ON THE TRANSITIVITY OF PERSPECTIVITY
IN CONTINUOUS GEOMETRIES*

BY

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Introduction. The class of finite dimensional projective geometries has been extended to include non-finite dimensional ones by J. von Neumann's remarkable discovery of continuous geometries.‡ In an axiomatic formulation of the geometry as an irreducible complemented modular lattice§ the finiteness of the dimensionality is guaranteed by a chain condition. Von Neumann drops this chain condition and, retaining explicitly only two of its weak consequences, namely, completeness of the geometry and a certain continuity of the lattice operations, succeeds in establishing the existence of an essentially unique real-valued dimension function which may have either a discrete bounded range (the classical finite dimensional projective geometries) or a continuous bounded range (the new continuous geometries). In every case it is understood that the dimension function $D(a)$ is to satisfy

$$D(a + b) + D(ab) = D(a) + D(b)$$

for all $a, b$.

It is clear that such a dimension function will be closely connected with perspectivities. For $a, b$ are said to be perspective if there exists a $c$ such that

$$a + c = b + c, \quad ac = bc;$$

and for such $a, b$ (1) implies

$$D(a) + D(c) = D(a + c) + D(ac) = D(b + c) + D(bc) = D(b) + D(c)$$

and hence, if $D(c)$ is finite, $D(a) = D(b)$. This motivates a definition of equidimensionality, namely, $a$ and $b$ are called equidimensional if and only if they are perspective. That this definition will lead to the desired dimension function (in an irreducible system) depends in an essential way on the funda-

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The transitivity of perspectivity has been established by von Neumann for reducible as well as irreducible systems* but partly by indirect methods which require the full force of the completeness and continuity axioms. Now while these axioms are indeed necessary for the existence of the dimension function (in irreducible systems), weaker ones will secure the transitivity of perspectivity (in reducible as well as irreducible systems), in fact, just those parts of von Neumann's axioms which involve at most countable sets of elements.†

The present paper is devoted chiefly to a proof of the transitivity of perspectivity which uses direct methods throughout and holds for all systems satisfying these weaker axioms. The paper is divided into six sections. The weakened set of axioms to be used is formulated in §1. We require parts of C.G., part I, usually in very specialized form, and for convenience these are collected (briefly) in §§2, 3, 4. The new material in the proof of the transitivity of perspectivity is contained in §5. The additivity and continuity properties of perspectivity are established in §6. The Lemma 5.1 in §5 may perhaps be not without some interest of its own.

1. The partially ordered system. We shall consider a system $L$ of elements $a, b, c, \ldots, x, y, u, v, \ldots, A, B, \ldots$ which is partially ordered, that is, we shall assume that a relation $a \preceq b$ (written equivalently $b \succeq a$) holds for certain pairs of elements of $L$ in such a way that

(i) $a \preceq b, b \preceq c$ together imply $a \preceq c$, and
(ii) $a \preceq b, b \preceq a$ are together equivalent to $a = b$.

The following axioms are postulated:

**Axiom I. Countable completeness.** For every finite or countably infinite set‡ of elements $a_1, a_2, \ldots$ there exist the following elements:

$I_1$. a sum element $a$ (written $\sum_n a_n$ or equivalently $a_1+a_2+\cdots$) such that for any $x$ of $L$, $x \succeq a$ if and only if $x \succeq a_n$ for every $n$,

$I_2$. an intersection element $a$ (written $\prod_n a_n$ or equivalently $a_1a_2\cdots$) such that for any $x$ of $L$, $x \preceq a$ if and only if $x \preceq a_n$ for every $n$.

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* For the general case see C.G., part III, p. 22, Theorem 2.3; the special (irreducible) case is also a consequence of the theorems of C.G., part I (see C.G., part I, p. 49, corollary to Theorem 5.16).
† That the "countable" axioms are really weaker than the original axioms of von Neumann can be shown by a simple example which satisfies the "countable" axioms but which has no zero, and hence is not complete.
‡ All sets considered in this paper will be non-void. Thus Axiom I does not imply the existence of a zero or of a unit element.
Axiom II. Countable continuity. Let $a_1, a_2, \ldots$ be any countably infinite sequence, and let $c$ be an arbitrary element of $L$. Then

\begin{align*}
\Pi_1. \quad & (\sum_{n=1}^\infty a_n)c = \sum_{p=1}^\infty \{ (\sum_{n=1}^p a_n)c \}; \\
\Pi_2. \quad & (\prod_{n=1}^\infty a_n) + c = \prod_{p=1}^\infty \{ (\prod_{n=1}^p a_n) + c \}.
\end{align*}

Axiom III. Modularity. For all $a$, $b$, $c$,

$$(a + b)c = \{ a + (a + c)b \}c,$$

or what is equivalent, $a \leq c$ implies $(a + b)c = a + bc$.

Axiom IV. Complementation. For any three $a$, $b$, $c$ such that $a \leq b \leq c$ there exists an element $d$ such that $b + d = c$, $bd = a$.

2. Independent sets of elements. We make the following definition:

Definition 2.1. A finite ($\geq 2$) or countably infinite set of elements $a_1, a_2, \ldots$ is independent (written $(a_n, n = 1, 2, \ldots \perp)$) if for every two mutually exclusive subsets $a_{i_1}, a_{i_2}, \ldots$ and $a_{j_1}, a_{j_2}, \ldots$

$$\left( \sum_n a_{in} \right) \left( \sum_n a_{jn} \right) = \prod_n a_n.$$

The $a_n$ are said to be independent over $\theta$ if all such $(\sum_n a_{in})(\sum_n a_{jn})$ equal $\theta$.

Lemma 2.1. If the $a_n$ are independent over $\theta$, then $\theta = \prod_n a_n$ and $(a_n, n = 1, 2, \ldots \perp)$.

**Proof.** Since $\prod_n a_n = a_1(\prod_{n\neq 1} a_n)$, the lemma follows from Definition 2.1.

Lemma 2.2. If $a_1, a_2, \ldots$ are independent over $\theta$, then every subset $a_{i_1}, a_{i_2}, \ldots$ is independent over $\theta$.

**Proof.** The lemma follows directly from Definition 2.1 and Lemma 2.1.

Lemma 2.3. If $a_1, a_2, \ldots$ are independent over $\theta$ and if $(a_{i_r}, r = 1, \ldots)$ are mutually exclusive subsets for $i = 1, 2, \ldots$, then $\sum_i a_{i_r}$, $i = 1, 2, \ldots$, are independent over $\theta$.

**Proof.** The lemma follows immediately from Definition 2.1.

Lemma 2.4. If $\theta$, $a_1, a_2, \ldots$ are such that for every two finite and mutually exclusive subsets $a_{i_1}, \ldots, a_{i_p}$ and $a_{j_1}, \ldots, a_{j_q}$

$$\left( \sum_{n=1}^p a_{in} \right) \left( \sum_{n=1}^q a_{jn} \right) = \theta,$$

then the $a_n$ are independent over $\theta$.

* If $\theta$ is a zero element of $L$, that is, if $a \leq \theta$ holds for every $a$ in $L$, then our independence over $\theta$ is precisely the notion of independence as used in C.G., part I, chap. 2.
Proof. Let \( a_{i_n}, (n=1, \cdots) \), and \( a_{j_n}, (n=1, \cdots) \), be any two mutually exclusive subsets of \( a_1, a_2, \cdots \). Then

\[
\left( \sum_{n} a_{i_n} \right) \left( \sum_{n} a_{j_n} \right) = \left( \sum_{n} a_{i_n} \right) \left\{ \sum_{q} \left( \sum_{n=1}^{q} a_{j_n} \right) \right\} \\
= \sum_{q} \left\{ \left( \sum_{n} a_{i_n} \right) \left( \sum_{n=1}^{q} a_{j_n} \right) \right\} \\
= \cdots = \sum_{q} \left\{ \sum_{p} \left( \sum_{n=1}^{p} a_{i_n} \right) \left( \sum_{n=1}^{q} a_{j_n} \right) \right\} \\
= \sum_{q} \left\{ \sum_{p} (\theta) \right\} = \theta;
\]

and the lemma follows from Definition 2.1.

Corollary. A countably infinite set of elements is independent over \( \theta \) if and only if every finite \((\geq 2)\) subset is independent over \( \theta \).

Proof. The corollary follows immediately from Lemmas 2.2 and 2.4.

