsberichte der Kaiserlichen Akademie der Wissenschaften, part IIa, vol. 123 (1914), p. 805.

§ Cf. W. Sierpiński, (II) *Sur l'équivalence de trois propriétés des ensembles abstraits*, Fundamenta Mathematicae, vol. 2 (1921), pp. 179–188; C. Kuratowski and W. Sierpiński, *Le théorème de Borel-Lebesgue dans la théorie des ensembles abstraits*, Fundamenta Mathematicae, vol. 2 (1921), pp. 172–178.

monotonic families of neighborhoods* the distributive property gives also local compactness and regularity. The author has been unable to prove that the Lindelöf property is among these necessary conditions; if it is, it is possible to state simple necessary and sufficient conditions for the distributive property (see Theorems 14 and 16).

LEMMA I. *In a Fréchet space H in order that every point set be separable, it is necessary and sufficient that* (1) *every closed set be separable and* (2) *if a point is a limit point of a point set, it is a limit point of a countable subset of the point set.*

LEMMA II. *In a Fréchet space V every monotonic family of neighborhoods of a point contains a sub-collection which is a well-ordered monotonic descending family of neighborhoods of the point.*

THEOREM 1. *A space satisfies the first countability axiom if each point in it has a monotonic family of neighborhoods and one of the following holds*: (A) *The space is a Hausdorff space in which every point set is α-compact in itself*; (B) *the space is a space H in which a point is a limit point of a point set if and only if it is a limit point of a countable subset of the point set.*

Proof. Consider first case (A). Let S be the set of all points in the space, P be a point in it, H be a well-ordered monotonic descending family of neighborhoods of P, and K be a well-ordering of the points of $S-P$. Let U_1 be an element of H, P_1 be the first point of K in U_1, and V_1 the first element in H which is a subset of U_1 and of which P_1 is not a point or a boundary point. Suppose that P is not an isolated point of the space. Suppose that U_x, P_x, and V_x have been defined for each ordinal x less than a definite ordinal α. Provided that there exist elements of H common to all V_x's for $x<\alpha$ we shall define U_α, P_α, and V_α as follows: U_α is the first element of H common to all the V_x's for $x<\alpha$; P_α is the first element of K in U_α; V_α is the first element of H which is a subset of U_α and of which P_α is not a point or limit point. Let G be the well-ordered sequence $(V_1, V_2, V_3, \cdots, V_\omega, \cdots, V_\alpha, \cdots)$, where α is such that U_α, P_α, and V_α exist; and let $E=(P_1, P_2, P_3, \cdots, P_\omega, \cdots, P_\alpha, \cdots)$. Clearly each point of E is an isolated point of E. It follows from our condition that E, and hence G, has each a finite or a countable number of elements.

We shall show that each element of H contains an element of G. Suppose this were not true; then, since H is monotonic, it would contain an element U

* A *complete* family of neighborhoods of a point is one that defines the operation of derivation at that point; cf. Fréchet, pp. 172–173. Such a family is *monotonic* provided that every pair of its elements has the property that one is a subset of the other, and said to be *monotonic descending* with reference to a definite ordering provided that if one element precedes another, the first contains the second. In connection with a space H Fréchet we shall consider the term neighborhood as equivalent to the term "open set"; cf. Fréchet, pp. 186–187.

which is a subset of all elements of G. There exists a first ordinal λ which is greater than all ordinals x such that there exists an element V_x of G. Since H is a well-ordered sequence, there exists a first one of its elements that is common to all elements of G, and this element is by definition U_λ. Then the first point of K in U_λ is P_λ. There exist open sets R_1 and R_2 containing P_λ and P respectively and having no common points. Then P is not a point or limit point of $S-U_\lambda$ or of $S-R_2$; hence, there exists a first element of H that is a subset of U_λ and does not have P_λ on its boundary. This element is by definition V_λ. But, this is contary to the definition of λ. Thus, every element of H contains an element of G; the converse is true. Since H is a family of neighborhoods of P, so is G.†

