RATIONAL EQUIVALENCE OF A FORM TO A SUM OF pTH POWERS*

BY

RUFUS OLDENBURGER

1. **Introduction.** In my earlier papers on p-way matrices and associated forms, I introduced new rank concepts of higher dimensional matrices with the aim of obtaining a fairly general theory of equivalence of ordinary forms and multilinear forms of arbitrary degree p, where this theory contains as a special case the well known theory of equivalence of quadratic and bilinear forms under non-singular linear transformations in a given field. In a paper in these Transactions in 1936 I gave such a development for multilinear forms. The present paper is devoted to a theory of equivalence of ordinary forms. Specifically, the problem of determining necessary and sufficient conditions for the equivalence of ordinary forms and multilinear forms of arbitrary degree to forms with "diagonal matrices" is solved in these two papers for the class of non-singular transformations in a field ϕ, where ϕ is subject to slight restrictions in the case of ordinary forms.

Throughout the present paper *we shall use the expression "a sum of pth powers" to denote a sum of the form*

$$\lambda_1 x_1^p + \cdots + \lambda_\sigma x_\sigma^p,$$

where $\lambda_1, \cdots, \lambda_\sigma$ are constant elements of a given field; the coefficients $\lambda_1, \cdots, \lambda_\sigma$ *need not be unity.*

Let $F = a_{ij\cdots m} x_i x_j \cdots x_m$ be a form of degree greater than or equal to 2 with coefficients in a field ϕ, where the matrix $A = (a_{ij\cdots m})$ of F is symmetric; that is, $a_{1\cdots 12} = a_{1\cdots 121} = \cdots = a_{21\cdots 1}$; $a_{1\cdots 122} = a_{1\cdots 1212} = \cdots = a_{221\cdots 1}$, for example. Let G denote a grouping of the indices of F into two classes P_1, P_2 of partitions (multipartite indices), where P_1 contains an even number, greater than or equal to 2, of partitions. Since the partitions in P_1 and P_2 play different roles in the general theory of forms, the partitions in P_1 are said to be *signant* while those of P_2 are *non-signant.* To every G there corresponds uniquely a non-negative integer D_G defined elsewhere in terms of generalized determinants associated with F. The integer D_G is not changed if ϕ is replaced by a field ψ which contains ϕ. To every grouping g of the indices

* Presented to the Society, April 10, 1936 and April 9, 1937; presented to the International Mathematical Congress, July 18, 1936; received by the editors August 19, 1937.

of F into at least two partitions there corresponds a unique non-negative integer F_g defined in terms of factorization properties of the matrix A associated with F. The integer F_g depends on the field ϕ in which the coefficients of F are embedded. The ranks of types D_G and F_g are called determinant ranks and factorization ranks, respectively. Under non-singular linear transformations $x_i = b_{i\alpha}x'_\alpha$, $x_j = b_{j\beta}x'_\beta$, $\cdots$, $x_m = b_{m\gamma}x'_\gamma$ on F these ranks are invariant.*

To give the theory of this paper completeness, if F is a linear form $a_ix_i \not\equiv 0$, we define for F one rank D_G and one rank F_g each equal to 1.

Let π_n, $(p; r, s, \cdots, t)$, be the number of distinct permutations of p integers chosen from $1, \cdots, n$, where r are alike, s are alike, $\cdots$, t are alike, and $r+s+\cdots+t=p$; and where there are at least two numbers in the set $r, s, \cdots, t$. Evidently,

$$\pi_n(p; r, s, \cdots, t) = \frac{p!}{r!s!\cdots t!}.$$

Let k_{np} be the class of all integers π_n, $(p; r, s, \cdots, t)$, for given n, p. In the case where $p \geqq 2$ a field ϕ will be said to be (n, p)-proper if its characteristic is different from all prime factors of the numbers in the class k_{np}. Every field is said to be $(n, 1)$-proper.

The symmetric matrix $A = (a_{ij\cdots m})$ of the n-ary form $F = a_{ij\cdots m}x_ix_j\cdots x_m$ of degree p with coefficients in a field ϕ is unique if and only if the field ϕ is (n, p)-proper.

An (n, p)-proper field is evidently (m, p)-proper, where $m \leqq n$.

If F is quadratic, there is only one determinant rank and one factorization rank, and they are equal. The latter rank is trivial in this case. If F is of higher degree, the ranks of F are not always equal. A class of equality and inequality relations which exist between the ranks of F have been obtained elsewhere,† but this class is not complete.

I have proved, for example, that all determinant ranks of binary forms of any degree are equal when these ranks have exactly two partitions signant.‡

If a certain determinant rank and a certain factorization rank are equal, all of the remaining ranks of F are equal. *The leading contribution of the present paper is Theorem 2 which states that the ranks of an n-ary form F of degree p*

---

* R. Oldenburger, *Composition and rank of n-way matrices and multilinear forms*, Annals of Mathematics, vol. 35 (1934), pp. 625–636, 645–646. We shall call this paper I.

R. Oldenburger, *Non-singular multilinear forms and certain p-way matrix factorizations*, these Transactions, vol. 39 (1936), pp. 422–456, especially p. 432. We shall call this paper II.

† See paper I and its supplement, pp. 622–657.

‡ R. Oldenburger, *Relations between ranks of a general matrix*, Annals of Mathematics, vol. 39 (1938), pp. 172–178.

with respect to an (n, p)-proper field ϕ, are equal if and only if F is equivalent under non-singular linear transformations in ϕ to a form

$$C = a_1 x_1^p + \cdots + a_\sigma x_\sigma^p .$$

By the equality of ranks of a quadratic form this theorem includes the well known result* that any quadratic form is equivalent in a field of characteristic not 2 to a form C with $p=2$. Here and in what follows "*equivalent*" *will be used for "equivalent under non-singular linear transformations."*

That a multilinear form $M = a_{ij\cdots m} x_i y_j \cdots z_m$ has equal ranks for a field ϕ if and only if M is equivalent in ϕ to a form $x_1 y_1 \cdots z_1 + \cdots + x_n y_n \cdots z_n$, was noted in an earlier paper.†

With every form F given by

$$a_{ij\cdots m} x_i x_j \cdots x_m, \qquad i, j, \cdots, m = 1, \cdots, n,$$

where the matrix $(a_{ij\cdots m})$ is symmetric, we can associate a multilinear form M given by

$$a_{ij\cdots m} x_i y_j \cdots z_m,$$

and uniquely determined by F. If F is equivalent under non-singular linear transformations in an (n, p)-proper field ϕ to a sum C of n pth powers, we shall say that F is *non-singular with respect to* ϕ. Theorem II of the present paper implies that *F is non-singular with respect to an (n, p)-proper field ϕ if and only if its associated multilinear form M is non-singular with respect to ϕ* (that is, M is non-singular in the sense of paper II). We have thus succeeded in generalizing the definitions of non-singularity of quadratic and bilinear forms to general forms in such a way that the above property, obviously valid for quadratic and bilinear forms, also holds for general forms F and M. The importance of my earlier Transactions paper is now more clearly brought out. The conditions obtained there for determining whether or not a given multilinear form is non-singular may be applied here to the form M to determine whether or not F is non-singular.

Since it is not always a simple matter to determine the factorization ranks of a given form F, other conditions are obtained for the equivalence of F to a sum of pth powers. It is proved that if p is odd and greater than or equal to 3 an n-ary form F is equivalent in an (n, p)-proper field ϕ to a sum of pth powers if and only if the generalized determinant $|x_i a_{ij\cdots m}|$, with $j, \cdots, m$ signant, is the product of linearly independent linear factors in ϕ, and under

* C. C. MacDuffee, *The Theory of Matrices*, Ergebnisse der Mathematik, vol. 2, part 5, 1933.

† R. Oldenburger, II, p. 432.

reduction in ϕ to $kx_1' \cdots x_n'$ the form F reduces covariantly to C. If p is even and greater than or equal to 4, the same statement applies except that $|x_i a_{ij\cdots m}|$ is replaced by $|x_i x_j a_{ijk\cdots m}|$, with $k, \cdots, m$ signant, and this determinant is a product of squares of linearly independent linear factors. Hočevar* sketched part of a proof of the property that a form factors into linear factors in the complex field if and only if it divides every third order minor of its Hessian, and he gave a method of finding these factors which involves, besides simply performed algebraic manipulations, only the solution of an algebraic equation E with an inequality side condition. Although the fact is not mentioned by Hočevar, this proof is valid only if the form has no repeated factors involving variables. Hence, assuming that the roots of E have been found subject to the side condition, we can determine directly the equivalence of a form to a sum of pth powers in the complex field in a finite number of steps. In this sense *the problem of determining whether or not a given form F is equivalent to a sum of pth powers is completely solved for the complex field.* If F is at most quaternary and of odd degree, the equation E is of the fourth degree or less and can be solved by well known methods. The same statement applies if F is n-ary, where $n \leqq 2$, and of even degree. Whether or not F is equivalent to a sum of pth powers can be determined directly, in these cases, in a finite number of steps.

There is a direct method, involving an induction process, of determining whether or not the ranks of a form F are equal. It is based upon the theorem that an n-ary form F of degree p, $(p \geqq 2)$, is equivalent to C in an (n, p)-proper field ϕ if and only if one determinant rank of F is n and the forms $F_1 = a_{ij\cdots m} x_j \cdots x_m, \cdots, F_n = a_{nj\cdots m} x_j \cdots x_m$ are simultaneously equivalent in ϕ to sums of $(p-1)$th powers. It is to be observed that $F \equiv x_1 F_1 + \cdots + x_n F_n$. The problem of simultaneous equivalence of $F_1, \cdots, F_n$ to sums of $(p-1)$th powers, where these forms are quadratic, is quite different from the problem for which these are cubic or of higher degree. This is due in part to the essential difference between transformations $x_i = b_{i\alpha} x_\alpha'$ which bring

$$C = \lambda_1 x_1^p + \cdots + \lambda_r x_r^p, \qquad \lambda_1, \cdots, \lambda_r \neq 0$$

into a similar form

$$C' = \lambda_1' x_1'^p + \cdots + \lambda_r' x_r'^p, \qquad \lambda_1', \cdots, \lambda_r' \neq 0,$$

for $p = 2$, and the transformations which bring C into C' for $p \geqq 3$. In fact, for

* Hočevar, *Sur les formes décomposables en facteurs linéaires*, Comptes Rendus de l'Académie des Sciences, vol. 138 (1904), pp. 745–747.

$p \geqq 3$ and an (n, p)-proper field ϕ the matrix $B=(b_{i\alpha})$ of the non-singular linear transformation $x_i = b_{i\alpha}x_\alpha'$, $(i, \alpha = 1, 2, \cdots, n, n \geqq r)$, bringing C into C' is of the type

$$B = \left\| \begin{matrix} B_{11} & 0 \\ B_{21} & B_{22} \end{matrix} \right\|, \tag{1}$$

where B_{11} is of order r, and has but one non-vanishing element in each row and column. The transformations which bring a quadratic form C into a similar form C' are thus more complicated than the transformations which bring a form C of the third degree or higher into a like form C'. There is an analogue of this for multilinear forms.*

It is to be noted that the shape of B in (1) depends on r and not on p. The solution of the general problem of equivalence in ϕ of $F_1, \cdots, F_n$ to sums of $(p-1)$th powers, $(p \geqq 4)$, depends primarily on the following statements:

1. A transformation with matrix B of (1) brings any sum of qth powers in $x_1, \cdots, x_r$ in an (n, q)-proper field into a sum of qth powers with the same number of non-vanishing coefficients.