Lemma 2.5. Let \( \theta, a_1, a_2, \cdots \) satisfy \( a_n \geq \theta \) for every \( n \). Let \( r_1, r_2, \cdots, r_m \) be distinct integers, and let \( S \) be any set of integers not containing \( r_m \). If \( a_{r_m} \sum_{n \neq r_m} a_n = \theta \), then

\[
\left( \sum_{n=1}^{m} a_{r_n} \right) \left( \sum_{n \in S} a_n \right) = \left( \sum_{n=1}^{m-1} a_{r_n} \right) \left( \sum_{n \in S} a_n \right).
\]

Proof.

\[
\left( \sum_{n=1}^{m} a_{r_n} \right) \left( \sum_{n \in S} a_n \right) = \left( \sum_{n=1}^{m-1} a_{r_n} + a_{r_m} \right) \left( \sum_{n \in S} a_n \right) \left( \sum_{n \in S} a_n \right) \\
= \left\{ \left( \sum_{n=1}^{m-1} a_{r_n} \right) + a_{r_m} \left( \sum_{n \notin r_m} a_n \right) \right\} \left( \sum_{n \in S} a_n \right) \\
= \left( \sum_{n=1}^{m-1} a_{r_n} \right) + \theta \left( \sum_{n \in S} a_n \right) = \left( \sum_{n=1}^{m-1} a_{r_n} \right) \left( \sum_{n \in S} a_n \right),
\]

which proves the lemma.

Lemma 2.6. If \( \theta, a_1, a_2, \cdots \) are such that \( a_{n+1}(a_1 + \cdots + a_n) = \theta \) for every \( n = 1, 2, \cdots \), then the \( a_n \) are independent over \( \theta \).

Proof. By Lemma 2.4 we need only show that

\[
\left( \sum_{n=1}^{p} a_{i_n} \right) \left( \sum_{n=1}^{q} a_{j_n} \right) = \theta
\]
for all finite \( p, q \) and different \( i_1, \ldots, i_p; j_1, \ldots, j_q \); and this follows from a finite number of applications of Lemma 2.5.

**Corollary.** If \( \theta, c, a_1, a_2, \ldots \) are such that

\[
a_n(a_{n+1} + \cdots + a_{n+p} + c) = \theta
\]

for all \( n \geq 1, p \geq 1 \), then the \( c, a_n \) are independent over \( \theta \).

**Proof.** By Lemma 2.6, \( c, a_{n+p}, a_{n+p-1}, \ldots, a_n \) are independent over \( \theta \) for all \( n \geq 1, p \geq 0 \). Therefore, by the corollary to Lemma 2.4, \( c, a_1, a_2, \ldots \) are independent over \( \theta \).

**Lemma 2.7.** Let \( a_1, a_2, \ldots \) be independent over \( \theta \). If \( S_1, S_2, \ldots \) are arbitrary subsets of the integers \( 1, 2, \ldots \) and \( S \) is the set of the integers common to all \( S_n \), then

\[
\prod_{t} \left( \sum_{n \in S_t} a_n \right) = \sum_{n \in S} a_n.
\]

**Proof.** Let \( T = (a_{r_1}, a_{r_2}, \ldots) \) be the set of the integers not in \( S \). Then

\[
\prod_{t} \left( \sum_{n \in S_t} a_n \right) = \left( \sum_{n \in S} a_n + \sum_{n \in T} a_n \right) \prod_{t} \left( \sum_{n \in S_t} a_n \right)
\]

\[
= \sum_{n \in S} a_n + \left( \sum_{n \in T} a_n \right) \prod_{t} \left( \sum_{n \in S_t} a_n \right)
\]

\[
= \sum_{n \in S} a_n + \sum_{m} \left\{ \left( \sum_{t=1}^{m} a_{r_t} \right) \prod_{t} \left( \sum_{n \in S_t} a_n \right) \right\}
\]

\[
= \sum_{n \in S} a_n + \sum_{m} \left( \theta \right) = \sum_{n \in S} a_n
\]

(by repeated use of Lemma 2.5) as required.

**Lemma 2.8.** If \( a_n, (n = 1, 2, \ldots), \) are independent over \( \theta \) and \( \theta \leq u_n \leq a_n \) for \( n = 1, \ldots, p \), and if \( \theta \leq v_n \leq a_n \) for \( n = 1, \ldots, q \), then

\[
\left( \sum_{n=1}^{p} u_n \right) \left( \sum_{n=1}^{q} v_n \right) = \sum_{n=1}^{\min(p,q)} (u_nv_n).
\]

**Proof.**

\[
(u_1 + u_2)(v_1 + v_2) = (u_1 + u_2)(u_1 + a_2)(a_1v_1 + v_2)
\]

\[
= (u_1 + u_2) \{v_2 + v_1a_1(u_1 + a_2)\}
\]

\[
= (u_1 + u_2) \{v_2 + v_1(u_1 + a_1a_2)\} = (u_1 + u_2)(v_2 + v_1u_1)
\]

\[
= \cdots = (u_1 + u_2)(u_1v_1 + u_2v_2) = u_1v_1 + u_2v_2.
\]

Thus the lemma holds for \( p = q = 2 \). But if the lemma holds for all \( p = q < m \),

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then it holds for \( p = q = m \) too. For
\[
\left( \sum_{n=1}^{m} u_n \right) \left( \sum_{n=1}^{m} v_n \right) = \left( \sum_{n=1}^{m-1} u_n \right) \left( \sum_{n=1}^{m-1} v_n \right) + u_m v_m = \sum_{n=1}^{m} (u_n v_n),
\]
since \( \theta \leq \sum_{n=1}^{m-1} u_n, \sum_{n=1}^{m-1} v_n \leq \sum_{n=1}^{m-1} a_n; \theta \leq u_m, v_m \leq a_m; \) and \( \sum_{n=1}^{m-1} a_n, a_m \) are independent over \( \theta \). Thus the lemma holds for all \( p = q \). If \( p \neq q \), say \( p < q \), we can set \( u_n = \theta \) for \( p < n \leq q \) and apply the result just proved for \( p = q \).

**Lemma 2.9.** If \( a_1, a_2, \ldots \) are independent over \( \theta \) and \( \theta \leq u_n, v_n \leq a_n; \) for \( n = 1, 2, \ldots \), then
\[
\left( \sum_{n=1}^{m} u_n \right) \left( \sum_{n=1}^{m} v_n \right) = \sum_{n=1}^{m} (u_n v_n).
\]

**Proof.**
\[
\left( \sum_{n=1}^{m} u_n \right) \left( \sum_{n=1}^{m} v_n \right) = \sum_{p} \left\{ \sum_{n=1}^{p} u_n \right\} \left\{ \sum_{q} \left( \sum_{n=1}^{q} v_n \right) \right\}
\]
\[
= \sum_{q} \left\{ \sum_{p} \left( \sum_{n=1}^{p} u_n \right) \left( \sum_{n=1}^{q} v_n \right) \right\}
\]
\[
= \sum_{n} \left\{ \sum_{p} \min(p, q) \right\} \left( \sum_{n=1}^{\min(p, q)} (u_n v_n) \right)
\]
\[
= \sum_{n} (u_n v_n)
\]
(by Lemma 2.8) as required.

**Lemma 2.10.** Let \( a_1, a_2, \ldots \) be independent over \( \theta \). If \( a_i \geq a_{ij} \geq \theta \) for all \( i, j \), and if the elements \( a_{ij}, j = 1, \ldots \), are independent over \( \theta \) (whenever there are at least two elements in the set) for every \( i = 1, 2, \ldots \), then the set of all \( a_{ij}, (i, j = 1, \ldots) \), is independent over \( \theta \).

**Proof.** If \( a_{i,r}, (r = 1, \ldots) \), and \( a_{k,i}, (s = 1, \ldots) \), are mutually exclusive subsets of the \( a_{ij} \), then
\[
\left( \sum_{r} a_{i,r} \right) \left( \sum_{s} a_{k,i} \right) = \left\{ \sum_{i_r} \left( \sum_{j_r} a_{i_r,j_r} \right) \right\} \left\{ \sum_{k_s} \left( \sum_{l_s} a_{k_s,l_s} \right) \right\}
\]
\[
= \sum_{n} \left\{ \sum_{i_r} a_{n,i_r} \right\} \left( \sum_{l_s} a_{n,l_s} \right)
\]
\[
= \sum_{n} \{ \theta \} = \theta
\]
(by Lemma 2.9), which proves the lemma.
Lemma 2.11. If \( \theta \leq a_0 \) and if \( a_n, a_n' \) are defined for \( n=1, 2, \ldots \) in such a way that

\[
a_{n-1} = a_n + a_n', \quad a_n a_n' = \theta,
\]

for \( n=1, 2, \ldots \), then \( (\prod_r a_r), a_n', (n=1, 2, \ldots) \), are independent over \( \theta \) and \( a_0 = \sum_n a_n' + \prod_r a_r \).