Consider next case (B). Let P be a limit point of the set of distinct points, $E=(P_1, P_2, P_3, \cdots)$, none of which is P, and let $H=[W]$ be a family of neighborhoods of P. For each n let W_n be an element of H containing no point of $P_1+P_2+P_3+\cdots+P_n$, and $G=(W_1, W_2, W_3, \cdots)$. Then each element of H contains an element of G; for, if some element U of H did not contain an element of G, it would be a subset of every element of G. Hence, if n is any integer, U is a subset of W_{n+1} and does not contain P_n. But this involves a contradiction, since P is a limit point of E. Thus every element of H contains an element of G, and conversely. It follows that G is a family of neighborhoods of P.†

THEOREM 2. *A locally compact Hausdorff space which has the Lindelöf property satisfies the first countability axiom.*

Proof. Let P be a point of our space T. For each point Q of $T-P$ let U_Q and V_Q be mutually exclusive open sets containing Q and P respectively. Then $T-P$ may be covered by a countable sequence $(U_{Q_1}, U_{Q_2}, U_{Q_3}, \cdots)$ of the elements of $[U_Q]$. Let R be an open set containing P such that $\overline{R}$ is compact; let $W_n=R\cdot(T-\sum_{i=1}^{i=n}\overline{U}_{Q_i})$; and let $F=(W_1, W_2, W_3, \cdots)$. Then F is a monotonic descending family of neighborhoods of P.

For, let M be a point set having points distinct from P in every element of F. It may be shown that M has a subset $N=(P_1, P_2, P_3, \cdots)$ of distinct points such that for $P_n \epsilon W_n$ each n. Since $R \supset W_n$, $\overline{W}_n$ is compact; hence N has a limit point X which is a point of $\overline{W}_1\cdot\overline{W}_2\cdot\overline{W}_3\cdot\cdots$. If X were a point of $T-P$, there would be an integer n such that X belongs to U_{Q_n}. Since $\overline{W}_n$ contains no point of U_{Q_n} we are involved in a contradiction.

Conversely if P is a limit point of a point set K, every element of F contains a point of K distinct from P.

LEMMA III. *In a space H the limiting set of a collection of point sets is closed.*

† Cf. Fréchet, p. 173.

THEOREM 3. *Every regular space H which satisfies the first countability axiom and has the distributive property is locally compact.*

This theorem may be proved by methods analogous to those used in the proof of Theorem 8 of the author's first paper, p. 677.

THEOREM 4. *Every space V which has the distributive property is α-compact.*

Proof. Suppose there exists a space which satisfies the hypothesis, but contains an uncountable point set M whose derived set is vacuous. Let K be a countable subset of M, $N=M-K$, and $P_1, P_2, P_3, \cdots$ be points of K. For each point x of N and each positive integer n let $g_{xn}=x+P_n$. Let G^* be the aggregate $[g_{xn}]$. For each point x of N there exists a neighborhood R_x of x which contains no point of $M-x$. Hence x is the limiting set of the aggregate $G_x^*=(g_{x1}, g_{x2}, g_{x3}, \cdots)$. Since N has no limit point, it is closed. It follows from our hypothesis that G^* contains a sub-collection G whose limiting set is N. Let $G_x=G\cdot G_x^*$. Since R_x contains no point of any element of $G-G_x$, G_x contains infinitely many distinct elements. Since N is an uncountable point set, and an element of G_y contains the point z of N only if $y=z$, G has uncountably many elements. Hence there exists an integer m such that P_m is common to infinitely many elements of G. This involves a contradiction with the fact that N is the limiting set of G.

THEOREM 4A. *In order that a metric space should have the distributive property, it is necessary and sufficient that it be locally compact and separable.*

This theorem is a consequence of Theorem 4, Gross, loc. cit., pp. 805–806, and Theorems 8 and 9 of the authors first paper, pp. 677–678.

A space S_1 is said to be a *sub-space* of a space S_2 provided that (1) every point of S_1 is a point of S_2, and (2) if P is an arbitrary point and M an arbitrary point set in S_1 then P is a limit point of M in S_1 if and only if it is a limit point of M in S_2.