2. For properly restricted fields the class of non-singular linear transformations bringing a form C with $p \geqq 3$ into a like form C' is identical with the class of transformations which brings a set of forms $L, M, \cdots, Q$ into a like set, where $L, M, \cdots, Q$ are sums of qth powers, $(q \geqq 3)$, in $x_1, \cdots, x_r$ and have the property that there exists for each variable x_σ in the set $x_1, \cdots, x_r$ at least one form in the set having a non-vanishing coefficient for x_σ^q.

3. If an n-ary form $F = a_{ij\cdots m}x_ix_j \cdots x_m$ is equivalent in an (n, p)-proper field ϕ to a sum of pth powers under transformation with matrix B of (1), the terms in F involving only $x_{r+1}, \cdots, x_n$ are equivalent in ϕ, under non-singular linear transformations on $x_{r+1}, \cdots, x_n$, to a sum of pth powers.

By these and other properties, the problem of equivalence of a form F to a sum of pth powers is solved by treating consecutively the equivalence of sets of subforms associated with F, where the forms in these sets are of lower degree than F. For the sake of brevity some of the more involved parts of the theory will be omitted.

In the study of non-singular multilinear forms carried through in an earlier paper, difficulties arose† in the process of determining the non-singularity of a given form. The analogue for multilinear forms of the above induction process avoids these difficulties entirely, so that it is a relatively simple

* R. Oldenburger, II, pp. 425–427, 442–443.

† R. Oldenburger, II, p. 431.

matter to determine whether or not a given form is non-singular. Since the analogue involves few new features, its presentation will be omitted.

The induction process when applied to binary forms leads to conditions, like those mentioned above, involving the generalized determinants $|x_i a_{ij\cdots m}|$, and $|x_i x_j a_{ijk\cdots m}|$, except that the conditions are much stronger in that the determinants are replaced by ordinary second order 2-way determinants, and no distinction is made between even and odd values of p. Since the binary case involves at most the determination of whether or not $(\epsilon)^{1/2}$ is in ϕ, given that ϵ is in ϕ, we may consider the problem of equivalence of a binary form of any degree to be completely solved for a $(2, p)$-proper field in the sense that this equivalence can be recognized by simple direct steps. This development can be used to give necessary and sufficient conditions for the equivalence of an algebraic equation $P(x)=0$ to the equation

$$y^p - A = 0$$

under linear fractional transformations in a $(2, p)$-proper field ϕ. This problem was considered for a finite field $GF(p)$ by Brahana.*

Bronowski† considered the problem of the equivalence under transformations, not necessarily non-singular, of a form of the pth degree to a form of the type

$$\sum_i y_i^p + \sum_i \lambda_i Y_i^p.$$

Bronowski translated the problem into one in geometry which has not been solved.

Sylvester proved‡ that a fairly broad class of binary forms of degree p can be written in the complex field as sums of pth powers of linear forms. These linear forms are, however, in general not linearly independent.

It will be proved elsewhere that *every form F with symmetric matrix can be written in a field with p or more elements as a sum*

$$a_1 L_1^p + \cdots + a_r L_r^p,$$

where r is finite, $L_1, \cdots, L_r$ are linear forms, and $a_1, \cdots, a_r$ are in the given field. For a field with less than p elements this representation is not in general possible. The forms which can be represented as above, where $L_1, \cdots, L_r$

* Brahana, *Note on irreducible quartic congruences*, these Transactions, vol. 38 (1935), pp. 395–400.

† Bronowski, *The sum of powers of a canonical expression*, Proceedings of the Cambridge Philosophical Society, vol. 29 (1933), pp. 69–82.

‡ J. J. Sylvester, Philosophical Magazine, 1851, p. 94.

are linearly independent, are the forms with which the present paper is concerned. In this case *r takes on its minimum value* for the class of forms of degree p in r variables, where the forms in this class cannot be written by means of non-singular linear transformations in terms of less than r variables, and these forms have symmetric matrices.

It is a significant thing that the representability of a form of odd degree in a field ϕ by a sum of powers

$$aL^p + bM^p + \cdots + dN^p$$

of linearly independent linear forms $L, M, \cdots, N$ depends directly on the factorability of a second form in ϕ into a product $kRS \cdots T$, of linearly independent linear factors $R, S, \cdots, T$.

The generalized *matrix method of approach* used here is a new one in the theory of forms of degree higher than quadratic.

Throughout the present paper *when we equate two forms of degree p we will assume that the field of coefficients has $p+1$ or more elements*, so that equality of forms implies equality of the corresponding coefficients.

2. **Ranks of general matrices and forms.** Hitchcock* studied certain ranks of a p-way matrix. In another paper I proved† that generalized determinant minors of the product

$$\left(\sum_m a_{ij\cdots rm} b_{m\alpha\cdots\beta} \right)$$

of two matrices $A = (a_{ij\cdots m})$ and $B = (b_{m\alpha\cdots\beta})$ are not always sums of products of determinant minors of A and B, but rather sums of products of determinant minors of matrices called "derivates" associated with A and B. In terms of determinant minors of A and derivates of A, I defined determinant ranks of A for any grouping of the indices into partitions and allowable signancy‡ of these partitions, and proved the invariance§ of these ranks under non-singular linear transformations on the form $a_{ij\cdots m}x_iy_j \cdots z_m$ associated with A. The definitions of a few determinant ranks essential to the argument will be given explicitly.

The matrix $A = (a_{ij\cdots m})$, $(i, j, \cdots, m = 1, \cdots, n)$, is said to be of *order n*. We shall assume in what follows that A is symmetric. The ranks of a form $F = a_{ij\cdots m}x_ix_j \cdots x_m$ are the ranks of its associated matrix A.

* F. L. Hitchcock, *Multiple invariants and generalized rank of a p-way matrix or tensor*, Journal of Mathematics and Physics, vol. 7 (1927), pp. 40–79.

† R. Oldenburger, I, p. 632.

‡ For a discussion of signancy of generalized determinants see L. H. Rice, *Introduction to higher determinants*, Journal of Mathematics and Physics, vol. 9 (1930), p. 48.

§ R. Oldenburger, I, pp. 633–635, 645–646.

The simplest determinant rank of the matrix A is the ordinary rank of the 2-way matrix $(a_{i\tau})$ obtained from A by letting i be the index of the rows of $(a_{i\tau})$, and τ the index of the columns of $(a_{i\tau})$, where τ is the partition of indices $j \cdots lm$ ranging over the values $1 \cdots 11$, $1 \cdots 12, \cdots, 1 \cdots 1n$, $1 \cdots 21, 1 \cdots 22, \cdots, 1 \cdots 2n, \cdots, n \cdots nn$, where n is the order of A. This rank will be called the *principal determinant rank of* A. It was used by Mayer in a paper in these Transactions.*

Let the indices of $A = (a_{ij\cdots m})$ be grouped into partitions $\rho, \sigma, \cdots, \tau$. The minimum value of ϵ for which A can be written in the form

$$(a_{ij\cdots m}) \equiv \left(\sum_{\alpha=1}^{\epsilon} b_{\alpha\rho} c_{\alpha\sigma} \cdots d_{\alpha\tau} \right),$$

where $b_{\alpha\rho}, \cdots, d_{\alpha\tau}$ are in a given field ϕ, is called the $(\rho\sigma \cdots \tau)$-factorization rank of A with respect to ϕ. The number ϵ is always finite. The $(ij \cdots m)$-factorization rank of A with respect to ϕ is called the *principal factorization rank*† of A with respect to ϕ. That this rank depends on ϕ is evident from the following example.

The form $x^3 - 3xy^2$ has a matrix $A = (a_{ijk})$ for which $a_{111} = 1$, $a_{122} = a_{212} = a_{221} = -1$ and all other elements vanish. The matrix A can be written as

$$\left(\sum_{\alpha=1}^{2} b_{\alpha i} b_{\alpha j} b_{\alpha k} \right),$$

where

$$(b_{\alpha i}) = \left\| \begin{matrix} \dfrac{1}{(2)^{1/3}} & \dfrac{I}{(2)^{1/3}} \\ \dfrac{1}{(2)^{1/3}} & \dfrac{-I}{(2)^{1/3}} \end{matrix} \right\|, \qquad I = (-1)^{1/2}.$$

Since the principal determinant rank of A is 2, the principal factorization rank is at least 2. Therefore, the principal factorization rank of A with respect to the complex field is 2. In the complex field $x^3 - 3xy^2$ is equivalent to $x^3 + y^3$ by the results of the present paper (Theorem I). In the field of reals the principal factorization rank of A is at least 3 since otherwise $x^3 - 3xy^2$ would be equivalent in the field of reals to $\lambda x^3 + \mu y^3$ by Theorem I of this paper. It follows from Theorem IIIa of the present paper that these forms are not equivalent for this field.

---

* W. Mayer, *Die Differentialgeometrie der Untermannigfaltigkeiten des R_n konstanter Krümmung*, these Transactions, vol. 38 (1935), pp. 274–310.

† R. Oldenburger, *On arithmetic invariants of binary cubic and binary trilinear forms*, Bulletin of the American Mathematical Society, vol. 42 (1936), pp. 871–873.

If the principal factorization and determinant ranks of $F = a_{ij\cdots m}x_i x_j \cdots x_m$ are equal for a field for which $(a_{ij\cdots m})$ is unique, by the definitions of these ranks the form F is equivalent at once, for some n, to a like form for which

$$(a_{ij\cdots m}) = \left(\sum_{\alpha=1}^{n} b_{\alpha i}c_{\alpha j}\cdots d_{\alpha m}\right),$$

where $(b_{\alpha i}), (c_{\alpha j}), \cdots, (d_{\alpha m})$ are non-singular matrices of order n. The associated multilinear form $M = a_{ij\cdots m}x_i y_j \cdots z_n$ is equivalent to $M' = x_1' y_1' \cdots z_1' + \cdots + x_n' y_n' \cdots z_n'$, the matrix of which is $(\delta_{\alpha i}\delta_{\alpha j}\cdots\delta_{\alpha m})$, where $(\delta_{\alpha i}) = (\delta_{\alpha j}) = \cdots = (\delta_{\alpha m})$ is the Kronecker delta of order n. It is a fairly simple matter to show that all of the ranks of M' equal n. Since the ranks of F are ranks of M, this proves the following lemma:

LEMMA I. *If the principal ranks of an n-ary form F of degree p are equal for an (n, p)-proper field, all ranks of F are equal.*

3. **Necessary and sufficient conditions for equality of ranks.** We shall prove the following theorem:

THEOREM I. *Let F be a given n-ary form of degree p, and let ϕ be an (n, p)-proper field. The principal determinant rank of F and the principal factorization rank of F in ϕ are equal to n if and only if F is equivalent in ϕ to a sum of n pth powers.*

It is obvious that the theorem is true for linear forms. The result for the quadratic case is well known.* We shall assume in what follows that $p \geqq 3$.

Let $F = a_{ij\cdots m}x_i x_j \cdots x_m$. From the definitions of ranks the principal ranks of F are equal to n if and only if the matrix of F can be written as

$$\left(\sum_{\alpha=1}^{n} a_{\alpha i}b_{\alpha j}\cdots d_{\alpha m}\right), \tag{2}$$

where $(a_{\alpha i}), \cdots, (d_{\alpha m})$ are non-singular.