Proof.

\[
a_n' \left( a_{n+1}' + a_{n+2}' + \cdots + a_{n+p}' + \prod_r a_r \right)
= a_n' a_n \left( a_{n+1} + a_{n+2} + \cdots + a_{n+p} + \prod_r a_r \right) = \theta
\]

for all \( n, p \). Hence \( (\prod_r a_r), a_1', a_2', \ldots \) are independent over \( \theta \) by the corollary to Lemma 2.6. Furthermore

\[
a_0 = a_1' + a_1 = a_1' + a_2' + a_2 = \cdots
= \sum_{n=1}^{\infty} a_n' + a_r = \sum_{n=1}^{\infty} a_n' + \prod_r a_r,
\]

for \( r=1, 2, \ldots \). Hence

\[
a_0 = \prod_r \left( \sum_{n=1}^{\infty} a_n' + a_r \right) = \sum_{n=1}^{\infty} a_n' + \prod_{r=1}^{\infty} a_r;
\]

and the lemma is proved.

3. Perspectivities and perspective mappings. We make the following definition:

Definition 3.1. \( a, b \) are perspective (written \( a \sim b \)) if there exists an element \( c \) such that

(i) \( a + c = b + c \),
(ii) \( ac = bc \).

Then \( c \) is called the axis of the perspectivity.

Lemma 3.1. If \( a, b \) are perspective and \( \theta \leq ab \), then there exists an element \( d \) such that

(i) \( a + d = b + d = a + b \), and
(ii) \( ad = bd = \theta \).

Proof. Let \( c \) be an axis of perspectivity for \( a \) and \( b \). Since \( \theta \leq ab \leq \{ c(a+b) + ab \} \), Axiom IV secures the existence of an element \( d \) such that

\[
d + ab = c(a + b) + ab, \quad dab = \theta.
\]
For this $d$ we have

$$a + d = a + c(a + b) + ab = (a + c)(a + b) = a + b.$$  

Similarly $b + d = a + b$, and (i) holds.

$$ad = d \{ c(a + b) + ab \} = d \{ ab + ac(a + b) \} = dab = 0,$$

since $ac(a + b) = ac \leq ab$. Similarly $bd = \theta$, and (ii) holds. Thus $d$ satisfies the requirements of the lemma.

**Definition 3.2.** The sublattice of the $x$ satisfying $x \leq a$ is denoted by $L(a)$. If $a \leq b$, the sublattice of the $x$ satisfying $a \leq x \leq b$ is denoted by $L(a, b)$.*

**Lemma 3.2.** If $a, b$ are perspective with axis $c$ and $\theta = ac = bc$, then a $(1, 1)$ correspondence between the elements of $L(\theta, a)$ and those of $L(\theta, b)$ which preserves the relation $\leq$ is defined by the inverse mappings

$$(P) \quad a_1 \rightarrow b_1 = (a_1 + c)b$$

$$(Q) \quad b_1 \rightarrow a_1 = (b_1 + c)a.$$  

**Proof.** If $\theta \leq x \leq a$ then under $(P) x \rightarrow (x + c)b$, and under $(Q)$

$$(x + c)b \rightarrow \{ (x + c)b + c \} = (x + c)(b + c)a$$

$$= \{ c + x(b + c) \} = ca + x(b + c)$$

$$= x(a + c) = x.$$  

Hence $(Q)$ is inverse to $(P)$. Similarly $(P)$ is inverse to $(Q)$. It follows that the correspondence is $(1, 1)$. The invariance of the relation $\leq$ is clear from the definition of $(P)$ and $(Q)$.

**Definition 3.3.** The mappings of Lemma 3.2 are called perspective mappings.

**Lemma 3.3.** If $a_1$ corresponds to $b_1$ under a perspective mapping, then $a_1 \sim b_1$.

**Proof.** Suppose $a_1$ corresponds to $b_1$ under a perspective mapping of $L(\theta, a)$ on $L(\theta, b)$ with axis $c$. Then $a_1$ is perspective to $b_1$ with axis $c$, for

$$a_1 + c = (b_1 + c)a + c = (b_1 + c)(a + c)$$

$$= (b_1 + c)(b + c) = b_1 + c,$$

$$a_1c = a(b_1 + c)c = ac = bc = b(a_1 + c)c = b_1c,$$

and conditions (i) and (ii) therefore hold.

* The Axioms I, II, III, IV hold in $L(a)$ and in $L(a, b)$. $L(a)$ has a unit (greatest) element, namely $a$, and $L(a, b)$ has a unit element $b$ and a zero (smallest) element $a$.  

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Lemma 3.4. If $P_i$ is a perspective mapping of $L(\theta, a_i)$ on $L(\theta, b_i)$ for $i = 1, \cdots, p$, where $a_{i+1} = b_i$ for $i = 1, \cdots, p - 1$, then the product mapping of the $P_i$ is a $(1, 1)$ mapping of $L(\theta, a_1)$ on $L(\theta, b_p)$ which preserves the relation $\leq$.

Proof. The lemma follows immediately from Definition 3.3.

Definition 3.4. The mapping of Lemma 3.4 is called a projective mapping.

4. Transitivity of perspectivity in special cases. We prove the following lemma:

Lemma 4.1. $a \sim b, b \sim c, (a, b, c) \perp$ together imply $a \sim c$.

Proof. By Lemma 3.1 $x, y$ exist such that

\[ a + x = b + x = a + b, \quad b + y = c + y = b + c, \]
\[ ax = bx = \theta, \quad by = cy = \theta, \]

where $\theta = abc$.

Then $a$ is perspective to $c$ with axis $d = (a+c)(x+y)$. For

\[ a + d = a + (a + c)(x + y) = (a + c)(a + x + y) \]
\[ = (a + c)(a + b + y) = (a + c)(a + b + c) = a + c. \]

Similarly $c+d=a+c$. Thus $a+d=c+d$, and (i) holds. Also

\[ ad = a(a + c)(x + y) = a(x + y) = a(a + b)(x + y) \]
\[ = a\{x + (a + b)y(b + c)\} = a\{x + \theta\} = \theta. \]

Similarly $cd=\theta$. Thus $ad=cd$, and (ii) holds.

Lemma 4.2. $a_n \sim b_n$, for $n = 1, 2, \cdots$, and $(a_n + b_n, n = 1, 2, \cdots) \perp$ together imply $\sum a_n \sim \sum b_n$.

Proof. By Lemma 3.1 we may assume the existence of elements $x_n$ such that

\[ a_n + x_n = b_n + x_n = a_n + b_n, \]
\[ a_n x_n = b_n x_n = \theta, \]

where $\theta = \prod_n (a_n b_n)$. Then $\sum_n a_n, \sum_n b_n$ are perspective with axis $\sum_n x_n$, for the relations

\[ \sum_n a_n + \sum_n x_n = \sum_n (a_n + x_n) \]
\[ = \sum_n (b_n + x_n) = \sum_n b_n + \sum_n x_n, \]
give property (i); and
\[
\left( \sum_n a_n \right) \left( \sum_n x_n \right) = \sum_n (a_n x_n)
\]

\[
= \sum_n (b_n x_n) = \left( \sum_n b_n \right) \left( \sum_n x_n \right)
\]

(by Lemma 2.9) gives property (ii).

**Lemma 4.3.** If an infinite independent sequence of elements \(a_0, a_1, \ldots\) satisfy \(a_n \sim a_{n+1}\), for \(n = 0, 1, \ldots\), then \(a_0 = \prod_n a_n\).