LEMMA IV. *Every subspace of a space S is a space S, and every subspace of a space H is a space H.*

THEOREM 5. *If a space S has the distributive property, then every regular, locally compact subspace of it has this property.*

Proof. Let S_1 be a space S having the distributive property, and let T be a regular, locally compact sub-space of it. In T let K be a closed point set and G be a collection of sets such that each point P of K belongs to a subset K_P of K which is the limiting set of a sub-collction G_P of G. Let L_P be the limiting set of G_P with respect to the space S_1. Then $K_P=L_P\cdot M$, where M is the set of all points belonging to T. Let N be the sum of all point sets L_P, where

the range of P is K; N' be the derived set of N with reference to the space S_1; and $\overline{N}=N+N'$. It follows from the definition of N that $N\cdot M=K$. Suppose that $N'\cdot M$ contains a point Q which does not belong to K. Since K is a closed point set with respect to T, Q is not a limit point of K in either T or S_1. Hence, there exists in T an open set R_1 containing Q such that if $\overline{R}_{1T}$ denotes the sum of R_1 and its limit points in T, then $K\cdot\overline{R}_{1T}$ is vacuous. Since T is locally compact, there exists in it an open set R_2 containing Q such that $\overline{R}_{2T}$ is compact. Let $R_3=R_1\cdot R_2$. Then R_3 is an open set in T and contains Q. Also $\overline{R}_{1T}\supset\overline{R}_{3T}$. It follows that Q is not a limit point of $M-R_3$ in the space S_1. Hence, there exists in S_1 an open set U which contains Q but contains no point of $M-R_3$. Since Q is a limit point of N, there exists in K a point x such that U contains a point y of L_x. Then U must contain points of infinitely many elements of G_x. Since the elements of G_x are subsets of M, and $U\cdot M$ is a subset of R_3, R_3 contains points of infinitely many elements of G_x. Since $\overline{R}_{3T}$ is compact, it must contain a point W which belongs to the limiting set of G_x; since $\overline{R}_{1T}\supset\overline{R}_{3T}$, the latter contains no point of K. This involves a contradiction with the fact that $K\supset K_x$. Hence the point Q does not exist, and $\overline{N}\cdot M=K$.

For each point x of $\overline{N}-K$ let $h_{1x}, h_{2x}, h_{3x}, \cdots$ be the pairs $(x, 1)$, $(x, 2)$, $(x, 3), \cdots$. Let H be the aggregate $[h_{ix}]$, where the range of i is the set of positive integers, and that of x is the point set $\overline{N}-K$. Since S_1 is a space S, $\overline{N}$ is closed, and for every point P of K the following holds: $\overline{N}\supset N\supset L_P$. It follows that each point of $\overline{N}$ belongs to a subset of $\overline{N}$ which is the limiting set in S_1 of a sub-collection of $G+H$. Since S_1 has the distributive property, $G+H$ has a sub-collection G_1+H_1 such that $\overline{N}$ is the limiting set of G_1+H_1 in S_1 and such that G_1 and H_1 respectively are sub-collections of G and H respectively.

Suppose that K is not a subset of K_1, where K_1 is the limiting set of G_1 in S_1. Then K must contain a point E which belongs to the limiting set of H_1. Let R_E be an open set in T containing E such that $\overline{R}_{ET}$ is compact. It follows from an analogous situation above that in S_1 the point E is not a limit point of $M-R_E$ and that there exists in S_1 an open set U_E which contains no point of $M-R_E$. Then U_E contains a point X of an element of H_1. Let B denote the limiting set of G in S_1. Since $B\supset N$, and B is closed, $B\supset\overline{N}$. Since X is a point of $\overline{N}-K=\overline{N}-\overline{N}\cdot M$, and all elements of G are sub-sets of M, X is the unique limit point in S_1 of an infinite subset of M, and U_E, which contains X, contains such a set E_X. Then $R_E\supset M\cdot U_E\supset E_X$. Since X is not a point of T, E_X has no limit point in T. This involves a contradiction with the fact that $\overline{R}_{ET}$ is a compact subset of T. Hence $K_1\supset K$. Since $M\supset K$, it follows that $M\cdot K_1\supset K$.