If F is equivalent in ϕ to a sum of pth powers, there exist non-vanishing constants λ_α in ϕ and a non-singular matrix $(g_{\alpha\beta})$ with elements in ϕ such that

$$F \equiv \lambda_\alpha g_{\alpha i}g_{\alpha j}\cdots g_{\alpha m}x_i x_j \cdots x_m.$$

The matrix $A = (a_{ij\cdots m})$ of F is now of the form (2) where $(a_{\alpha i}) = (\lambda_\alpha g_{\alpha i})$,

$$(b_{\alpha j}) = (g_{\alpha j}), \cdots, (d_{\alpha m}) = (g_{\alpha m}), \quad \alpha \text{ not summed}.$$

Since $(g_{\alpha\beta})$ is non-singular, $(a_{\alpha i}), \cdots, (d_{\alpha m})$ are non-singular.

* C. C. MacDuffee, *The Theory of Matrices.*

Conversely, assume that the matrix A of F is in the form (2). Let $(A^{i\beta})$ be the inverse of $(a_{\alpha i})$. Applying the transformation

$$x_i = A^{i\beta} x'_\beta$$

we obtain a form whose matrix* is of the form

$$(\delta_{\alpha\beta} b'_{\alpha j} \cdots d'_{\alpha m}),$$

where $(\delta_{\alpha\beta})$ is a Kronecker delta. It is therefore no restriction on the generality of the method to assume at the start that $(a_{\alpha i}) = (\delta_{\alpha i})$. We shall prove that the form F is then a sum of pth powers.

Write $A = (\delta_{\alpha i} b_{\alpha j} c_{\alpha k} q_{\alpha l} \cdots e_{\alpha t} d_{\alpha m})$, and $B = (b_{\alpha j})$, $C = (c_{\alpha k})$, $\cdots$, $D = (d_{\alpha m})$. Assume that $b_{11} = 0$. Since B is non-singular, $(b_{12}, \cdots, b_{1n}) \neq (0, \cdots, 0)$. The symmetry of A and the invariance of symmetry imply that

$$\begin{aligned} b_{12} c_{1k} \cdots e_{1t} d_{11} &= b_{11} c_{1k} \cdots e_{1t} d_{12}, \\ &\cdots\cdots\cdots\cdots\cdots \\ b_{1n} c_{1k} \cdots e_{1t} d_{11} &= b_{11} c_{1k} \cdots e_{1t} d_{1n}, \end{aligned} \tag{3}$$

for all values of $k, \cdots, t$. The right-hand members of (3) vanish. The direct product $H = C \times \cdots \times E$ of $C, \cdots, E$ is the matrix $(c_{jk} \cdots e_{mt})$ whose elements are the possible products of elements of $C, \cdots, E$.

Display H as an ordinary 2-way matrix $(h_{\rho\sigma})$, where ρ is the partition $j \cdots m$ and σ is the partition $k \cdots t$. By a lemma proved elsewhere† the determinant of $(h_{\rho\sigma})$ is a product of powers of determinants of $C, \cdots, E$. Since these are non-singular, H is non-singular as displayed. Since the products $c_{1k} \cdots e_{1t}$ are the elements of a row of $(h_{\rho\sigma})$, and since this matrix is non-singular, these products cannot all vanish. It follows from (3) that $d_{11} = 0$. By the same argument $c_{11} = \cdots = e_{11} = 0$.

The symmetry of A implies that

$$b_{11} \cdots e_{11} d_{1m} = b_{m1} \cdots e_{m1} d_{m1} \tag{4}$$

for every m. Since the left member of (4) vanishes, for every m some quantity in the set $b_{m1}, \cdots, d_{m1}$ vanishes. For $m = 2$ it is no restriction to take this to be b_{21}. Since

$$b_{21} c_{2j} q_{21} \cdots d_{21} = b_{2j} c_{21} \cdots d_{21}$$

---

* The matrix can then be written as $(b_{\beta j} \cdots d_{\beta m})$ where β is not summed. Necessary and sufficient conditions for the factorability of a matrix in this form were given in II, p. 452.

† R. Oldenburger, I, p. 625; II, p. 442.

for every j, and since by the non-singularity of B $(b_{22}, \cdots, b_{2n}) \neq (0, \cdots, 0)$, it follows that

$$c_{21} \cdots d_{21} = 0.$$

Since one of the quantities $c_{21}, \cdots, d_{21}$ vanishes, it is no restriction to take this to be c_{21}. Since

$$b_{21}c_{2j}q_{2i}f_{21} \cdots d_{21} = b_{2i}c_{2j}q_{21}f_{21} \cdots d_{21}, \tag{5}$$

for all i, j, and $b_{2i}c_{2j}$ cannot be zero for all i, j (by the non-singularity of B, C), and since the left-hand member of (5) vanishes,

$$q_{21} \cdots d_{21} = 0.$$

It is seen by induction that $q_{21} = \cdots = d_{21} = 0$. By the same argument we prove that

$$b_{m1} = \cdots = d_{m1} = 0$$

for every m. Since $B, \cdots, D$ are non-singular, the elements $b_{11}, \cdots, d_{11}$ cannot be zero. It follows that no diagonal elements of $B, \cdots, D$ vanish.

We shall now prove that the non-diagonal elements of $B, \cdots, D$ vanish.

If some non-diagonal element of B vanishes, we may take it to be b_{12}. By the symmetry of A

$$b_{11}c_{12}q_{11} \cdots d_{11} = b_{12}c_{11}q_{11} \cdots d_{11} = b_{21}c_{21}q_{21} \cdots d_{21}; \tag{6}$$

whence $c_{12} = 0$. Hence $q_{12} = \cdots = d_{12} = 0$. By (6) some element in the set $b_{21}, \cdots, d_{21}$ vanishes. It is no restriction on the generality of the method to take this to be b_{21}. From

$$b_{21}c_{22}q_{22} \cdots d_{22} = b_{22}c_{21}q_{22} \cdots d_{22}$$

it follows that $c_{21} = 0$. Hence $q_{21} = \cdots = d_{21} = 0$. We have proved that if $b_{rs} = 0$ for a given r, s, where $r \neq s$, then $b_{sr}, c_{rs}, c_{sr}, \cdots, d_{rs}, d_{sr} = 0$.

If elements other than b_{12} in the first row of B are zero, it is no restriction to take them to be $b_{13}, \cdots, b_{1r}$ and to assume that $b_{1,r+1}, \cdots, b_{1n} \neq 0$. Then

$$b_{1j} = c_{1j} = \cdots = d_{1j} = b_{j1} = c_{j1} = \cdots = d_{j1} = 0, \qquad j = 2, \cdots, r,$$

and

$$b_{1j}, b_{j1}, c_{1j}, c_{j1}, \cdots, d_{1j}, d_{j1} \neq 0, \qquad j = r+1, \cdots, n.$$

By the symmetry of A,

$$b_{r+1,1}c_{r+1,1} \cdots e_{r+1,1}d_{r+1,j} = b_{1j}c_{11} \cdots e_{11}d_{1,r+1} = 0,$$

for $j = 2, \cdots, r$. Hence

$$d_{r+1,2} = \cdots = d_{r+1,r} = 0.$$

It follows at once that $d_{\alpha m}=0$, for $\alpha=r+1, \cdots, n$; $m=2, \cdots, r$, and $\alpha=2, \cdots, r$; $m=r+1, \cdots, n$. Evidently, also $b_{\alpha m}= \cdots =e_{\alpha m}=0$ for the same pairs of ranges of α and m.

We have proved that

$$B = \left\| \begin{matrix} b_{11} & 0 & \cdots & 0 & b_{1,r+1} & \cdots & b_{1n} \\ 0 & b_{22} & \cdots & b_{2r} & 0 & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot & \cdot & \cdots & \cdot \\ 0 & b_{r2} & \cdots & b_{rr} & 0 & \cdots & 0 \\ b_{r+1,1} & 0 & \cdots & 0 & b_{r+1,r+1} & \cdots & b_{r+1,n} \\ \cdot & \cdot & \cdots & \cdot & \cdot & \cdots & \cdot \\ b_{n1} & 0 & \cdots & 0 & b_{n,r+1} & \cdots & b_{nn} \end{matrix} \right\| ; \tag{7}$$

and $C, \cdots, D$ are of the same form. If no element of B vanishes, B is of the form (7) where $r+1=2$, and the minor

$$\left\| \begin{matrix} b_{22} & \cdots & b_{2r} \\ \cdot & \cdots & \cdot \\ b_{r2} & \cdots & b_{rr} \end{matrix} \right\|$$

does not occur. The representation (7) hence includes all cases.

If some elements of the first row of

$$B^* = \left\| \begin{matrix} b_{r+1,r+1} & \cdots & b_{r+1,n} \\ \cdot & \cdots & \cdot \\ b_{n,r+1} & \cdots & b_{nn} \end{matrix} \right\|$$

vanish, let these be $b_{r+1,r+\sigma}, \cdots, b_{r+1,n}$, $(\sigma \geqq 2)$. By the above reasoning $b_{r+\sigma,1}$ vanishes. Since this contradicts an earlier assumption, it follows that no elements of B^* are zero.

By simultaneous interchanges of the rows and columns of $B, \cdots, D$, corresponding to permutation of the variables of F, we can bring B into the form

$$B = \left\| \begin{matrix} b_{11} & \cdots & b_{1s} & 0 & \cdots & 0 \\ \cdot & \cdots & \cdot & \cdot & \cdots & \cdot \\ b_{s1} & \cdots & b_{ss} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & b_{s+1,s+1} & \cdots & b_{s+1,n} \\ \cdot & \cdots & \cdot & \cdot & \cdots & \cdot \\ 0 & \cdots & 0 & b_{n,s+1} & \cdots & b_{nn} \end{matrix} \right\|, \qquad s = n - r + 1,$$

and $C, \cdots, D$ into similar forms where the elements of

$$\left\| \begin{matrix} b_{11} \cdots b_{1s} \\ \cdot \quad \cdots \cdot \\ b_{s1} \cdots b_{ss} \end{matrix} \right\|, \cdots, \left\| \begin{matrix} d_{11} \cdots d_{1s} \\ \cdot \quad \cdots \cdot \\ d_{s1} \cdots d_{ss} \end{matrix} \right\|$$

are all non-zero.

Applying the above argument to the set of minors

$$\left\| \begin{matrix} b_{s+1,s+1} \cdots b_{s+1,n} \\ \cdot \quad \cdots \quad \cdot \\ b_{n,s+1} \quad \cdots b_{nn} \end{matrix} \right\|, \cdots, \left\| \begin{matrix} d_{s+1,s+1} \cdots d_{s+1,n} \\ \cdot \quad \cdots \quad \cdot \\ d_{n,s+1} \quad \cdots d_{nn} \end{matrix} \right\|$$

we conclude that $B, \cdots, D$ can be written in the form

$$B = \left\| \begin{matrix} B_1 & 0 & \cdots 0 \\ 0 & B_2 & \cdots 0 \\ \cdot & \cdot & \cdots \cdot \\ 0 & 0 & \cdots B_w \end{matrix} \right\|, \cdots, \quad D = \left\| \begin{matrix} D_1 & 0 & \cdots 0 \\ 0 & D_2 & \cdots 0 \\ \cdot & \cdot & \cdots \cdot \\ 0 & 0 & \cdots B_w \end{matrix} \right\|, \tag{8}$$

where the elements of the minors $B_1, \cdots, B_w, \cdots, D_1, \cdots, D_w$ are not zero, and $B_i, \cdots, D_i$ are minors of the same order for every i.

Assuming that $B_1, \cdots, B_w$ are not all of order 1, we may write $B, \cdots, D$ in the form (8), where $B_1, \cdots, D_1$ are of order $R \geqq 2$.