**Proof.** From Lemmas 2.2 and 4.1, \(a_n \sim a_n\) for all \(n\). By Lemma 3.1 we may therefore assume the existence of elements \(x_n\), \((n = 1, 2, \ldots)\), such that

\[
a_0 + x_n = a_n + x_n = a_0 + a_n,
\]

\[
a_0 x_n = a_n x_n = \theta,
\]

where \(\theta = \prod_n a_n\). We deduce successively

\[
a_0 \leq a_n + x_n, \quad n = 1, 2, \ldots,
\]

\[
a_0 \leq \left( \sum_{n=p}^{\infty} a_n \right) + \left( \sum_{n=1}^{\infty} x_n \right), \quad p = 1, 2, \ldots
\]

\[
a_0 \leq \prod_p \left\{ \left( \sum_{n=p}^{\infty} a_n \right) + \left( \sum_{n=1}^{\infty} x_n \right) \right\} = \prod_p \left( \sum_{n=p}^{\infty} a_n \right) + \left( \sum_{n=1}^{\infty} x_n \right)
\]

\[
= \sum_{n=1}^{\infty} x_n
\]

by Lemma 2.7. Hence

\[
\begin{align*}
a_0 &= a_0 \sum_{n=1}^{\infty} x_n = a_0 \left( \sum_{p=1}^{\infty} \sum_{n=1}^{p} x_n \right) = \sum_{p=1}^{\infty} \left( a_0 \sum_{n=1}^{p} x_n \right) = \theta,
\end{align*}
\]

if only \(a_0 \sum_{n=1}^{p} x_n = \theta\) for \(p = 1, 2, \ldots\). Now for any fixed \(p\),

\[
a_0 \sum_{n=1}^{p} x_n = a_0 \left\{ \sum_{n=1}^{p-1} x_n + \left( a_0 + \sum_{n=1}^{p-1} x_n \right) x_p \right\},
\]

and, since

\[
x_p \left( a_0 + \sum_{n=1}^{p-1} x_n \right) = x_p (a_0 + a_p) \left( a_0 + \sum_{n=1}^{p-1} a_n \right) \left( a_0 + \sum_{n=1}^{p-1} x_n \right) = x_p a_0 = \theta,
\]

therefore

\[
a_0 \sum_{n=1}^{p} x_n = a_0 \sum_{n=1}^{p-1} x_n.
\]

A finite number of such reductions gives
\[ a_0 \sum_{n=1}^{p} x_n = a_0 x_1 = \theta, \]
as required, and the lemma is proved.

**Lemma 4.4.** \( a \sim x, x \sim a_1, ax \leq a_1 \leq a \) together imply \( a_1 = a \).

**Proof.** Let \( ax = a_1 x = \theta \). Since \( \theta \leq a_1 \leq a \), Axiom IV secures the existence of an element \( a'_1 \) such that

\[ a_1 + a'_1 = a, \quad a_1 a'_1 = \theta. \]

By Lemmas 3.1 and 3.2 there exist perspective mappings \( T_1 \) of \( L(\theta, a) \) and \( T_2 \) of \( L(\theta, x) \) on \( L(\theta, a_1) \). Define by induction on \( n \)

\[ x_n = T_1(a_n), \quad x'_n = T_1(a'_n), \quad a_{n+1} = T_2(x_n), \quad a'_{n+1} = T_2(x'_n). \]

Then Lemma 2.11, the relation \( ax = \theta \), and Lemma 2.10, give \( (a_n', x_n', n = 1, 2, \ldots) \) \( \bot \). Lemma 2.2 then gives \( (a'_n, x'_n, a'_{n+1}) \) \( \bot \), and Lemmas 3.3 and 4.1 give \( a_n' \sim a'_{n+1} \), for \( n = 1, 2, \ldots \). Lemma 2.2 shows that \( (a_n', n = 1, 2, \ldots) \) \( \bot \); hence by Lemma 4.3 \( a'_1 = \prod n a'_n = \theta \). Thus \( a_1 = a_1 + \theta = a_1 + a'_1 = a \); and the lemma is proved.

**Definition 4.1.** If \( \theta \) has been defined, we sometimes write

\[ \sum_n (\oplus x_n) \]
(or the equivalent \( x_1 \oplus x_2 \oplus \cdots \)) in place of \( \sum_n x_n \) (or \( x_1 + x_2 + \cdots \)), provided the \( x_n \) are independent over \( \theta \). If \( \theta \leq u \leq v \), then \([v-u]\) will denote an element (fixed) such that \( u \oplus [v-u] = v \). (Such an element exists by Axiom IV.)

**Lemma 4.5.** \( a \sim x, x \sim b, ab \leq x \) together imply \( a \sim b \).

**Proof.** (a) Consider the special case where \( ab = bx = ax \) and \( x \leq a+b \). Let \( b_1 = b(a+x) \). Then \( b_1 \) is perspective to \( x \) with axis \( a \), for

\[ b_1 + a = b(a + x) + a = (b + a)(a + x) = x + a, \]
hence relation (i) holds; and

\[ b_1 a = b(a + x)a = ba = xa, \]
hence (ii) holds.

Since \( b \sim x \) and \( b_1 x = b(a + x) x = bx, bx \leq b_1 \leq b \); and Lemma 4.4 gives \( b_1 = b \). Hence \( b \leq a + x \). Similarly \( a \leq b + x \). It follows that \( a \) is perspective to \( b \) with axis \( x \), for
\[ a + x = a + x + b = b + x; \]
hence relation (i) holds, and \( ax = bx \) (by the special hypotheses of (α)) hence relation (ii) holds. This proves Lemma 4.5 in the special case (α).

(β) Suppose now only that \( ab = bx = ax \) (equal, say \( θ \)). Let \( T_1, T_2 \) be perspective mappings of \( L(θ, x) \) on \( L(θ, a) \) and on \( L(θ, b) \), respectively. Set \( a_0 = a, \ x_0 = x, \ b_0 = b \), and define \( a_n, a'_n, x_n, x'_n, b_n, b'_n \), for \( n = 1, 2, \cdots \), by induction on \( n \) as follows:

\[
x_n = x_{n-1}(a_{n-1} + b_{n-1}), \quad a_n = T_1(x_n), \quad b_n = T_2(x_n),
\]
\[
x'_n = [x_{n-1} - x_n], \quad a'_n = T_1(x'_n), \quad b'_n = T_2(x'_n).
\]
If we set \( \bar{a} = \prod_n a_n, \bar{x} = \prod_n x_n, \bar{b} = \prod_n b_n \), then Lemma 2.11, the relation \( ab = θ \), and Lemma 2.3 give \( \bar{a} = \bar{a} + \sum_n a'_n \), \( \bar{b} = \bar{b} + \sum_n b'_n \); and \( \bar{a} + \bar{b}, \ a'_n + b'_n, \ (n = 1, 2, \cdots) \), are independent over \( θ \). Lemma 3.3 shows that \( a'_n \sim x'_n \) and \( x'_n \sim b'_n \). Since \( a'_n b'_n = θ \) and \( x'_n (a'_n + b'_n) = θ \), Lemmas 2.6 and 4.1 show that \( a'_n \sim b'_n \), for \( n = 1, 2, \cdots \). Now Lemma 4.2 shows that \( a \sim b \) if only \( \bar{a} \sim \bar{b} \).

\( \bar{a}, \ \bar{x}, \ \bar{b} \) satisfy the hypotheses of Lemma 4.5 and the special conditions of (α), for \( \bar{a} = T_1(\bar{x}), \ \bar{b} = T_2(\bar{x}) \) imply \( \bar{a} \sim \bar{x}, \ \bar{x} \sim \bar{b}; \ \bar{a} \bar{x} = \bar{a} \bar{a} \bar{x} = θ, \ \bar{b} \bar{x} = \bar{b} \bar{b} \bar{x} = θ, \ \bar{a} \bar{b} = \bar{a} \bar{a} \bar{b} \bar{b} = θ; \) and, since \( x_n \leq a_n - b_n = 1 \) for all \( n \),

\[
\bar{x} \leq \prod_n (a_n + b_n) = \prod_n \prod_p \left( \prod_{n=1}^p a_n + \prod_{n=1}^q b_n \right) = \prod_n a_n + \prod_n b_n = \bar{a} + \bar{b}
\]
by two applications of Axiom II. Hence \( \bar{a} \sim \bar{b} \); and Lemma 4.5 is proved for the special case (β).

(γ) Suppose now only \( ab = bx \) (equal, say \( θ \)). Let \( T_1, T_2 \) be perspective mappings of \( L(θ, a) \) on \( L(θ, x) \) and of \( L(θ, x) \) on \( L(θ, b) \), respectively. Set

\[
\begin{align*}
a_1 &= ax, & x_1 &= T_1(a_1) = a_1, & b_1 &= T_2(x_1), \\
a_2 &= [a - ax], & x_2 &= T_1(a_2), & b_2 &= T_2(x_2).
\end{align*}
\]
Then \( a = a_1 \oplus a_2, \ x = x_1 \oplus x_2, \ b = b_1 \oplus b_2 \). By Lemma 3.3, \( a_1 = x_1 \sim b_1 \). Since the hypotheses of Lemma 4.5 and the special conditions of (β) are clearly satisfied by \( a_2, x_2, b_2, a_2 \sim b_2 \). Lemma 4.2 now gives \( a \sim b \), and Lemma 4.5 is established for the special case (γ).