Since $\overline{N}\supset K_1$, $K=\overline{N}\cdot M\supset K_1\cdot M$. From $K\supset K_1\cdot M$ and $K_1\cdot M\supset K$, it fol-

lows that $K = K_1 \cdot M$. Hence K is the limiting set of G_1 in T. Thus we have shown that T has the distributive property.

A space H is said to be *nearly a space L* provided that if in that space P is any point, M is any point set, and P is a limit point of M, then P is the derived set of a subset of M.

THEOREM 5A. *Every regular, locally compact subspace of a space H which has the distributive property and is nearly a space L has the distributive property.*

The proof is the same as that for Theorem 5.

THEOREM 6. *In a space H which has the distributive property every point set is α-compact in itself.*

Proof. Suppose the theorem is not true and that S_1 is a space H which has the distributive property but contains an uncountable point set M which contains no limit point of itself. Let T be the subspace of S_1 whose points are the points of M. To show that T has the distributive property adopt the notation of the proof of Theorem 5 and follow this proof to the place where the existence of the collections G_1 and H_1 is established, and suppose as there that K contains a point E not belonging to the limiting set of G_1. Define M' as the derived set of M in S_1. Since the points of elements of H_1 are points of $\overline{N} - K$, it follows that E is a limit point of $M' - M \cdot M'$ in S_1. Since S_1 is a space H, derived sets in it are closed, and E is a point of M' and hence a limit point of M. Since E is a limit point of M, we are involved in a contradiction. Thus E does not exist, and the argument of Theorem 5 shows that T has the distributive property.

By Theorem 4 the set M must have a limit point in T. But this again is contrary to the definition of M. Thus the supposition that the theorem is not true leads to a contradiction.

Note. When the space of our hypothesis is a space S the proof may be simplified. Let S_1 be our space and define M and T as above; then T is a regular, locally compact subspace of S_1, since all its points are isolated. By Theorem 5, T has the distributive property, and by Theorem 4, we are involved in a contradiction.

THEOREM 7. *In order that in a space H each point set either be condensed in itself or be separable, it is necessary and sufficient that every point set be α-compact in itself.*†

Proof. Obviously the condition is necessary. Suppose that it is not sufficient, and that the space contains a point set E which neither contains a con-

† In part our proof of Theorem 7 follows methods used by Sierpiński; cf. Sierpiński (II). See also the introduction for a discussion of the relation of Theorem 7 to some results by Sierpiński and Kuratowski.

densation point of itself nor is separable. Then, for each point P of E there exists a countable subset $D(P)$ of E such that P is not a point or a limit point of $E-D(P)$. Let T be a well-ordered sequence $(p_1, p_2, p_3, \cdots, p_\omega, p_{\omega+1}, \cdots, p_\alpha, \cdots)$ of the points of E. We shall now define a sequence of the type δ, where δ is the smallest transfinite ordinal of the third class; $U=(q_1, q_2, q_3, \cdots q_\omega, q_{\omega+1}, \cdots, q_\beta, \cdots)$, where $\beta<\delta$.

Proceed as follows: Let $q_1=p_1$. Let β be a definite ordinal less than δ. Suppose that q_x has been defined for all ordinals $x<\beta$, and let U_β be the set of all q_x's for such x's. Let S_β be the sum of all point sets $D(q_y)$, where q_y is an element of U_β. Let q_β be the first point of T which is not a point or a limit point of S_β.

We shall now show that q_β exists for every ordinal β less than δ. For, if q_β does not exist for all such ordinals β, there must be a first such ordinal λ for which it does not exist. Then it follows from our definitions that each point of E is either a point or a limit point of S_λ. But S_λ is the sum of all point sets $D(q_z)$, where q_z ranges over U_λ; thus S_λ is the sum of a countable number of countable sets and is countable; then E is separable. Thus, the supposition that λ exists leads to a contradiction.