The determinant of B is

$$|B| = |B_1| \cdot |B_2| \cdot \cdots \cdot |B_w|.$$

We can write

$$|B_1| = \frac{1}{c_{12}c_{21}q_{11} \cdots d_{11}q_{21} \cdots d_{21}} \begin{vmatrix} b_{11}c_{12}q_{11} \cdots d_{11} & b_{12}c_{12}q_{11} \cdots d_{11} \cdots & b_{1R}c_{12}q_{11} \cdots d_{11} \\ b_{21}c_{21}q_{21} \cdots d_{21} & b_{22}c_{21}q_{21} \cdots d_{21} \cdots & b_{2R}c_{21}q_{21} \cdots d_{21} \\ b_{31} & b_{32} \quad \cdots & b_{3R} \\ \cdot & \cdot \quad \cdots & \cdot \\ b_{R1} & b_{R2} \quad \cdots & b_{RR} \end{vmatrix},$$

since the denominator is not zero.

By the symmetry of A

$$\begin{aligned} b_{11}c_{12}q_{11} \cdots d_{11} &= b_{21}c_{21}q_{21} \cdots d_{21}, \\ b_{12}c_{12}q_{11} \cdots d_{11} &= b_{22}c_{21}q_{21} \cdots d_{21}, \\ &\cdots\cdots\cdots\cdots\cdots, \\ b_{1R}c_{12}q_{11} \cdots d_{11} &= b_{2R}c_{21}q_{21} \cdots d_{21}, \end{aligned}$$

whence the first two rows of the above determinant are identical. Hence

$|B_1| = 0$, whence B is singular. This gives a contradiction. Minors of (8) are of order 1 and

$$B = \left\| \begin{matrix} b_{11} & 0 \cdots 0 & 0 \\ \cdot & \cdots\cdots & \cdot \\ 0 & 0 \cdots 0 & b_{nn} \end{matrix} \right\|, \cdots, \quad D = \left\| \begin{matrix} d_{11} & 0 \cdots 0 & 0 \\ \cdot & \cdots\cdots & \cdot \\ 0 & 0 \cdots 0 & d_{nn} \end{matrix} \right\|.$$

Now

$$F = \delta_{\alpha i} b_{\alpha j} \cdots d_{\alpha m} x_i x_j \cdots x_m = b_{11} \cdots d_{11} x_1^p + \cdots + b_{nn} \cdots d_{nn} x_n^p,$$

which completes the proof of the theorem.

The above argument is valid if there are only two matrices in the set $B, \cdots, D$.

By Lemma I we can state Theorem I in the following form:

THEOREM Ia. *Let F and ϕ be given as in Theorem* I. *The ranks of a form F of degree p are equal in ϕ if and only if F is equivalent in ϕ to a sum of pth powers.*

If the characteristic of ϕ is 2 and F is quadratic of rank n, F is always a sum of $r \leqq n$ squares since $a_{ij} + a_{ji} = 2a_{ij} = 0$. This generalizes for p-ic forms, $(p \geqq 3)$. We therefore have the theorem:

THEOREM II. *Let ϕ be a field whose characteristic is a factor of all numbers in the set k_{np} of* §1, *or an (n, p)-proper field. The ranks (or the principal ranks) of an n-ary form $F = a_{ij\cdots m} x_i x_j \cdots x_m$ of degree p with respect to a field ϕ are equal if and only if F is equivalent in ϕ to a sum of pth powers.*

4. A condition for the equivalence of forms of odd degree to sums of pth powers. Let $A = (a_{ij\cdots m})$ denote a p-way matrix, where $i, j, \cdots, m = 1, \cdots, n$, and p is even. Let $m_1, \cdots, m_n$ denote n distinct values of n; $j_1, \cdots, j_n$ distinct values of j; and so on. Let $(e^{j_1 \cdots j_n})$ be the *generalized Kronecker delta* for which $e^{j_1 \cdots j_n} = 1$ if $(j_1, \cdots, j_n)$ is an even permutation of $1, \cdots, n$, and equal to -1 if $(j_1, \cdots, j_n)$ is an odd permutation of $1, \cdots, n$. The determinant of A with all indices signant is defined* to be

$$|A| = \sum e^{j_1 \cdots j_n} \cdots e^{m_1 \cdots m_n} a_{1 j_1 \cdots m_1} \cdots a_{n j_n \cdots m_n}, \tag{9}$$

where the summation is over all distinct permutations of the numbers in the sets $(j_1, \cdots, j_n), \cdots, (m_1, \cdots, m_n)$. It is of order n.

Let an n-ary form $F = a_{ij\cdots m} x_i x_j \cdots x_m$ be of odd degree p. The *i-characteristic determinant* of F is defined to be the determinant $|x_i a_{ij\cdots m}|$ with all indices $[j, \cdots, m]$ signant. If F is of degree 3, this is the Hessian of F except for a constant factor. We shall prove the following theorem:

* R. Oldenburger, I, p. 631.

THEOREM III. *An n-ary form F of odd degree $p \geqq 3$ is equivalent in an (n, p)-proper field ϕ to a sum of n pth powers, if and only if the i-characteristic determinant D of F factors in ϕ into linearly independent linear factors, and under reduction in ϕ of D to canonical form $Kx_1, \cdots, x_n$ F transforms covariantly to a sum of pth powers.*

The i-characteristic determinant of $G = \lambda_i x_i^p$ is the determinant

$$D = \left| x_i \lambda_i \delta_{ij \cdots m} \right|,$$

where $\delta = (\delta_{ij \cdots m})$ is the generalized Kronecker delta whose only non-vanishing elements are

$$\delta_{1 \cdots 1} = \cdots = \delta_{n \cdots n} = 1.$$

Evidently

$$D = Kx_1 \cdots x_n, \tag{10}$$

where $K = \lambda_1 \cdots \lambda_n$. Under non-singular transformations

$$x_i = p_{ij} x_j' \tag{11}$$

on G to give a form F, D transforms by a general theorem on products of determinants* into

$$D' = \left| P \right|^{p-1} K \prod^{i} p_{ij} x_j',$$

where $P = (p_{ij})$. The determinant D is thus a covariant of G. It follows that the i-characteristic determinant of any form F, equivalent to G, factors in ϕ into linearly independent linear factors. This is therefore a necessary condition for the equivalence in ϕ of a form to a sum of pth powers.

Assume that the i-characteristic determinant of F factors in ϕ into linearly independent linear factors $L, M, \cdots, N$. Applying the transformations

$$x_1' = L, \cdots, x_n' = N$$

to F we obtain a form F' whose i-characteristic determinant is of the form

$$Kx_1' \cdots x_n'.$$

We therefore consider only forms F where D satisfies (10). We shall prove that such a form is equivalent to a sum of pth powers if and only if it is already a sum of pth powers. If F is equivalent to a form G, then G is equivalent to F under a transformation (11) which brings (10) covariantly into $qx_1' \cdots x_n'$ for some q. For such a transformation

* R. Oldenburger, I, p. 632.

$$|P|^{p-1}K\prod^{i}(p_{ij}x_j') \equiv qx_1' \cdots x_n'.$$

By the unique factorization property, similarly used elsewhere,* the linear expressions $p_{1j}x_j', \cdots, p_{nj}x_j'$ are equal in some order to the products of $x_1', \cdots, x_n'$ by constants in ϕ. The transformations on G which bring D into $qx_1' \cdots x_n'$ are of the form

$$x_i = \alpha_j x_j', \qquad i, j = 1, \cdots, n; j \text{ not summed}, \tag{12}$$

where there is one j for every i, and conversely. Since (12) transforms G into a sum of pth powers, F is a sum of pth powers.

In the binary cubic case we have simply the following theorem:

THEOREM IIIa. *A binary cubic form $F = a_{ijk}x_ix_jx_k$ is equivalent in a field ϕ with characteristic not 3 to a sum of two cubes if and only if the i-characteristic determinant of F factors in ϕ into distinct linear factors.*

To prove this theorem it is only necessary to show that if the i-characteristic determinant D of F factors in ϕ into kx_1x_2 for some $k \neq 0$, then F is a sum of cubes. It is to be noted that ϕ is (2, 3)-proper if and only if its characteristic is not 3. Let $a_{111} = \alpha$, $a_{222} = \beta$, $a_{112} = \gamma$, $a_{122} = \delta$. Then a simple direct calculation shows that the i-characteristic determinant of F is $(\alpha\delta - \gamma^2)x_1^2 + (\alpha\beta - \gamma\delta)x_1x_2 + (\gamma\beta - \delta^2)x_2^2$. If this is to be of the form kx_1x_2, we must have $\alpha\delta - \gamma^2 = \gamma\beta - \delta^2 = 0$, $\alpha\beta - \gamma\delta \neq 0$. It follows readily that $\gamma = \delta = 0$, $\alpha, \beta \neq 0$; thus F is of the required form.

That the factorization of the i-characteristic determinant into linearly independent linear factors is not in general enough to insure the equivalence of an n-ary form F of degree p to a sum of n pth powers follows from an example:

Example. Let $F = 6x_1x_2x_3$ with matrix $A = (a_{ijk})$. The only non-vanishing elements of A are $a_{123} = a_{132} = a_{213} = a_{231} = a_{312} = a_{321} = 1$. Let ϕ be a (3, 3)-proper field. The i-characteristic determinant of F is $D = 2x_1x_2x_3$. Except for an integral factor it is the Hessian of F. Now D is a product of linearly independent linear factors in ϕ. Since D is in canonical form, if F is equivalent to a sum of three cubes, by Theorem III, F is such a sum. The form F is not such a sum.

The determinant $|A|$ of a p-way matrix $A = (a_{ij\cdots m})$ of order n, (p odd), where $j, \cdots, m$ are signant, is given by (9). We may have $|A| \neq 0$, even if $a_{\alpha j\cdots m} \equiv a_{\beta j\cdots m}$ for $\alpha \neq \beta$. The determinant is said to be of order n. The $(j \cdots m, i)$ derivate of A is the matrix obtained by adjoining $(n-1)$ matrices all equal to A in the direction associated with the index i in the p-space

* R. Oldenburger, *Equivalence of multilinear forms singular on one index*, Duke Mathematical Journal, vol. 2 (1936), p. 674. See also B. L. van der Waerden, *Moderne Algebra*, Springer, 1930, vol. 1, pp. 63–66.

representation of A. The *rank* $r[j \cdots m, i]$ of A and $F = a_{ij\cdots m}x_ix_j \cdots x_m$ is the upper bound of the orders of the non-vanishing determinant minors of this derivate, expanded with $j, \cdots, m$ signant.*

For an (n, n)- and (n, p)-proper field the i-characteristic determinant of F can be written as

$$\sum_{\Gamma} k_\Gamma x_{i_1} \cdots x_{i_n} \mid a_{ij\cdots m} \mid, \tag{13}$$

where the summation is over all distinct choices Γ of the values of $i_1 \cdots i_n$ from the set $1, 2, \cdots, n$; $k_\Gamma = 1/\alpha!\beta! \cdots \gamma!$, where, for example, α of the indices in Γ are equal, β other indices are equal, and $|a_{ij\cdots m}|$ is a determinant minor of the $(j \cdots m, i)$ derivate of A with $j, \cdots, m$ signant and all minors of this derivate occur in (13). There follows the theorem:

THEOREM IV. *Let ϕ be an (n, n)- and (n, p)-proper field. The i-characteristic determinant of the n-ary form $F = a_{ij\cdots m}x_ix_j \cdots x_m$ of odd degree p, $(p \geqq 3)$, is not identically zero if and only if the rank $r[j \cdots m, i]$ of F is n.*

By a theorem proved elsewhere,† the principal determinant rank of F is greater than or equal to $r[j \cdots m, i]$. Hence we have the following theorem:

THEOREM V. *Let ϕ be a field satisfying the assumptions of Theorem* IV. *If the i-characteristic determinant of an n-ary form F of odd degree greater than or equal to 3 factors into linearly independent linear factors in ϕ, the principal determinant rank of F is n.*

5. The equivalence of forms of even degree to sums of pth powers. With an n-ary form

$$F = a_{ij\cdots m}x_ix_j \cdots x_m$$

of even degree p, $(p \geqq 4)$, associate the determinant

$$E = | x_ix_ja_{ijk\cdots m} |$$

with all indices signant. We shall call this the *ij-characteristic determinant of F.*

We shall prove the following theorem:

THEOREM VI. *An n-ary form F of even degree p, $(p \geqq 4)$, is equivalent in an (n, p)-proper field ϕ to a sum of pth powers if and only if the ij-characteristic determinant E of F factors in ϕ into squares of linearly independent linear factors, and under reduction of E in ϕ to canonical form $Kx_1^2 \cdots x_n^2$, F transforms covariantly to a sum of pth powers.*

* R. Oldenburger, I, p. 633.