(δ) Suppose finally only the hypotheses of Lemma 4.5. The method by which (γ) was deduced from (β) can be applied in the same way to deduce (δ) from (γ). Thus Lemma 4.5 is proved.

**Corollary.** If \( T_1, T_2 \) are perspective mappings of \( L(θ, a) \) on \( L(θ, x) \), and of \( L(θ, x) \) on \( L(θ, b) \), respectively, and if \( T_2 T_1 \) maps \( a_1 \) on \( b_1 \), then \( a_1 b_1 = θ \) implies \( a_1 \sim b_1 \).
Proof. Set $x_1 = T_1(a_1)$. Lemmas 3.3 and 4.5 applied to $a_1$, $x_1$, $b_1$, give the desired result.

5. Transitivity of perspectivity. We prove the following lemma:

Lemma 5.1. If $a_1 \geq a_2 \geq \cdots$ and $c$ are given, and if $\theta = \prod_n (a_n c)$, then there exist decompositions

$$a_n = a_n c \oplus a_n'',$$

for $n = 1, 2, \cdots$, such that $a_1' \geq a_2' \geq \cdots$.

Proof. Let $I_n = [a_n - (a_n c + a_{n+1})]$, for $n = 1, 2, \cdots$, and let $\bar{a} = \prod_n a_n$, $I = [\bar{a} - \bar{a}c]$. Then

$$a_1 = I_1 \oplus (a_1 c + a_2),$$
$$a_2 = I_2 \oplus (a_2 c + a_3),$$
$$\cdots \cdots \cdots \cdots \cdots ,$$
$$a_n = I_n \oplus (a_n c + a_{n+1}),$$
$$\cdots \cdots \cdots \cdots \cdots ,$$
$$\bar{a} = I \oplus \bar{a}c;$$

and clearly $cI = cI\bar{a}c = \theta$, $cI_r = cI_{r+1}(a_rc + a_{r+1}) = \theta$, $I_{r+k} \leq a_{r+k} \leq a_r$, for all $r \geq 1$, $k \geq 0$.

We can now prove that $a_r = a_rc \oplus I \oplus \sum_{n=r}^\infty I_n$, for $r = 1, 2, \cdots$. In the first place, $a_rc$, $I$, $I_n$, $(n = r, r+1, \cdots)$ are independent over $\theta$ by the corollary to Lemma 2.6 since $a_rcI = \theta$; and for all $p > n \geq r$,

$$I_n (a_rc + I + \sum_{m=n+1}^p I_m) = I_n a_n (a_rc + a_{n+1}) (a_rc + I + \sum_{m=n+1}^p I_m)$$
$$= I_n (a_n c + a_{n+1}) (a_rc + I + \sum_{m=n+1}^p I_m)$$
$$= \theta.$$

Secondly,

$$a_rc + I + \sum_{n=r} I_n = a_rc + \prod_{n=r} a_m + \sum_{n=r} I_n$$
$$= \prod_{m=r} (a_rc + a_m + \sum_{n=r} I_n)$$
$$= \prod_{m=r} (a_rc + a_m + \sum_{n=r} I_n).$$

Now if $m \geq r$,
Thus \( a_r c + I \oplus I \sum_{m=1}^{\infty} I_r = \prod_{m=1}^{\infty} (a_r) = a_r \) as required. It is now clear that if we set

\[
a'_n = I \oplus \sum_{r=1}^{\infty} I_r,
\]

we will obtain the decompositions required by the lemma.

**Corollary.** In Lemma 5.1, \( I, I_n, (n=1, 2, \ldots) \), are independent over \( \theta \).

**Proof.** This is immediate from the proof of Lemma 5.1.

**Theorem 5.1. Transitivity of perspectivity.** \( a \sim x, x \sim b \) together imply \( a \sim b \).

**Proof.** (I) Let \( \theta = abx \), and let \( T_1, T_2 \) be perspective mappings of \( L(\theta, a) \) on \( L(\theta, x) \) and of \( L(\theta, x) \) on \( L(\theta, b) \), respectively. Let \( T = T_2 T_1 \) be the product mapping of \( L(\theta, a) \) on \( L(\theta, b) \), and let \( T^{-1} \) be the inverse mapping to \( T \). We shall use the notation \( a_1 \rightarrow b_1 \), or the equivalent \( b_1 = T(a_1) \), to denote that \( b_1 \) is the map of \( a_1 \) under \( T \).

(II) Let \( c = ab \), and let \( a_0 = [a - c] \), where \( a_0 \) is restricted to satisfy a certain condition which will be stated precisely later (see (III) below). Set \( b_1 = (T(a_0))_c \), \( b'_1 = [T(a_0) - b_1] \), \( a_1 = T^{-1}(b_1) \), \( a'_1 = T^{-1}(b'_1) \). Then

\[
a = a_0 \oplus c,
\]

\[
a_0 = a_1 \oplus a'_1 , \quad a_1 \rightarrow b_1 \leq c , \quad a'_1 \rightarrow b'_1 , \quad (b'_1 c = \theta).
\]

Since \( \theta \leq b_1 \leq a \), \( T(b_1) \) is defined. Let \( \{ T(b_1) \} c = b_2 , \ b'_2 = [T(b_1) - b_2] , \ b_{12} = T^{-1}(b_2) , \ a_2 = T^{-1}(b_2) , \ b_{12}' = T^{-1}(b'_2) , \ a_2' = T^{-1}(b_{12}') \). Then

\[
a_1 = a_2 \oplus a'_2 , \quad a_2 \rightarrow b_{12} \rightarrow b_2 \leq c , \quad a'_2 \rightarrow b_{12}' \rightarrow b'_2 , \quad (b'_2 c = \theta).
\]

Similarly, obtain the table

\[
a_0 = a_1 \oplus a'_1 , \quad a_1 \rightarrow b_1 \leq c , \quad a'_1 \rightarrow b'_1 , \quad (b'_1 c = \theta),
\]

\[
a_1 = a_2 \oplus a'_2 , \quad a_2 \rightarrow b_{12} \rightarrow b_2 \leq c , \quad a'_2 \rightarrow b_{12}' \rightarrow b'_2 , \quad (b'_2 c = \theta),
\]

\[
a_{n-1} = a_n \oplus a'_n , \quad a_n \rightarrow b_{12} \ldots n \rightarrow b_{23} \ldots n \rightarrow \cdots \rightarrow b_n \leq c , \quad a'_n \rightarrow b_{12} \ldots n \rightarrow b_{23} \ldots n \rightarrow \cdots \rightarrow b'_n , \quad (b'_n c = \theta),
\]

\[
\cdots \cdots \cdots .
\]
We can now prove the following statements:

(α) \(\prod_{n=1}^{\infty} a_n = \theta\) and \(a_0 = \sum_{n=1}^{\infty} (\oplus a_n')\).

(β) \(a'_n, b'_{12}, \ldots, b'_{2n}, \ldots, b'_n\) are independent over \(\theta\), for \(n = 1, 2, \ldots\).

(γ) If we set \(d_n = a'_n + b'_{12}, \ldots, n, b'_{23}, \ldots, + b'_n\), then \(d_1, d_2, \ldots\) are independent over \(\theta\).

(δ) All the primed elements \(a'_1, a'_2, \ldots, b'_1, b'_{12}, \ldots\) are independent over \(\theta\).

Proof of (α). Let \(\prod_{n} a_n = \bar{a}\). Then \(T^n(\bar{a}) = TT \cdots T(\bar{a})\) \((n\) factors \(T\)) is defined for \(n = 1, 2, \ldots\); and \(T(\bar{a}), T^2(\bar{a}), \ldots\) are independent over \(\theta\) by the corollary to Lemma 2.6 since

\[\{ T^n(\bar{a}) \} \{ T^{n+1}(\bar{a}) + \cdots + T^{n+p}(\bar{a}) \}\]

has a \(T^{-n}\) map which is \(\leq a_0b = a_0ab = a_0c = \theta\), for all \(n, p \geq 1\). Since \(T^n(\bar{a}) \sim T^{n+1}(\bar{a})\) by Lemma 3.3, Lemma 4.3 shows that \(\bar{a} = \theta\). The statement (α) now follows from Lemma 2.11.

Proof of (β).