Next we shall show that q_β is an isolated point of U. By definition q_β is not a point or a limit point of U_β. Further, $S_{\beta+1}$, which contains $D(q_\beta)$, contains no point of $U-U_{\beta+1}$. Since $U-q_\beta=U_\beta+(U-U_{\beta+1})$, it follows that q_β is an isolated point of U.

Thus, every point of the uncountable sequence U is an isolated point of U. By our hypothesis, however, U must contain a limit point of itself. Thus, the supposition that our condition is not sufficient has led to a contradiction.

THEOREM 8. *In order that for each infinite collection of point sets in a space H it be true that at most a countable number of its elements fail to be subsets of its limiting set, it is necessary and sufficient that every point set in the space be α-compact in itself.*

Proof. We shall first show that the condition is sufficient. Suppose that it is not and that there exists in our space a point set K and a collection G of point sets such that K is the limiting set of G, but that G contains an uncountable sub-collection G_1, none of whose elements are subsets of K. For each element g of G_1 let P_g be a point which does not belong to K. It follows by our hypothesis that the set $[P_g]$ *contains* a point Q, every neighborhood of which contains infinitely many elements of $[P_g]$. Then Q belongs to the limiting set of G, and we are involved in a contradiction.

Conversely, let M be an uncountable point set in a space which satisfies our condition. Let G_2 be a collection of point sets whose elements are the

points of M, no two elements being the same point. Then M' is the limiting set of G, and M' and M have uncountably many points in common. Thus, the condition is necessary.

Theorems 8 and 9 are generalizations of Theorems 2 and 4, respectively, of our first paper, and are of interest in connection with Theorem 7, and also with Theorems 6 and 10A, in that they indicate consequences of the distributive property.

THEOREM 9. *In order that for a space H every infinite collection of sets should contain a countable sub-collection having the same limiting set as the collection itself, it is necessary and sufficient that every point set in the space be separable.*

Proof. To prove the necessity of the condition proceed as follows: Let N be a point set and $M=\overline{N}$. Let G be a collection such that for each point x of N there exists a collection of elements of G, $g_{1x}, g_{2x}, g_{3x}, \cdots$, where g_{nx} is the pair (n, x). Now proceed by methods analogous to those used in the proof of Theorem 4 of the author's first paper.

Consider next the sufficiency. Let G be a collection of sets and K be the limiting set of G. By our condition K contains a countable subset $N=P_1+P_2+P_3+\cdots$ such that $\overline{N}=K$. Then *for each n the point P_n belongs to the limiting set of some countable sub-collection of G.* Suppose that for some definite $P_j=Q$ this is not true. Let g_0 be a definite element of G. For n greater than zero suppose that g_k has been defined for $k<n$. Let $G_n=G-(g_0+g_1+g_2+\cdots+g_{n-1})$; let H_n be the sum of all elements of G_n; and let $F_n=Q_1+Q_2+Q_3+\cdots$ be a countable set such that $\overline{F}_n \supset H_n \supset F_n$. Then Q is a point of F_n'.

For, let R be an open set containing Q. Since Q belongs to the limiting set of G, and hence of G_n, R contains points of infinitely many elements of G_n. If Q were common to infinitely many elements of G_n, it would be common to a countable infinity of such elements and would belong to the limiting set of this countable collection; this, however, is contrary to the definition of Q. Thus R contains points of H_n distinct from Q, and hence points of F_n. Thus $Q \epsilon F_n'$. For each positive integer k let t_k denote a definite element of G_n that contains Q_k. Let T denote the aggregate $(t_1, t_2, t_3, \cdots)$. Since T has a finite or a countable number of elements, Q does not belong to its limiting set. Hence, there must exist an open set U containing Q which contains points of at most a finite number of the elements of T, say of $t_{k_1}, t_{k_2}, \cdots, t_{k_j}$. Then $U\cdot F_n=\sum_{i=1}^{i=j} U\cdot F_n\cdot t_{k_i}$. Since Q belongs to the derived set of $U\cdot F_n$, it follows that for some i, $0<i<j+1$, Q is a limit point of $U\cdot F_n\cdot t_{k_i}$, that is of t_{k_i}. Let g_n be defined as the sum of n and such a t_{k_i}, and $E=(g_1, g_2, g_3, \cdots)$. Then Q is a limit point of each element of E, and hence belongs to the limiting set of E.