† R. Oldenburger, I, p. 641.

The proof of this theorem is very similar to that of Theorem III. The transformation from F to

$$F' = a_{ij\cdots m} b_{i\alpha} b_{j\beta} \cdots b_{m\gamma} x_\alpha' x_\beta' \cdots x_\gamma'$$

under the non-singular transformation $x_i = b_{i\alpha} x_\alpha'$ corresponds, by a theorem on determinants proved elsewhere,* to the transition from E to

$$| a_{ijk\cdots m} b_{i\alpha} b_{j\beta} x_\alpha' x_\beta' | \, | b_{k\gamma} |^{p-2}.$$

The determinant E of $G = \lambda_i x_i^p$, $(i = 1, \cdots, n; \lambda_i \neq 0)$, is $\lambda_1 \cdots \lambda_n x_1^2 \cdots x_n^2$. Evidently, the ij-characteristic determinant of a form F equivalent to G is of the form $KL^2 \cdots N^2$, where $L, \cdots, N$ are linearly independent linear factors.

Assume that the determinant E of F factors, as required in the theorem, and that F is equivalent to G. Letting $x_1' = L, \cdots, x_n' = N$ we can transform the form F into a new form for which $E = q x_1'^2 \cdots x_n'^2$. Since G is equivalent to F, there is a non-singular transformation $x_i = b_{i\alpha} x_\alpha'$ such that

$$| b_{i\alpha} |^{p-2} \lambda_1 \cdots \lambda_n (b_{1\alpha} x_\alpha')^2 \cdots (b_{n\alpha} x_\alpha')^2 \equiv q x_1'^2 \cdots x_n'^2.$$

It follows from this identity that the transformation $x_i = b_{i\alpha} x_\alpha'$ is of the form (12). Since the transformation (12) brings G into a sum of pth powers, F is a sum of pth powers.

Let $A = (a_{ijk\cdots m})$ be a p-way matrix, ($p \geqq 4$ and even), of order n. The determinant of A with $k, \cdots, m$ signant is the expansion (9) except that $e^{i_1 \cdots i_n}$ does not occur. The indices i and j need not range over distinct values. The $(k \cdots m, ij)$ derivate of A is a matrix obtained from A by adjoining matrices equal to A in the directions of p-space associated with the indices i and j. The rank† $r[k \cdots m, ij]$ of A and $F = a_{ij\cdots m} x_i x_j \cdots x_m$ is the upper bound of the orders of the non-vanishing determinant minors of the $(k \cdots m, ij)$ derivate of A with $k, \cdots, m$ signant in these determinants. For an (n, p)-proper field ϕ with a characteristic not equal to $2, 3, \cdots, \xi_n$, where ξ_n is a positive prime depending on n, the coefficients in the expansion of the ij-characteristic determinant of F can be written as sums of nth order determinant minors of the $(k \cdots m, ij)$ derivate of A with $k, \cdots, m$ signant. We therefore have the following theorem:

THEOREM VII. *For a field ϕ with characteristic different from $2, 3, \cdots, \xi_n$, where ξ_n is finite and large enough, the ij-characteristic determinant of an n-ary form $F = a_{ij\cdots m} x_i x_j \cdots x_m$ of even degree p, ($p \geqq 4$), is identically zero if the rank $r[k \cdots m, ij] < n$.*

* R. Oldenburger, I, p. 632.

† R. Oldenburger, I, p. 633.

Since $r[k \cdots m, ij]$ is not greater than the principal determinant rank of F,* we have the following analogue of Theorem V.

THEOREM VIII. *Let ϕ be a field with the properties of Theorem* VI. *If the ij-characteristic determinant of an n-ary form F of even degree greater than or equal to* 3 *factors into squares of linearly independent linear factors in ϕ, the principal determinant rank of F is n.*

6. Method for obtaining linear factors. Let an n-ary form $F = a_{ij\cdots m}x_ix_j \cdots x_m$, $(i, j, \cdots, m = 1, \cdots, n)$, of degree p, $(p \geqq 2)$, with elements in the complex field, be denoted by $F(x_1, \cdots, x_n)$.

Making the non-singular transformations $x_1 = x_1'$, $x_2 = \lambda x_1' + x_2'$, $x_3 = \mu x_1' + x_3'$, $\cdots$, $x_n = \nu x_1' + x_n'$ on F we obtain $F(x_1', \lambda x_1' + x_2', \cdots, \nu x_1' + x_n')$. Assuming that $F \not\equiv 0$, by the continuity of F, we have $F(1, \lambda, \cdots, \nu) \not\equiv 0$ for $\lambda, \cdots, \nu \neq 0$. Since this is the coefficient of $x_1'^p$ we have the following lemma:

LEMMA II. *A non-identically vanishing form F of degree p in $x_1, \cdots, x_n$ is equivalent in the complex field to a form for which the coefficient of x_1^p is not zero.*

Assume that F has a term in x_1^p and does not contain a repeated irreducible factor.† The last assumption implies that the resultant of F and $\partial F/\partial x_1$ involving $x_2, \cdots, x_n$ is not identically zero in $x_2, \cdots, x_n$. We can therefore choose values $a_2, \cdots, a_n$ such that

$$F(x_1, a_2, \cdots, a_n) = 0, \qquad \frac{\partial F(x_1, a_2, \cdots, a_n)}{\partial x_1} = 0$$

have no root x_1 in common.

By a theorem from implicit function theory it follows that x_1 is represented analytically by distinct power series $P_1(x_2, \cdots, x_n), \cdots, P_p(x_2, \cdots, x_n)$ at the distinct points $(\alpha_1, a_2, \cdots, a_n), \cdots, (\alpha_p, a_2, \cdots, a_n)$ for which

$$F(\alpha_i, a_2, \cdots, a_n) = 0, \quad \left[\frac{\partial F}{dx_1}\right]_{(\alpha_i, a_2, \cdots, a_n)} \neq 0.$$

According to Hočevar‡ the derivatives of x_1 with respect to $x_2, \cdots, x_n$ of order higher than one involve the third order determinant minors of the Hessian H_F of F in such a way that when these minors vanish simultaneously for a point $(x_1, \cdots, x_n)$ where $\partial F/\partial x_1 \neq 0$, these derivatives also vanish. It

* R. Oldenburger, I, p. 641.

† For the case of two variables see G. A. Bliss, *Algebraic Functions*, American Mathematical Society Colloquium Publications, vol. 16, New York, 1933, pp. 29–30.

‡ Hočevar, op. cit., pp. 745–747.

follows that if F is a factor of these minors of H_F, the derivatives of x_1 of higher order than the first vanish at all points for which $F=0$, $\partial F/\partial x_1 \neq 0$. In this case the series $P_1, \cdots, P_p$ involve only terms of the first degree in $x_2, \cdots, x_n$. Except for constant multipliers, the polynomial factors of F are the expressions $x_1-P_1, \cdots, x_1-P_p$ which are linear.

Conversely, if F factors into linear factors, it can be shown at once that the third order determinant minors contain F as a factor.

Except for constant multipliers, the expressions x_i-P_i, $(i=1, \cdots, p)$, are the same as

$$x_1\left[\frac{\partial F}{\partial x_1}\right]_{p_i}+\cdots+x_n\left[\frac{\partial F}{\partial x_n}\right]_{p_i},$$

where p_i is the point $(\alpha_i, a_2, \cdots, a_n)$. These expressions are linearly independent if and only if the matrix

$$M=\left(\left[\frac{\partial F}{\partial x_i}\right]_{p_j}\right)$$

is of rank greater than or equal to p. For the D of Theorem III, and $p=n$, this condition can be stated in terms of the non-singularity of M. We have the theorem:

THEOREM IX. *Let F be a form in $x_1, \cdots, x_n$ of degree n. The form F factors in the complex field into linearly independent linear factors if and only if the following set of conditions is satisfied:*

(1) *F has no repeated factors.*

(2) *F divides the third order determinant minors of its Hessian.*

(3) *At points p_1 $(\alpha_1, a_2, \cdots, a_n), \cdots, p_n$ $(\alpha_n, a_2, \cdots, a_n)$ on $F=0$ for which there are distinct power series expansions for x_1 in terms of $x_2, \cdots, x_n$, the matrix following is non-singular:*

$$\left\|\begin{matrix}\left[\frac{\partial F}{\partial x_1}\right]_{p_1} & \cdots & \left[\frac{\partial F}{\partial x_n}\right]_{p_1}\\ \cdot & \cdots & \cdot\\ \left[\frac{\partial F}{\partial x_1}\right]_{p_n} & \cdots & \left[\frac{\partial F}{\partial x_n}\right]_{p_n}\end{matrix}\right\|.$$

7. **An induction process.** We shall prove the following theorem:

THEOREM X. *Let*

$$F=a_{ij\cdots m}x_ix_j\cdots x_m, \qquad i, j, \cdots, m=1, \cdots, n,$$

be a form of degree $p \geqq 2$, and let ϕ be (n, p)-proper. The form F is equivalent in ϕ to a sum of n pth powers if and only if

(1) *the principal determinant rank of F is n;*

(2) *the subforms $F_1 = a_{1j\cdots m}x_j \cdots x_m, \cdots, F_n = a_{nj\cdots m}x_j \cdots x_m$ are simultaneously equivalent in ϕ to a set of forms of the type*

$$G_1 = \mu_{11}y_1^{p-1} + \cdots + \mu_{1n}y_n^{p-1}, \cdots, G_n = \mu_{n1}y_1^{p-1} + \cdots + \mu_{nn}y_n^{p-1}.$$

If $p=2$ the theorem is evidently true. Assume that $p \geqq 3$. If F is equivalent to

$$G = \lambda_1 y_1^p + \cdots + \lambda_n y_n^p,$$

there exists a non-singular linear transformation

$$x_i = b_{i\alpha}y_\alpha, \qquad i, \alpha = 1, \cdots, n, \tag{14}$$

such that

$$a_{ij\cdots k}b_{i\alpha}b_{j\beta}\cdots b_{k\gamma}y_\alpha y_\beta \cdots y_\gamma = G, \tag{15}$$

where $(b_{i\alpha}) = (b_{j\beta}) = \cdots = (b_{k\gamma})$. Equation (15) implies the following matrix identity:

$$(a_{ij\cdots k}b_{i\alpha}b_{j\beta}\cdots b_{k\gamma}) = (c_{\alpha\beta\cdots\gamma}), \tag{15'}$$

where $C = (c_{\alpha\beta\cdots\gamma})$ is the matrix of G taken with all elements zero except $c_{\cdots i} = \lambda_i$, $(i = 1, \cdots, n)$. Let the inverse $(b_{i\alpha})^{-1}$ of $(b_{i\alpha})$ be denoted by $(B^{\alpha i})$. The identity now implies

$$(a_{ij\cdots k}b_{j\beta}\cdots b_{k\gamma}) = (c_{\alpha\beta\cdots\gamma}B^{\alpha i}). \tag{16}$$