\(b_{r(r+1)} \cdots n(b_{r(r+1)} + b_{r(r+3)} + \cdots + b'_{n}) = \theta\)

since it has a \(T^{-r}\) map which is \(\leq a_0b = \theta\). The statement (β) now follows from the corollary to Lemma 2.6.

Proof of (γ). \(\theta \leq d_n(d_{n+1} + d_{n+2} + \cdots + d_{n+p})\)

\[\leq (a' + b'_{12} + \cdots + b'_n) \left\{ \begin{array}{c} a'_{n+1} + b'_{12} \cdots (n+1) + \cdots + b'_{n+1} \\ + a'_{n+2} + b'_{12} \cdots (n+2) + \cdots + b'_{n+2} \\ + \cdots \\ + a'_{n+p} + b'_{12} \cdots (n+p) + \cdots + b'_{n+p} \end{array} \right\} \]

\[= a'(a'_{n+1} + a'_{n+2} + \cdots + a'_{n+p}) + (b'_{12} + \cdots + b'_n) \left\{ \begin{array}{c} b'_{12} \cdots (n+1) + \cdots + b'_{n+1} \\ + b'_{12} \cdots (n+3) + \cdots + b'_{n+2} \\ + \cdots \\ + b'_{12} \cdots (n+p) + \cdots + b'_{n+p} \end{array} \right\} \]

\[= \theta + T \begin{pmatrix} a_n + b_{12} + \cdots + b_{(n-1)n} \\ + a'_{n+1} + b'_{12} \cdots (n+1) + \cdots + b'_{n(n+1)} \\ + a'_{n+2} + b'_{12} \cdots (n+2) + \cdots + b'_{n(n+1)(n+2)} \\ + \cdots \end{pmatrix} \]

\[\leq \cdots \leq T^{n+1}(\theta) = \theta.\]

The statement (γ) now follows from the corollary to Lemma 2.6.

Proof of (δ). The statement (δ) follows from (β) and (γ) by Lemma 2.10.

(III) By the corollary to Lemma 4.5 \(a'_n \sim b'_{12}, \ldots, n, b'_{23}, \ldots, b'_{(n-1)n} \sim b'_n\). Since \(a'_n, b'_{12}, \ldots, b'_n\) are independent, repeated application
of Lemma 4.1 shows that \( a_n' \sim b_n' \) for \( n = 1, 2, \cdots \). Now set

\[
A = \sum_{n=1}^{\infty} \left( \bigoplus a_n' \right) \oplus \sum_{r \neq 0} \left( \bigoplus b_{r(r+1)} \cdots \right),
\]

\[
B = \sum_{n=1}^{\infty} \left( \bigoplus b_n' \right) \oplus \sum_{r \neq 0} \left( \bigoplus b_{r(r+1)} \cdots \right).
\]

By Lemma 4.2, \( A \sim B \). Furthermore \( T(A) = B \).

Now suppose that \( a_0 \) was chosen (in (II) above) in such a way that, the \( b_n' \)
having been defined as above, we should have \( \left( \sum_{n=1}^{\infty} b_n' \right) c = \theta \) (that such an \( a_0 \)
exists will be shown in (V) below). Setting

\[
g = \left[ c - \sum_{r \neq 0} \left( \bigoplus b_{r(r+1)} \cdots \right) \right]
\]

we have \( Bg = Bgc = g \sum_{r \neq 0} \left( \bigoplus b_{r(r+1)} \cdots \right) = \theta \), and we can define

\[
h = \left[ b - (B \oplus g) \right].
\]

Then we clearly have

\[
c = \sum_{r \neq 0} \left( \bigoplus b_{r(r+1)} \cdots \right) \oplus g,
\]

\[
a = A \oplus g,
\]

\[
b = B \oplus g \oplus h.
\]

By Lemma 4.2, \( a \sim b \) if only \( h = \theta \).

(IV) We proceed to show that \( h = \theta \). Let \( g' = T^{-1}(g + h) \). Since

\[
A \oplus g = a = T^{-1}(b) = T^{-1}(B \oplus g \oplus h) = A \oplus g',
\]

it follows that there exists a perspective mapping \( S \) of \( L(\theta, g') \) on \( L(\theta, g) \).

Now set \( h_0 = h \) and define \( h_n', h_n \), for \( n = 1, 2, \cdots \), by induction on \( n \) as follows:

\[
h_n' = T^{-1}(h_{n-1}), \quad h_n = S(h_n').
\]

Then \( h_0, h_1, \cdots \) are independent over \( \theta \) by the corollary to Lemma 2.6 since

\[
h_n(h_{n+1} + \cdots + h_{n+p}) = (ST^{-1})^n \left\{ h(h_1 + \cdots + h_p) \right\}
\]

\[
= (ST^{-1})^n \left\{ hg(h_1 + \cdots + h_p) \right\}
\]

\[
= (ST^{-1})^n(\theta) = \theta.
\]

Since \( h_{n-1} \sim h_n', h_n' \sim h_n \), and \( h_{n-1} h_n = \theta \leq h_n' \), Lemma 4.5 shows that \( h_{n-1} \sim h_n \).

Now Lemma 4.3 gives \( h = h_0 = \theta \) as required.
To complete the proof of Theorem 5.1 we need only show that the
\( a_0 = [a - ab] = [a - c] \) defined in (II) above could be chosen in such a way that
\( \sum_{n=1}^{\infty} b_n' = \theta \) will hold in (III) above. We first note that if we set
\[
T^{-1}(c), v_2 = T^{-1}(v_1 c), \ldots, v_{n+1} = T^{-1}(v_n c), \ldots
\]
then it is sufficient to choose \( a_0 \) so that \( a = a_0 \oplus c \) and
\[
v_n c = (v_n + a_0) c,
\]
for \( n = 1, 2, \ldots \). For if \( a_0 \) is so chosen, then
\[
\theta \leq b_{n+1}'(c + b_1' + b_2' + \cdots + b_n')
\]
\[
= T\{b_{n(n+1)}' T^{-1}(c) + a_1' + b_{12}' + \cdots + b_{(n-1)n}'\}
\]
\[
\leq T\{b_{n(n+1)}' c(v_1 + a_0 + b_{12}' + \cdots + b_{(n-1)n}')\}
\]
\[
= T\{b_{n(n+1)}' (v_1 c + b_{12}' + \cdots + b_{(n-1)n}')\}
\]
\[
\leq T^2\{b_{(n-1)(n+1)}' (v_2 + a_0 + b_{123}' + \cdots + b_{(n-2)(n-1)n}')\}
\]
\[
\leq \cdots \leq T^n(b_{123\cdots(n+1)}' v_n c) \leq b_{n+1}' c = \theta;
\]
hence
\[
b_{n+1}'(c + b_1' + b_2' + \cdots + b_n') = \theta,
\]
for \( n = 1, 2, \ldots \). By Lemma 2.6, \( c, b_1', b_2', \ldots \) are independent over \( \theta \); hence \( \sum_{n=1}^{\infty} b_n' = \theta \) as required.

Thus we have only to construct an \( a_0 \) such that
\[
a = a_0 \oplus c, \quad v_n c = (v_n + a_0) c,
\]
for \( n = 1, 2, \ldots \). Apply Lemma 5.1 to \( v_1 \geq v_2 \geq \cdots \) and \( c \), and obtain \( I, I_n, n = 1, 2, \ldots \), as in the proof of Lemma 5.1. Then \( v_n = v_n c \oplus I \oplus \sum_{m=n}^{\infty} I_m \).
Let \( u = [a - (v_1 + c)] \). Then
\[
u \left( I + \sum_{m=1}^{\infty} I_m \right) = uv_1 \left( I + \sum_{m=1}^{\infty} I_m \right) = \theta,
\]
and we can set
\[
a_0 = I \oplus \sum_{m=1}^{\infty} (\oplus I_m) \oplus u.
\]
This \( a_0 \) satisfies our requirements, for
\[
a_0 + c = c + I + \sum_{m=1}^{\infty} I_m + u = c + v_1 + u = a,
\]
\[ a_0c = a_0(v_1 + c)c = \left( I + \sum_{m=1}^{\infty} I_m \right)c \]
\[ = \left( I + \sum_{m=1}^{\infty} I_m \right)v_1c = \theta. \]

Hence \( a_0 \oplus c = a \). And
\[ (v_n + a_0)c = \left( v_n c + I + \sum_{m=1}^{\infty} I_m + u \right)c \]
\[ = (v_n c + a_0)c \]
\[ = v_n c + a_0 c = v_n c + \theta = v_n c. \]

This completes the proof of Theorem 5.1.