Thus, each point P_n of N is a point of the limiting set of some countable sub-collection M_n of G. Let $M=\sum_{t=1}^{t=\infty} M_n$, and let L be the limiting set of M. Then $K=\overline{N}\supset L\supset N$. Since in a space H derived sets are closed, it follows that $L=\overline{N}=K$. But, M is a countable sub-collection of G.

THEOREM 10. *In a Hausdorff space having the distributive property every closed point set is separable.*

Proof. Suppose that a space S_1 satisfies the hypothesis but not the conclusion of our theorem. Then there exists in it an uncountable, closed, non-separable point set E. By Theorems 6 and 7 the set E contains a point of condensation of itself Q; similarly $E-Q$ contains a point of condensation of itself P. Let U and V be mutually exclusive open sets containing Q and P, respectively. Let $M=U\cdot E$, $N=V\cdot E$, and $H=[S_1-(U+V)]\cdot E$. Then one of the three point sets M, N, or H is non-separable; for otherwise E, their sum, would be separable. Consider the two cases: (I) Either M or N is not separable; (II) both M and N are separable. Consider first case (I) and suppose that it is N that is not separable. Let $K=E-\overline{M}$; then $H+N=E-M\supset K\supset N$, and K is not separable. For, suppose that K is separable and has a countable subset K_1 such that $\overline{K}_1\supset K$. By definition H is the product of the two closed sets E and $S_1-(U+V)$, and thus is closed. It follows that every point of $N=K-K\cdot H$ is either a point or a limit point of $N\cdot K_1$. Thus, the supposition that K is separable, involves a contradiction with the hypothesis that N is not separable.

We shall now define certain sequences by an induction process. Let z_0 be a point of $M-Q$, $R_0=z_0$, and $U_0=U$. Now suppose that U_k, R_k, and z_k have been defined for all non-negative integers k less than the definite integer n. Let U_n be an open set containing Q such that $U_{n-1}-\overline{R}_{n-1}\cdot U_{n-1}\supset U_n$, let z_n be a point of $M\cdot(U_n-Q)$, and let R_n be an open set containing z_n such that $\overline{U}_n-Q\supset\overline{R}_n$. Let $F=z_1+z_2+z_3+\cdots$. The existence of U_n, z_n, and R_n for every positive integer n may be shown by making use, in particular, of Hausdorff's Axiom D. Since the open sets $R_1, R_2, R_3, \cdots$ are mutually exclusive, it follows that $F\cdot F'$ is vacuous.

For each point t of $\overline{K}$ and each positive integer n let $g_{tn}=t+z_n$, and G be the aggregate of all such g_{tn}'s. Since $z_1, z_2, z_3, \cdots$ are distinct points, the limiting set of the aggregate $(g_{t1}, g_{t2}, g_{t3}, \cdots)$ is $t+F'$. Thus, each point of $\overline{K}+F'$ belongs to a subset of $\overline{K}+F'$ which is the limiting set of a sub-collection of G. Hence, G has a sub-collection G_1 whose limiting set is $\overline{K}+F'$. Let W be the product of $\overline{K}$ and the sum of the elements of G_1. Since $E-\overline{M}=K$, and $M\supset F$, it follows that $K\cdot\overline{F}$ is vacuous. Then every point of K is a point or a limit point of W, and so is every point of $\overline{K}$. If the elements of G_1 were count-

able, so would be the points of W; then $\overline{K}$ would be separable. This is impossible, since K is not separable, and no point of K is a limit point of $\overline{K} \quad K$. Hence, there are uncountably many elements of G_1, and there must exist an integer j such that z_j belongs to uncountably many elements of G_1. Then z_j belongs to the limiting set of G_1, that is to $\overline{K}+F'$. But, z_j belongs to neither $\overline{K}$ nor F'. Thus, case (I) involves a contradiction.