Setting $i = 1, \cdots, n$ in (16) we obtain the following relations between minors of the matrices in (16):

$$\begin{aligned} (a_{1j\cdots k}b_{j\beta}\cdots b_{k\gamma}) &= (c_{\alpha\beta\cdots\gamma}B^{\alpha 1}), \\ \cdots\cdots\cdots\cdots\cdots\cdots &, \\ (a_{nj\cdots k}b_{j\beta}\cdots b_{k\gamma}) &= (c_{\alpha\beta\cdots\gamma}B^{\alpha n}). \end{aligned}$$

These equations yield

$$\begin{aligned} a_{1j\cdots k}b_{j\beta}\cdots b_{k\gamma}y_\beta \cdots y_\gamma &= c_{\alpha\beta\cdots\gamma}B^{\alpha 1}y_\beta \cdots y_\gamma, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots &, \\ a_{nj\cdots k}b_{j\beta}\cdots b_{k\gamma}y_\beta \cdots y_\gamma &= c_{\alpha\beta\cdots\gamma}B^{\alpha n}y_\beta \cdots y_\gamma. \end{aligned} \tag{17}$$

If we let $\mu_{rs} = c_{s\cdots s}B^{sr}$, the right members of (17) become $G_1, \cdots, G_n$.

Since the principal determinant rank of G is obviously n, this rank of F is n if F is equivalent to G.

We have proved the necessity of the conditions. To prove the sufficiency we assume that the principal determinant rank of F is n, and that there exist non-singular linear transformations (14) in ϕ and a choice of μ_{ij}, $(i, j=1, \cdots, n)$, such that

$$
\begin{aligned}
&a_{1j\cdots k}b_{j\beta}\cdots b_{k\gamma}y_\beta\cdots y_\gamma = G_1,\\
&\cdots\cdots\cdots\cdots\cdots\cdots,\\
&a_{nj\cdots k}b_{j\beta}\cdots b_{k\gamma}y_\beta\cdots y_\gamma = G_n.
\end{aligned}
\tag{18}
$$

By (18)

$$
a_{ij\cdots k}b_{j\beta}\cdots b_{k\gamma}x_iy_\beta\cdots y_\gamma = x_\xi G_\xi, \qquad i, \cdots, \xi = 1, \cdots, n. \tag{19}
$$

Substituting (14) in (19) we obtain

$$
a_{ij\cdots k}b_{i\alpha}b_{j\beta}\cdots b_{k\gamma}y_\alpha y_\beta\cdots y_\gamma = G_\xi b_{\xi\alpha}y_\alpha.
$$

Since A is symmetric, the matrix of $G_\xi b_{\xi\alpha}y_\alpha$ is also symmetric. Denote this matrix by $D=(d_{\alpha\beta\cdots\gamma})$. Then

$$
d_{\alpha\beta\cdots\beta} = \mu_{\xi\beta}b_{\xi\alpha}
$$

(the repeated index on the left does not indicate summation) while all other elements of D vanish. By symmetry $d_{\alpha\beta\cdots\beta}=d_{\beta\cdots\beta\alpha}$. If $\alpha\neq\beta$, then $d_{\beta\cdots\beta\alpha}=0$, whence $d_{\alpha\beta\cdots\beta}=0$ also. Hence at most $d_{1\cdots 1}, \cdots, d_{n\cdots n}$ are non-zero. Since D is of principal determinant rank n, these are all non-zero. It follows that F is equivalent in ϕ to a form of type G, and that the *transformation which reduces $F_1, \cdots, F_n$ to $G_1, \cdots, G_n$ reduces F to G.*

The problem now resolves into the simultaneous equivalence in ϕ of $F_1, \cdots, F_n$ to $G_1, \cdots, G_n$.

8. **Transformations which bring sums of pth powers, $p\geqq 3$, into sums of pth powers.** We shall prove the following theorem:

THEOREM XI. *Let ϕ be an (n, p)-proper field. If the non-singular linear transformation $x_i=b_{ij}y_j$ brings a sum of pth powers $a_ix_i^p$, $(a_i\neq 0, i=1, \cdots, n; p\geqq 3)$, into a sum of pth powers $c_iy_i^p$, $(c_i\neq 0; i=1, \cdots, n)$, the matrix $B=(b_{ij})$ of order n has exactly one element in each row and column different from zero.*

Applying the transformation $x_i=b_{ij}y_j$ to $F=a_ix_i^p$ and equating to $c_iy_i^p$ we obtain

$$
a_i(b_{ij}y_j)^p = c_iy_i^p. \tag{20}
$$

Since ϕ is (n, p)-proper we can write (20) in the matrix form

$$
(a_ib_{ij}b_{ik}\cdots b_{im}b_{iq}) = (c_i\delta_{ij}\delta_{ik}\cdots\delta_{iq}), \tag{21}
$$

where $(\delta_{ij})=(\delta_{ik})= \cdots =(\delta_{iq})=I$, and I is a Kronecker delta. Now

$$(a_i b_{i1} b_{ik} \cdots b_{iq}) = (c_i \delta_{i1} \delta_{ik} \cdots \delta_{iq}). \tag{22}$$

Since the element $c_1\delta_{11}\delta_{11} \cdots \delta_{11}$ is the only non-vanishing element of the matrix on the right of (22), it is of principal determinant rank 1, therefore this is true of the left matrix of (22). We can write the latter matrix as the product

$$(a_i b_{i1} b_{ik} \cdots b_{im})(b_{iq}) \tag{23}$$

of ordinary 2-way matrices, where i is the column index of

$$N = (a_i b_{i1} b_{ik} \cdots b_{im}),$$

and $k \cdots m$ is the partition of indices associated with the rows of N (in (b_{iq}) i and q are row and column indices respectively). Since B is non-singular, (23) implies that N is of rank 1. Every second order minor of N can be written in the form

$$a_r b_{r1} a_s b_{s1} \left\| \begin{matrix} b_{rk_1} \cdots b_{rm_1} & b_{rk_2} \cdots b_{rm_2} \\ b_{sk_1} \cdots b_{sm_1} & b_{sk_2} \cdots b_{sm_2} \end{matrix} \right\|. \tag{24}$$

By a lemma of another paper* since B is non-singular, the 2-way matrix $Q=(b_{jk} \cdots b_{lm})$ is non-singular, where the partitions $\tau=j \cdots l$ and $\sigma=k \cdots m$ are the partitions of the rows and columns of Q. Since the determinants in (24) for a given r, s, $(r\neq s)$, form the class of all determinant minors of the rows of Q obtained by setting $j= \cdots =l=r$, s it follows that all of these determinant minors cannot vanish. Hence

$$a_r b_{r1} a_s b_{s1} = 0$$

for $r\neq s$. Since a_r, $a_s\neq 0$, we have $b_{r1}b_{s1}=0$ for $r\neq s$. Similarly

$$b_{r2}b_{s2} = \cdots = b_{rn}b_{sn} = 0$$

for $r\neq s$. Take $b_{i1}\neq 0$ for i equal to some value t. Then $b_{i1}=0$ for $i\neq t$. If $b_{i2}\neq 0$ for $i=f$, where $f\neq t$, then $b_{i2}=0$ for $i\neq f$; similar conclusions hold for $b_{i3}, \cdots, b_{in}$.

It is to be noted that if *p is a prime and ϕ has characteristic p, any non-singular linear transformation brings F into a sum of pth powers.*

Let $B=(b_{ij})$ be chosen so that it satisfies the preceding theorem. Apply the transformation $x_i=b_{ij}y_j$ to a form $c_i x_i^q$. We obtain $c_i(b_{ij}y_j)^q$ which is a sum of qth powers. We have proved the following theorem:

* R. Oldenburger, I, p. 625. See also G. W. King, *The indeterminate and composite products of matrices*, Journal of Mathematics and Physics, vol. 13 (1934), p. 445.

THEOREM XII. *Let ϕ be an (n, p)-proper field. If a non-singular linear transformation in ϕ brings a sum of n pth powers of $x_1, \cdots, x_n$ into such a sum, where $p \geqq 3$, this transformation brings a sum of qth powers of $x_1, \cdots, x_n$, for any q, into a sum of qth powers with the same number of non-vanishing coefficients.*

This implies the following corollary:

COROLLARY I. *Let ϕ be an (n, p)-proper field. Let F be a sum of n pth powers of $x_1, \cdots, x_n$ where $p \geqq 3$. A set of forms $F, R, S, \cdots, T$ in $x_1, \cdots, x_n$ are simultaneously equivalent in ϕ to sums of powers of $x_1, \cdots, x_n$ if and only if $R, S, \cdots, T$ are already sums of powers in $x_1, \cdots, x_n$.*

The reader should compare the simplicity of the theory of equivalence of r-tuples of forms in $x_1, \cdots, x_n$, where one of the forms is equivalent to $a_i x_i^p$, $(a_i \neq 0; i = 1, \cdots, n; p \geqq 3)$, with the corresponding theory for the quadratic case.

We are now able to prove the following theorem:

THEOREM XIII. *Let ϕ be an (n, p)-proper field. If the non-singular transformation $x_i = b_{ij} y_j$, $(i, j = 1, \cdots, n)$, brings a sum of r, $(r \leqq n)$, pth powers $F = a_i x_i^p$, $(i = 1, \cdots, r;\ a_i \neq 0)$, into a sum of r pth powers $F' = b_i y_i^p$, $(i = 1, \cdots, r;\ b_i \neq 0)$, then the matrix $B = (b_{ij})$ is of the form*

$$\left\| \begin{matrix} B_{11} & 0 \\ B_{21} & B_{22} \end{matrix} \right\|,$$

where B_{11} is a minor of order r with exactly one non-vanishing element in each row and column.

The matrix of F is unique since ϕ is also (r, p)-proper. Applying the transformation with matrix B to F we obtain

$$\sum_{i=1}^{r} a_i (b_{ij} y_j)^p, \qquad i = 1, \cdots, n.$$

Equating to F', we can write

$$\sum_{i=1}^{r} a_i b_{ij} b_{ik} \cdots b_{iq} y_j y_k \cdots y_q = \sum_{i=1}^{r} b_i y_i^p, \qquad j, \cdots, q = 1, \cdots, n. \tag{25}$$

Since there are no terms in $y_{r+1}, \cdots, y_n$ on the right of (25) we obtain, by the restrictions on ϕ,

$$\sum_{i=1}^{r} a_i b_{ij} b_{ik} \cdots b_{iq} = 0,$$

for $j=r+1, \cdots, n; k, \cdots, q=1, \cdots, n$. Setting $j=r+1$ in the above equation and writing it in matrix form we get

$$\left(\sum_{i=1}^{r} a_i b_{i,r+1} b_{ik} \cdots b_{im}\right)(b_{iq}) = 0, \tag{26}$$

where $k \cdots m$ is the partition of the rows of the left factor in (26), i is the index of the columns, and i, q are the row and column indices of (b_{iq}). Since B is non-singular the left factor in (26) is of rank zero; whence $a_i b_{i,r+1} b_{ik} \cdots b_{im}=0$ (i not summed) for $i=1, \cdots, r; k, \cdots, m=1, \cdots, n$. By an argument used in the proof of Theorem XI, the 2-way matrix $Q=(b_{ik} \cdots b_{lm})$ is non-singular where $\tau=j \cdots l$ is the partition of the rows of Q and $\sigma=k \cdots m$ the partition of the columns. The products $b_{ik} \cdots b_{im}$, i not summed, are the elements in the row of Q obtained by setting $j= \cdots =l=i$. Since Q is non-singular, these do not all vanish for a given i. Hence $a_i b_{i,r+1}=0$ for every i. Similarly $a_i b_{ij}=0$ (i not summed) for $j=r+2, \cdots, n$. Since $a_i \neq 0$ for every i, we have $b_{ij}=0$ for $i=1, \cdots, r$; $j=r+1, \cdots, n$. If B is of this form, the terms on the left of (25) involve only $y_1, \cdots, y_r$. Theorem XI now implies that B_{11} has exactly one element not zero in each row and column.