6. Additivity and continuity properties of perspectivity. We prove the following lemma:

**Lemma 6.1.** If \( \theta \) is defined and if
\[ a = a_1 \oplus a_1' = a_2 \oplus a_2', \]
then \( a_1 \sim a_2 \) implies \( a_1' \sim a_2' \).

**Proof.** The perspectivity \( a_1 \sim a_2 \) implies, by Lemma 3.1, the existence of an \( x \) for which
\[ a_1 + x = a_2 + x = a_1 + a_2, \]
\[ a_1 x = a_2 x = \theta. \]
Let \( c = [a - (a_1 + a_2)] \). Then \( a_1' \) is perspective to \( (x+c) \) with axis \( a_1 \), for we have the relation
\[ a_1' + a_1 = a = c + (a_1 + a_2) = c + (a_1 + x) \]
\[ = (x + c) + a_1, \]
hence relation (i) holds; and
\[ a_1' a_1 = \theta = xa_1 = (x + \theta)a_1 = \{ x + (a_1 + a_2)c \} a_1 \]
\[ = \{ x + (a_1 + x)c \} a_1 = (x + c)a_1, \]
hence relation (ii) holds.

Similarly \( a_2' \) is perspective to \( (x+c) \). Theorem 5.1 then proves that \( a_1' \sim a_2' \) as required.

**Lemma 6.2.** If \( \theta \) is defined, and if
\[ a = a_1 \oplus a_1', \quad b = b_1 \oplus b_1', \]
then \( a \sim b, a_1 \sim b_1 \) together imply \( a_1' \sim b_1' \).
Proof. Let $T$ be a perspective mapping of $L(\theta, a)$ on $L(\theta, b)$, and let $u = T(a), v = T(a')$; then $a_1 \sim u, a'_1 \sim v$, and $b = u \oplus v$. Since $a_1 \sim b_1$, Theorem 5.1 gives $u \sim b_1$ and Lemma 6.1 gives $v \sim b'_1$. Since $a'_1 \sim v$ and $v \sim b'_1$, Theorem 5.1 gives $a'_1 \sim b'_1$.

Lemma 6.3. $a_1 \sim b_1, a_2 \sim b_2, a_1a_2 = b_1b_2$ together imply $(a_1 + a_2) \sim (b_1 + b_2)$.

Proof. Let $\theta = a_1a_2 = b_1b_2, d = a_1 + a_2 + b_1 + b_2$, and define $a' = [d - (a_1 \oplus a_2)], b' = [d - (b_1 \oplus b_2)]$. Then

$$(a' \oplus a_2) \oplus a_1 = d = (b' \oplus b_2) \oplus b_1.$$ 

By Lemma 6.1, $(a' \oplus a_2) \sim (b' \oplus b_2)$; hence by Lemma 6.2, $a' \sim b'$. Since $a' \oplus (a_1 \oplus a_2) = b' \oplus (b_1 \oplus b_2)$, Lemma 6.1 proves $(a_1 + a_2) \sim (b_1 + b_2)$ as required.

Lemma 6.4. If $\theta$ is defined, and if $a_1, \ldots, a_p$ and $b_1, \ldots, b_p$ are two sets of elements, each independent over $\theta$, with $a_r \sim b_r$ for $r = 1, \ldots, p$, where $p = 1, 2, \ldots$, then

$$\left( \sum_{r=1}^{p} (\oplus a_r) \right) \sim \left( \sum_{r=1}^{p} (\oplus b_r) \right).$$

Proof. Suppose the lemma established for $p = n$ for some fixed $n = 1, 2, \ldots$. Then

$$(a_1 \oplus \cdots \oplus a_n) \sim (b_1 \oplus \cdots \oplus b_n), \quad a_{n+1} \sim b_{n+1}$$

imply, by Lemma 6.3, $(a_1 + \cdots + a_{n+1}) \sim (b_1 + \cdots + b_{n+1})$; and the lemma will hold for $p = n + 1$. Since the lemma is trivially true for $p = 1$, it holds, by induction, for all $p = 1, 2, \ldots$.

We now define a relation $a \preceq b$ as follows:

Definition 6.1. $a \preceq b$ if $a \sim b_1$ for some $b_1 \leq b$.

Lemma 6.5. (I) $a \leq b$ implies $a \preceq b$.

(II) $a \leq b, b \preceq c$ together imply $a \preceq c$.

(III) $a \preceq b, b \preceq c$ together imply $a \preceq c$.

(IV) $a \preceq b, b \preceq a$ together imply $a \sim b$.

Proof. (I) follows from $a \sim a$.

(II): Let $\theta = ac$, and let $T$ be a perspective mapping of $L(\theta, b)$ on $L(\theta, c)$. Then $T(a)$ is defined, $a \sim T(a)$, and $T(a) \leq c$. Hence $a \preceq c$.

(III): $a \preceq b$ means $a \sim b_1$ for some $b_1 \leq b$. Since $b \preceq c$, (II) gives $b_1 \sim c_1$ for some $c_1 \leq c$. Theorem 5.1 then gives $a \sim c_1$. Hence $a \preceq c$ as required.

(IV): $a \preceq b$ means $a \sim b_1$ for some $b_1 \leq b$. If $b \preceq a$, then $b \sim a_1$ for some $a_1 \leq a$, and $b_1 \sim a_2$ for some $a_2 \leq a_1$, by (II). Then, by Theorem 5.1, $a \sim a_2$; and,
since \(a_2 \leq a\), Lemma 4.4 implies \(a_2 = a\). Since \(a_2 \leq a_1 \leq a\), we have \(a = a_1 \sim b\); that is \(a \sim b\) as required.

**Lemma 6.6.** If \(c \lt a\) and \(d = ca\), there exists an element \(a_1\) with \(d \leq a_1 \leq a\) and \(c \sim a_1\).

**Proof.** \(c \lt a\) means \(c \sim a_2\) for some \(a_2 \leq a\). Let \(\theta = ad\); then \(\theta \leq ca_2\), and there exists, by Lemmas 3.1 and 3.2, a perspective mapping \(T\) of \(L(\theta, c)\) on \(L(\theta, a_2)\). Define \(c' = [c-d]\) and \(c' = T(c')\); then \(c' \leq a_2 \leq a\), \(c'd = \theta\), \(c' \sim c\), and \(c'd = c'a_2d\) = \(\theta\). By Lemma 6.3,
\[
c = (c' \oplus d) \sim (c' \oplus d) \leq a,
\]
and \(a_1 = c' \oplus d\) satisfies all the requirements of the lemma.

**Lemma 6.7.** If \(\theta\) is defined and if \(a \oplus a_1 \geq a' \oplus a_2\), then \(a \sim a'\) implies \(a_2 \sim a_1\).

**Proof.** Let \(v = [(a \oplus a_1) - (a' \oplus a_2)]\). Then \(a \oplus a_1 = a' \oplus a_2 \oplus v\), and Lemma 6.1 implies \(a_2 \sim (a_2 \oplus v)\); thus \(a_2 \sim a_1\) as required.

**Lemma 6.8.** If \(\theta\) is defined and if \(a \oplus a_2 \sim a \oplus a_1\), then \(a_2 \sim a_1\).

**Proof.** By Lemma 6.6 (with the \(\theta\) of the present lemma in the place of the \(d\) of Lemma 6.6) \((a \oplus a_2) \sim u\), where \(\theta \leq u \leq (a + a_1)\). Let \(T\) be a perspective mapping of \(L(\theta, a + a_2)\) on \(L(\theta, u)\), and let \(aT = T(a)\) and \(aT = T(a_2)\); then \(a \oplus a_1 \geq aT \oplus a_2\) and \(a \sim aT\). By Lemma 6.7, \(a_2 \sim a_1\), and since \(a_2 \sim a_2\), Lemma 6.5 (III) implies \(a_2 \sim a_1\), which proves the lemma.

**Lemma 6.9.** If \(\theta\) is defined, and if
\[
u \oplus v_1 \oplus \cdots \oplus v_p \sim a,
\]
where \(u \leq a\) and \((v_1 \oplus \cdots \oplus v_p)a = \theta\), \((p = 1, 2, \ldots)\), then there exist elements
\(v_1', \ldots, v_p'\) with \(v_r \sim v_r'\), \((r = 1, \ldots, p)\), \(u, v_1, \ldots, v_p, v_1', \ldots, v_p'\) independent over \(\theta\), and \(u \oplus v_1' \oplus \cdots \oplus v_p' \leq a\).