Consider next case (II). Since both M and N are separable, so are $\overline{M}$ and $\overline{N}$. Let $K=E-(\overline{M}+\overline{N})$. Then K is not separable. Define first F and then G precisely as in the proof of case (I); by following this proof we again arrive at a contradiction. Thus, the supposition that the theorem is not true is untenable.

THEOREM 10A. *If a Hausdorff space is nearly a space L and has the distributive property, every point set in it is separable.*

This is a consequence of Theorems 10 and 6 and Lemma I.

THEOREM 11. *A space H which satisfies the first countability axiom is a Hausdorff space if and only if it is a space S.*†

THEOREM 12. *A locally compact space S (Hausdorff space) which satisfies the first countability axiom is regular.*‡

THEOREM 12A. *If a Hausdorff space is locally compact at one of its points P, and P has a countable family of neighborhoods, the space is regular at P.*

THEOREM 13. *A space S (Hausdorff space) which has the distributive property and satisfies the first countability axiom is regular.*

Proof. Suppose that S_1 is a space S which satisfies the hypothesis of the theorem but contains a point P at which it is not regular. Then there exists in S_1 an open set R containing P such that if R_i is any open set whatever containing P, then $\overline{R}_i$ is not a subset of R. Let $(V_1, V_2, V_3, \cdots)$ be a countable family of neighborhoods containing P such that $R \supset V_1 \supset V_2 \supset V_3 \supset \cdots$. Let $U_1=V_1$, $m_1=1$, and M_1 be a countable subset of V_1-P which has a unique limit point in $R'-R$, say P_1. By an induction process we shall now define U_n, M_n, m_n, and P_n for every positive integer n. Proceed as follows: Suppose they have been defined for all n's less than a definite integer k. Let m_k be the first integer greater than m_{k-1} such that V_{m_k} contains no point of $\sum_{j=1}^{j=k-1} M_j$, let $U_k=V_{m_k}$, and let M_k be a compact countable sequence of points belonging to U_k-P and having a unique limit point P_k belonging to $R'-R$.

† Cf. Hausdorff (I), pp. 263–265, and Fréchet, *Démonstration de quelques propriétés des ensembles abstraits*, American Journal of Mathematics, vol. 50 (1928), p. 65.

‡ Cf. Alexandroff and Urysohn, *Mémoire sur les espaces topologique compacts*, Verhandelingen, Koninklijke Akademie van Wetenschappen, Amsterdam, vol. 14 (1929), pp. 28–29.

We shall now prove that U_k, M_k, m_k, and P_k exist for all positive integers k; the argument suggests, in particular, the proof for the case $k=1$. Suppose that our proposition has been established for all n's less than k, where $1<k$. Then each of the sequences $M_1, M_2, M_3, \cdots, M_{k-1}$ has exactly one limit point; the derived set of their sum is $P_1+P_2+\cdots+P_{k-1}$, which is a subset of $R'-R$. Then P is not a limit point of the closed point set $\sum_{j=1}^{j=k-1} M_j$, and there exists an integer m_k greater than m_{k-1} such that V_{m_k} contains no point of this point set. Hence U_k exists. It follows from the definition of R that $\bar{U}_k$ is not a subset of R, and that U_k has a limit point P_k in $R'-R$. Since our space is a space S, U_k-P has a countable subset M_k such that P_k is the unique limit point of every infinite subset of M_k. Our existence theorem may thus be established by mathematical induction.