COROLLARY II. *Let ϕ be an (n, p)-proper field. If the non-singular transformation $x_i=b_{ij}y_j$, $(i, j=1, \cdots, n)$, brings a sum of pth powers $a_i x_i^p$, $(p \geqq 3$; $i=1, \cdots, r$; $a_i \neq 0)$, into a sum of pth powers $b_i y_i^p$, $(i=1, \cdots, r$; $b_i \neq 0)$, then the transformation brings any sum of qth powers $F=c_i x_i^q$, $(i=1, \cdots, r)$, into a sum of qth powers $F'=d_i x_i^q$, $(i=1, \cdots, r)$, where F and F' have the same number of non-vanishing coefficients.*

THEOREM XIV. *Let G be a sum of pth powers, $(p \geqq 3)$, of $x_1, \cdots, x_r$ with non-vanishing coefficients. Let F_i, $(i=1, \cdots, s)$, be a sum of qth powers of $x_1, \cdots, x_r$ for $i=1, \cdots, s$, where $q \geqq 3$ and the coefficients of x_i^p for any i are not all zero in $F_1, \cdots, F_s$. Let ϕ be an (n, p)- and (n, q)-proper field. The class of non-singular transformations $x_i=b_{ij}y_j$, $(i, j=1, \cdots, (n))$, $(n \geqq r)$, which bring G in ϕ into a like sum of powers G', is identical with the class of transformations which bring $F_1, \cdots, F_s$ into a like set $F_1', \cdots, F_s'$.*

Theorem XIII implies that the transformations bringing G into G' bring $F_1, \cdots, F_s$ into $F_1', \cdots, F_s'$. To prove the converse, assume that ϕ contains an infinite number of elements. Write

$$F_i = \mu_{ij} x_j^q, \qquad F_i' = \nu_{ij} y_j^q, \qquad\qquad i = 1, \cdots, s,$$

where by our assumptions $\mu_{1j}, \cdots, \mu_{sj}$ do not all vanish for a given j. Similar properties hold for $V_{1j}, \cdots, V_{sj}$. Write

$$L_j(z) \equiv \sum_{i=1}^{s} \mu_{ij} z_i, \qquad L_j'(z) = \sum_{i=1}^{s} v_{ij} z_i, \qquad j = 1, \cdots, r.$$

Choose a set of values $\alpha_1, \cdots, \alpha_s$ of $z_1, \cdots, z_s$ in ϕ such that* $L_j(\alpha)$, $L_j'(\alpha) \neq 0$, $(j=1, \cdots, r)$. Let $F \equiv \alpha_1 F_1 + \cdots + \alpha_s F_s \equiv L_1(\alpha) x_1^q + \cdots + L_r(\alpha) x_r^q$. The transform $F' = \alpha_1 F_1' + \cdots + \alpha_s F_s'$ of F is $L_1'(\alpha) y_1^q + \cdots + L_r'(\alpha) y_r^q$. Since none of the coefficients of $x_1^q, \cdots, x_r^q, y_1^q, \cdots, y_r^q$ in F and F' vanish, by Theorem XIII, G transforms into a like form G'.

If ϕ is a finite field, embed ϕ in an algebraically closed field ψ containing it. The above argument now applies for ψ. Since the coefficients of G and the b_{ij} are in ϕ, the coefficients of G' are in ϕ.

9. **Equivalence of forms of degree p, $p \geqq 4$, to sums of pth powers.** We shall prove the following theorem:

THEOREM XV. *Let ϕ be an (n, p)-proper field. Let $F = a_{i \cdots tq \cdots m} x_i \cdots x_t x_q \cdots x_m$ be an n-ary form of degree p, $(p \geqq 4)$, for which a subform $F_{\alpha \cdots \xi} = a_{\alpha \cdots \xi q \cdots m} x_q \cdots x_m$ of degree s, $(s \geqq 3)$, is equivalent in ϕ to a sum $S = \mu_i x_i^s$, $(\mu_i \neq 0)$, of n pth powers for some set of values $\alpha, \cdots, \xi$ of $i, \cdots, t$. The form F is equivalent in ϕ to a sum of n sth powers if and only if under reduction of $F_{\alpha \cdots \xi}$ to S in ϕ, F transforms covariantly into a sum of pth powers.*

Let Σ denote the set of values $\alpha, \beta, \cdots, \delta, \rho, \tau, \xi$ of the leading indices $i, \cdots, t$ of F. A study of minors of (a_{iT}), where T is the partition $j \cdots m$, reveals that the principal determinant rank of any subform F_σ (σ here denotes a fixed partition) *is greater than or equal to* the principal determinant rank of a subform F_Σ if σ is contained in Σ. By Theorem I the principal determinant rank of $F_{\alpha \cdots \xi}$ is n. Hence this rank of F is n. By Theorem X, F is equivalent in ϕ to a sum of n pth powers if and only if $F_1, \cdots, F_n$ are simultaneously equivalent in ϕ to sums of pth powers. Since α is contained in the set $\alpha, \cdots, \xi$, the principal determinant rank of F_α is n. Now F_α occurs in the set $F_1, \cdots, F_n$. If F is equivalent in ϕ to a sum of n pth powers, F_α is equivalent in ϕ to a sum of $(p-1)$th powers. There are n such powers because the principal determinant rank of F_α is n. By Theorem X, F_α is equivalent to such a sum if and only if $F_{\alpha 1}, \cdots, F_{\alpha n}$ are simultaneously equivalent in ϕ to sums of $(p-2)$th powers. The subform $F_{\alpha\beta}$ has principal determinant rank n and occurs in this set. Continuing this process, we find that if F is equivalent in ϕ to a sum of pth powers, $F_{\alpha\beta \cdots \tau}$ is equivalent in ϕ to a sum of n $(s+1)$th powers. By Theorem X, this equivalence is valid if and only if $F_{\alpha\beta \cdots \tau 1}, \cdots, F_{\alpha\beta \cdots \tau n}$ are simultaneously equivalent in ϕ to sums of sth powers. The subform $F_{\alpha\beta \cdots \tau\xi}$ occurs in this set.

* O. Haupt, *Einführung in die Algebra*, vol. 1, p. 166.

Reduce $F_{\alpha\beta\cdots\tau\xi}$ non-singularly to a sum of sth powers S. Simultaneously the remaining forms in the set $F_{\alpha\beta\cdots\tau 1}, \cdots, F_{\alpha\beta\cdots\tau n}$ are transformed into a new set $F'_{\alpha\beta\cdots\tau 1}, \cdots, F'_{\alpha\beta\cdots\tau n}$. By the corollary of Theorem XII, these forms are simultaneously equivalent in ϕ to a sum of qth powers if and only if they are already such sums. Assume that they are such sums. By a remark near the end of §7, the transformation T that reduces $F_{\alpha\beta\cdots\tau 1}, \cdots, F_{\alpha\beta\cdots\tau n}$, to sums of sth powers reduces $F_{\alpha\beta\cdots\tau}$ to a sum $F'_{\alpha\beta\cdots\tau}$ of $(s+1)$th powers. Let the result of applying T to $F_{\alpha\beta\cdots\rho 1}, \cdots, F_{\alpha\beta\cdots\rho n}$ be denoted by $F'_{\alpha\beta\cdots\rho 1}, \cdots, F'_{\alpha\beta\cdots\rho n}$. Since $F'_{\alpha\beta\cdots\tau}$ is in this set and is already a sum of n $(s+1)$th powers, the forms $F'_{\alpha\beta\cdots\rho 1}, \cdots, F'_{\alpha\beta\cdots\rho n}$ are, by the corollary of Theorem XII, simultaneously equivalent to sums of $(s+1)$th powers if and only if they are already such sums. If they are such sums, by the remark of §7 referred to above, the transformation T brings $F_{\alpha\beta\cdots\rho}$ into a form $F'_{\alpha\beta\cdots\rho}$, which is a sum of $(s+2)$th powers. Continuing this chain of reasoning we see that the T transforms $F_1', \cdots, F_n'$ of $F_1, \cdots, F_n$ are simultaneously equivlent in ϕ to sums of $(p-1)$th powers if and only if they are already such sums. The transformation T then brings F into a sum of pth powers F'.

Hence if F is equivalent in ϕ to a sum of n pth powers, the transformation T which reduces $F_{\alpha\cdots\xi}$ to a sum of n sth powers must reduce F to a sum of n pth powers. The sufficiency of this condition is obvious.

10. **Simultaneous equivalence of forms of degree p, $(p \geqq 3)$, to sums of pth powers.** The assumption made in Theorem XV concerning the existence of a subform $F_{\alpha\cdots\xi}$ of degree s, $(s \geqq 3)$, equivalent to a sum of n sth powers is evidently not a necessary condition for the equivalence of a form F to a sum of n pth powers. It is therefore convenient to have a method of determining whether or not a set of forms in n variables of degree p, $(p \geqq 3)$, are simultaneously equivalent to sums of pth powers where no form in the set is equivalent to a sum of n pth powers. I have developed such a theory which involves the theorems of §§7–9, Theorem XVI below, and other considerations. Since parts of the method are complicated, the development will be left to the reader. Essential to this theory is the following theorem:

THEOREM XVI. *Let*

$$B = (b_{ij}) = \begin{Vmatrix} B_{11} & 0 \\ B_{21} & B_{22} \end{Vmatrix},$$

where B_{11} is a minor of order r with exactly one non-vanishing element in each row and column. For the form

$$F = a_{ij\cdots km}x_i x_j \cdots x_k x_m, \quad i, \cdots, m = 1, \cdots, n,$$

of degree p, $(p \geqq 3)$, to be equivalent in an (n, p)-proper field ϕ under the non-singular transformation

$$x_i = b_{ij}y_j, \qquad i, j = 1, \cdots, n,$$

with matrix B to a sum of pth powers

$$F' = \lambda_\alpha y_\alpha^p, \qquad \alpha = 1, \cdots, r+g, \qquad \lambda_{r+1}, \cdots, \lambda_{r+g} \neq 0,$$

for some g, it is necessary that the form

$$L = a_{ij\cdots km}x_i x_j \cdots x_k x_m, \qquad i, \cdots, m = r+1, \cdots, n,$$

be equivalent in ϕ under the non-singular transformation

$$x_i = b_{ij}y_j, \qquad i, j = r+1, \cdots, n, \tag{27}$$

with matrix B_{22} to a form of the type

$$L' = \lambda_\alpha y_\alpha^p, \qquad \alpha = r+1, \cdots, r+g.$$

Assume that F' is equivalent in ϕ to F for some choice of the λ's and g under the transformation

$$y_i = b_{ij}x_j, \qquad i, j = 1, \cdots, n, \tag{28}$$

with matrix B. This implies that there exist a transformation (28), λ's, and a g such that

$$\sum_{\alpha=1}^{r+g} \lambda_\alpha b_{\alpha i} b_{\alpha j} \cdots b_{\alpha m} x_i x_j \cdots x_m = F, \qquad i, j, \cdots, m = 1, \cdots, n. \tag{29}$$

Since $b_{\alpha i} = 0$ for $\alpha = 1, \cdots, r$ and $i = r+1, \cdots, n$, we derive from (29) the relations

$$\sum_{\alpha=r+1}^{r+g} \lambda_\alpha b_{\alpha i} b_{\alpha j} \cdots b_{\alpha m} x_i x_j \cdots x_m = L, \qquad i, \cdots, m = r+1, \cdots, n. \tag{30}$$

If (30) is satisfied, there exists a non-singular transformation

$$y_\alpha = b_{\alpha i}x_i, \qquad \alpha, i = r+1, \cdots, n,$$

and a choice of L' such that L' reduces to L. Hence there exists a non-singular transformation (27) bringing L into L'.

It is to be noted that if $F_1, \cdots, F_t$ are sums of pth powers, then a set of forms $F_1, \cdots, F_t, F_{t+1}, \cdots, F_s$ are simultaneously equivalent in a field ϕ to sums of pth powers if and only if $F_{t+1}, \cdots, F_s$ are equivalent to sums of pth powers under transformations which bring $F_1, \cdots, F_t$ into sums of pth powers. These transformations can be written down by Theorems XIII and XIV.

11. **Equivalence of a binary form to a sum of two pth powers.** We shall prove a modification of Theorem XV. In this section we shall denote the subform

$$a_{i\,1\cdots 1\,eh\cdots gf\,2\cdots 2}\, x_i x_e x_h \cdots x_g x_f, \qquad i, \cdots, f = 1, 2,$$

of $F = a_{i\cdots m} x_i \cdots x_m$, $(i, \cdots, m = 1, 2)$, by $F_{\rho\tau}$. In $F_{\rho\tau}$ there are ρ fixed consecutive subscripts equal to 1 and τ equal to 2, the remaining subscripts being free. We shall need the following lemma:

LEMMA III. *Let ϕ be a $(2, p)$-proper field. Let $F = a_{ij\cdots m} x_i x_j \cdots x_m$ be a binary form of degree p with principal determinant rank 2. If the principal determinant rank of a subform $F_{\rho\tau} = x_1 F_{\rho+1,\tau} + x_2 F_{\rho,\tau+1}$ of degree greater than or equal to 3 is 2, while this rank is 1 for each of the subforms $F_{\rho+1,\tau}$, $F_{\rho,\tau+1}$, the form F is equivalent in ϕ to a sum of pth powers if and only if F is already a sum of pth powers; then $F_{\rho\tau} \equiv F$.*

The columns of the ordinary 2-way matrix $(a_{i\tau})$, where τ is the partition $j \cdots m$, are, by the symmetry of $(a_{i\cdots m})$, the columns of the matrix

$$A' = \left\| \begin{matrix} a_{11\cdots 1} & a_{11\cdots 12} & a_{11\cdots 122} & a_{11\cdots 1222} \cdots a_{12\cdots 2} \\ a_{21\cdots 1} & a_{21\cdots 12} & a_{21\cdots 122} & a_{21\cdots 1222} \cdots a_{22\cdots 2} \end{matrix} \right\|;$$

whence the rank of A' is the principal determinant rank of F. Since this rank of F is 2, some minor

$$N = \left\| \begin{matrix} a & b \\ c & d \end{matrix} \right\| = \left\| \begin{matrix} a_{1\,1\cdots 1\,2\cdots 2\,2\cdots 2} & a_{1\,1\cdots 1\,1\cdots 1\,2\cdots 2} \\ a_{2\,1\cdots 1\,2\cdots 2\,2\cdots 2} & a_{2\,1\cdots 1\,1\cdots 1\,2\cdots 2} \end{matrix} \right\|$$

of A' is non-singular, where for each element of the minor matrix N the first subscript is followed by sets of equal numbers, the first containing ρ 1's, the second σ 1's or 2's, and the last τ 2's. N is a minor of the matrix which is given by

$$M = \left\| \begin{matrix} a_{1\,1\cdots 1\,uv\cdots w\,2\cdots 2} \\ a_{2\,1\cdots 1\,uv\cdots w\,2\cdots 2} \end{matrix} \right\|, \qquad u, v, \cdots, w = 1, 2, \tag{31}$$

where only a typical column of M is displayed in (31). Here the first set of equal subscripts contains ρ 1's and the last τ 2's. Setting $u = 1$ and $u = 2$ in M we obtain minors M_1 and M_2 which can be written as

$$M_1 = \left\| \begin{matrix} a_{1\,1\cdots 1\,v\cdots w\,2\cdots 2} \\ a_{2\,1\cdots 1\,v\cdots w\,2\cdots 2} \end{matrix} \right\|, \qquad M_2 = \left\| \begin{matrix} a_{1\,1\cdots 1\,v\cdots w\,2\cdots 2} \\ a_{2\,1\cdots 1\,v\cdots w\,2\cdots 2} \end{matrix} \right\|.$$

The matrix M is composed of the columns of M_1 and M_2. The first subscript for each element of M_1 is followed by $(\rho+1)$ 1's, and the set of sub-

scripts ends with a set of τ 2's. For the matrix M_2 the sets have ρ 1's and $(\tau+1)$ 2's, respectively. The only columns that M_1 and M_2 do not have in common are the columns of N. Write

$$M_1 = (\xi_1 \cdots \xi_g), \qquad M_2 = (\xi_{g+1} \cdots \xi_{2g}), \qquad N = (\xi_{2g}\xi_1),$$

where the ξ's are columns of M_1, M_2, N. The matrices M, M_1, M_2 are the matrices of the forms $F_{\rho\tau}$, $F_{\rho+1,\tau}$, and $F_{\rho,\tau+1}$ of the lemma. We assume that the principal determinant ranks of $F_{\rho+1,\tau}$, $F_{\rho,\tau+1}$ are 1; whence the ranks of M_1 and M_2 equal 1. Assume that $\xi_2 \neq 0$. Since M_1 is of rank 1, ξ_1, ξ_2 are linearly dependent. Since ξ_2 also occurs among the vectors $\xi_{g+1}, \cdots, \xi_{2g-1}$, and M_2 is of rank 1, ξ_2 and ξ_{2g} are linearly dependent. It follows that ξ_1, ξ_{2g} are linearly dependent contrary to assumption.

Since $\xi_2 = \cdots = \xi_g = 0$, whence $\xi_{g+1} = \cdots = \xi_{2g-1} = 0$, we make use of the assumption of the lemma that the degree of $F_{\rho\tau}$ is greater than or equal to 3, and it therefore follows that there are at least two variables in the set $u, v, \cdots, w$. The elements

$$a_{1\cdots 1\, 2\cdots 2}, \qquad a_{2\, 1\cdots 1\, 2\cdots 2},$$

where the set of subscripts of the first element ends with $\tau+1$ 2's and the second element has $\rho+1$ subscripts equal to 1, now occur among the vectors $\xi_2, \cdots, \xi_g$. These elements therefore vanish. Since they are the same as the elements

$$a_{1\, 1\cdots 1\, 2\cdots 2\, 2\cdots 2}, \qquad a_{2\, 1\cdots 1\, 1\cdots 1\, 2\cdots 2}$$

of ξ_1, ξ_{2g}, where the subscripts fall into groups as in N, the form $F_{\rho\tau}$ is given by the equation

$$F_{\rho\tau} = a_{1\, 1\cdots 1\, 1\cdots 1\, 2\cdots 2}\, x_1^{\sigma+1} + a_{2\, 1\cdots 1\, 2\cdots 2\, 2\cdots 2}\, x_2^{\sigma+1}. \tag{32}$$

Since $F_{\rho\tau}$ is thus a sum of $(\sigma+1)$th powers, $(\sigma+1 \geqq 3)$, by Theorem XV, F is equivalent to a sum of pth powers if and only if it is already such a sum.

To complete the proof of the lemma we note that if F is a sum of two pth powers, the principal determinant rank of every subform $F_{\rho\tau} \not\equiv F$ is 1 or 0; whence the form $F_{\rho\tau}$ of the lemma is F.

The *characteristic determinant* of two quadratic forms F, G in the same variables x_1, x_2 with matrices A, B, respectively, is the determinant $|x_1A + x_2B|$.

If F is equivalent to a sum of 2 pth powers and there is no subform $F_{\rho\tau}$ satisfying the conditions of Lemma III, there is a quadratic form $F_{\rho\tau}$ with rank 2. For a form F having such a subform we shall prove the following strengthened form of Theorems III and VI. It is to be noted that *no distinction is made between even and odd values of p.*

THEOREM XVII. *Let ϕ be a $(2, p)$-proper field. Let $F = a_{ij\cdots km}x_i x_j \cdots x_k x_m$, $(i, \cdots, m = 1, 2)$, be a binary form of degree p, $(p \geqq 3)$, not already a sum of pth powers. The form F is equivalent in ϕ to a sum of 2 pth powers if and only if for some ρ, τ the quadratic subform $F_{\rho\tau}$ is of rank 2, the characteristic determinant D of $F_{\rho\tau}$, $F_{\rho-1,\tau+1}$ or $F_{\rho\tau}$, $F_{\rho+1,\tau-1}$ factors in ϕ into distinct linear factors, and under reduction of D to canonical form $Kx_1'x_2'$ in ϕ F transforms covariantly into a sum of pth powers.*

That the rank of $F_{\rho\tau}$ equals 2 for some ρ, τ was noted above to be a necessary condition. By the argument used in the proof of Theorem XV the simultaneous equivalence in ϕ of $F_{\rho\tau}$, $F_{\rho-1,\tau+1}$ and $F_{\rho\tau}$, $F_{\rho+1,\tau-1}$ to sums of squares is also a necessary condition. By Theorem X these pairs of forms are simultaneously equvalent in ϕ to sums of squares if and only if the cubic forms $F_{\rho-1,\tau} = x_1F_{\rho\tau} + x_2F_{\rho-1,\tau+1}$ and $F_{\rho,\tau-1} = x_1F_{\rho+1,\tau-1} + x_2F_{\rho\tau}$ are respectively equivalent to sums of cubes. By Theorem IIIa this equivalence is possible if and only if the characteristic determinants of $F_{\rho\tau}$, $F_{\rho-1,\tau+1}$ and $F_{\rho\tau}$, $F_{\rho+1,\tau-1}$, respectively, factor in ϕ into distinct linear factors. If it is assumed that this necessary condition is satisfied for one of the pairs of quadratic subforms, one of the cubic forms $F_{\rho-1,\tau}$ or $F_{\rho,\tau-1}$ is equivalent in ϕ to a sum of 2 cubes. By Theorem III the transformation in ϕ, which reduces D to canonical form $Kx_1'x_2'$, simultaneously reduces $F_{\rho-1,\tau}$ or $F_{\rho,\tau-1}$, as the case may be, to sums of cubes. By Theorem XV, F is now equivalent in ϕ to a sum of pth powers if and only if under reduction of D to $Kx_1'x_2'$. F transforms into a sum of pth powers.

ARMOUR INSTITUTE OF TECHNOLOGY,
CHICAGO, ILL.
THE INSTITUTE FOR ADVANCED STUDY,
PRINCETON, N. J.