The sequence $P_1, P_2, P_3, \cdots$ contains a sub-sequence $P_{n_1}, P_{n_2}, P_{n_3}, \cdots$, having not more than one limit point, such that $n_1<n_2<n_3<\cdots$. If this sub-sequence has a limiting set, let it be denoted by the symbol Q; otherwise, let Q be the null set. Let $P_{n_k}=Q_k$; $M_{n_k}=N_k$; let $O_{1k}, O_{2k}, O_{3k}, \cdots$ be the points of N_k; $g_{jk}=Q+O_{k1}+O_{jk}$; let G_k be the sequence $(g_{1k}, g_{2k}, g_{3k}, \cdots)$; $G^*=\sum_{j=1}^{j=\infty} G_j$; and $K=Q+N_1+\sum_{k=1}^{k=\infty} Q_k$. The limiting set of G_k is $Q+O_{k1}+Q_k$. Since K is closed and the space has the distributive property, G^* contains a sub-collection G whose limiting set is K. Suppose that G contains elements in common with at most a finite number of the elements of the aggregates $G_1, G_2, G_3, \cdots$, say with those having subscripts not greater than a definite integer t. Then, contrary to the fact that K contains infinitely many distinct points, the limiting set of G is a subset of $Q+\sum_{k=1}^{k=t}(O_{k1}+Q_k)$. Hence, for infinitely many values of k there exist elements of G which contain points of N_k. Thus, every element of $(V_1, V_2, V_3, \cdots)$ contains points in common with infinitely many distinct elements of G, and P belongs to the limiting set of G. Thus, the supposition that our space is not regular has led to a contradiction.

THEOREM 13A. *If a space S has the distributive property and a point P in it has a countable family of neighborhoods, the space is regular at P.*

This theorem may be proved by the methods used for Theorem 13. It follows by Theorem 3 that the space is locally compact at P.

Note. The statements of Theorems 12 and 13 differ only in that in the hypothesis of the one the distributive property takes the place of local compactness in that of the other; they are stated for both spaces S and Hausdorff spaces. Theorems 12A and 13A are analogous generalizations of Theorems 12 and 13 respectively; but 12A is stated for a Hausdorff space, while 13A is stated for a space S. The question arises as to whether each of the Theorems 12A and 13A hold for both types of spaces. The author has not found the

answer for the case of 13A; for 12A the answer is in the negative, as may be seen by considering a space T, when points and limit points are those of $P+K+\sum_{i=1}^{i=\infty} N_i$ of the proof of Theorem 13.

THEOREM 14. *If a Hausdorff space or a space S has the distributive property and each point in it has a monotonic family of neighborhoods, then the following properties hold for the space*: (1) *It satisfies the first countability axiom*; (2) *it is both a Hausdorff space and a space S*; (3) *every point set in it is separable*; *and* (4) *it is regular and locally compact.*

Proof. By Theorems 6 and 1 our space satisfies the first countability axiom; by Theorem 11 it is both a space S and a Hausdorff space; by Theorem 10 and Lemma I every point set in it is separable; by Theorem 13 it is regular; and by Theorem 3 it is locally compact.

Note. Theorems 14 and 15 may be regarded as a summary of results of this paper with regard to conditions necessary for the distributive property. Theorem 16 deals with sufficient conditions; Theorems 15 and 17 with necessary and sufficient conditions.

THEOREM 15. *If a space S (Hausdorff space) satisfies the first countability axiom and has the distributive property, then in order that one of its sub-spaces have the distributive property it is necessary and sufficient that the sub-space be regular and locally compact.*

Proof. The necessity follows from Theorem 14; the sufficiency from Theorem 5.

THEOREM 16. *A sufficient condition that a Hausdorff space have the distributive property is that it be locally compact and have the Lindelöf property, and that every closed point set in it be separable.*

Proof. By Theorem 2 our space satisfies the first countability axiom; by Theorem 12 it is regular; by Theorem 11 and Lemma I every point set in it is separable. The proof may be completed by following the methods for Theorem 9 of the author's first paper, p. 678. Theorem 17 follows from Theorems 14 and 16.

THEOREM 17. *In order that a Hausdorff space which has the Lindelöf property and in which every point has a monotonic family of neighborhoods should have the distributive property, it is necessary and sufficient that the space be locally compact and that every closed set in it be separable.*

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