**Proof.** Define \(a_1 = [a-u]\); then
\[
a = u \oplus a_1, u \oplus (v_1 \oplus \cdots \oplus v_p) \sim u \oplus a_1,
\]
and Lemma 6.8 implies \((v_1 \oplus \cdots \oplus v_p) \sim a_1). Thus there exists, by Lemma 6.6, a perspective mapping \(T\) of \(L(\theta, (v_1 \oplus \cdots \oplus v_p))\) on \(L(\theta, a_2)\) for some \(a_2\) satisfying \(\theta \leq a_2 \leq a_1\). Let \(v_r' = T(v_r)\) for \(r = 1, \ldots, p\). Then \(v_r \sim v_r'\), for \(r = 1, \ldots, p; \sum_{r=1}^{p} (\oplus v_r') \leq a_2 \leq a_1 \leq a\); and \(v_r', (r = 1, \ldots, p)\), are independent over \(\theta\). Since
\[
\left(\sum_{r=1}^{p} (\oplus v_r')\right) \left(u \oplus \sum_{r=1}^{p} (\oplus v_r)\right) = \left(\sum_{r=1}^{p} (\oplus v_r')\right) a_1 a \left(u \oplus \sum_{r=1}^{p} (\oplus v_r)\right) = \left(\sum_{r=1}^{p} (\oplus v_r')\right) a_1 u = \theta,
\]
\(u, v_1, \ldots, v_p, v'_1, \ldots, v'_p\) are independent over \(\theta\) by Lemma 2.10. Thus \(v'_1, \ldots, v'_p\) satisfy all the requirements of the lemma.

**Lemma 6.10.** Let \(a_1 \geq a_2 \geq \cdots\) be an infinite set of elements, and let \(c\) satisfy \(\theta \leq c \leq a_1\), where \(\theta = \prod_n a_n\). Suppose further that \(c \propto a_n\) for \(n = 1, 2, \ldots\). Then there exists an element \(c' \leq a_1\) such that \(c \sim c', cc' = \theta\), and \((c \oplus c') \propto a_n\) for \(n = 1, 2, \ldots\).

**Proof.** Define \(c_t = [ca_t - ca_{t+1}]\) for \(t = 1, 2, \ldots\); then \(ca_t = ca_{t+1} \oplus c_t\), and

\[
c = ca_1 = \sum_{t=1}^{\infty} (\oplus c_t) \oplus \prod_{t=1}^{\infty} (ca_t)
= \sum_{t=1}^{\infty} (\oplus c_t) + \theta = \sum_{t=1}^{\infty} (\oplus c_t).
\]

Suppose that \(c_{r_1r_2\ldots r_n}\) have been defined, for all \(1 \leq r_1 < r_2 < \cdots < r_n < t < \infty\) with \(1 \leq r_n < p\) for some fixed \(p = 1, 2, \ldots\) (\(n\) taking all values possible), in such a way that the following conditions are satisfied:

1. \((\alpha)_p\) \(c_t, t = 1, 2, \ldots; c_{r_1r_2\ldots r_n}, 1 \leq r_1 < r_2 < \cdots < r_n < t\), for \(r_n < p\), are independent over \(\theta\),

2. \((\beta)_p\) If we set

\[
cr = c_{r_1r_2\ldots r_n} = \sum_{t=r_n+1}^{\infty} (\oplus c_{r_1r_2\ldots r_n})
\]

then

\[
c_{r_1r_2\ldots r_n} = [c'_{r_1r_2\ldots r_n} - c_{r_1r_2\ldots r_n}a_{t+1}]
\]

and

\[
c_{r_1r_2\ldots r_n} \sim c'_{r_1r_2\ldots r_n}, \quad 1 \leq r_1 < r_2 < \cdots < r_n < t < \infty, \quad r_n < p.
\]

3. \((\gamma)_p\)

\[
c \sim \sum_{t=p}^{\infty} (\oplus c_t) \oplus \sum_{t=p}^{\infty} (\oplus c_{r_1r_2\ldots r_n}),
\]

where in the last summation \(r_1, r_2, \ldots, r_n\) take on all possible values with \(r_n < p\).

Then \(c \propto a_{p+1}\), and

\[
c \sim \left( \sum_{t=p+1}^{\infty} (\oplus c_t) \oplus \sum_{t=p+1, r_n < p}^{\infty} (\oplus c_{r_1r_2\ldots r_n}) \right) \oplus \left( c_p \oplus \sum_{r_n < p} \left( \oplus c_{r_1r_2\ldots r_n} \right) \right);
\]

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and Lemma 6.9 secures the existence of elements $c_p', c_{r_1 r_2} \cdots r_{n_P}, (1 \leq r_1 < r_2 < \cdots < r_n < p)$, such that $\theta \leq c_{p'}, \theta \leq c_{r_1 r_2} \cdots r_{n_P} \leq a_{p+1}$, $c_p \leq c_{p'}, c_{r_1 r_2} \cdots r_{n_P} \sim c_{r_1 r_2} \cdots r_{n_P}$, and $c_t, (t = 1, 2, \cdots)$, $c_{r_1 r_2} \cdots r_{n_t}$, $(1 \leq r_1 < r_2 < \cdots < r_n < \theta < \infty, r_n < p), c_p, c_{r_1 r_2} \cdots r_{n_P}, (1 \leq r_1 < r_2 < \cdots < r_n < p)$, are independent over $\theta$. Hence if we define $c_{p+1} = [c_{p'} - c_p a_{p+1}]$ and $c_{r_1 r_2} \cdots r_{n_P} = [c_{r_1 r_2} \cdots r_{n_P} - c_{r_1 r_2} \cdots r_{n_P} a_{p+1}]$, then $(\alpha)_{p+1}, (\beta)_{p+1},$ and $(\gamma)_{p+1}$ will be satisfied. Since $(\alpha)_1, (\beta)_1,$ and $(\gamma)_1$ are trivially satisfied, it follows that we can define the $c_{r_1 r_2} \cdots r_{n_t}, (for all 1 \leq r_1 < r_2 < \cdots < r_n < \theta < \infty)$, so that $(\alpha)_p, (\beta)_p,$ and $(\gamma)_p$ are satisfied for all $p = 1, 2, \cdots$. Then $c_1, c_2, \cdots, c'_1, c'_2, \cdots$ are independent over $\theta$, and $c_n \sim c'_n$ for $n = 1, 2, \cdots$. Lemmas 2.3 and 4.2 now imply that $c = \sum_{n=1}^{\infty} (\theta c_n) \sim \sum_{n=1}^{\infty} (\theta c'_n)$. If we set $c' = \sum_{n=1}^{\infty} (\theta c'_n)$, we will have $c \sim c', c' = \theta$, and $(\theta c') \leq a_1$.

Finally, since $c \leq a_p$, we have $c \sim c_p$ for some $c_p$ with $\theta \leq c_p \leq a_p$ by Lemma 6.6. Applying the reasoning of the preceding paragraph to $c_p$ and $a_p \leq a_{p+1} \cdots \cdots$ we obtain $c_p'$ such that $c_p \sim c_p'$ and $(c_p \oplus c_p') \leq a_p$. Then $c' \sim c$, $c \sim c_p$, $c_p \sim c_p'$, imply, by Theorem 5.1, $c' \sim c_p'$ and hence, by Lemma 6.3, $(c \oplus c') \sim (c_p \oplus c_p')$. Lemma 6.5 (III) now implies that $(c \oplus c') \sim a_p$ for all $p = 1, 2, \cdots$, and $c'$ satisfies all the requirements of the lemma.

**Lemma 6.11.** The hypotheses of Lemma 6.10 imply $c = \theta$.

**Proof.** Suppose that $c_1, c_2, \cdots, c_p^p$ have been defined for some fixed $p = 0, 1, \cdots$ in such a way that, if we write $c(p)$ for $c_1 + \cdots + c_p$, then $(\lambda)_p c \sim c_r$ for $r = 1, 2, \cdots$, $2^p$; $(\mu)_p c_r, r = 1, \cdots, 2^p$, are independent over $\theta$; $(\nu)_p c(p) \leq a_1$ and $c(p) \sim a_n$ for $n = 1, 2, \cdots$. Then there exists, by Lemma 6.10, an element $c' \leq a_1$ such that $c(p) \sim c'$, $(c(p) \oplus c') \sim \theta$, and $(c(p) \oplus c') \sim a_n$ for $n = 1, 2, \cdots$. Let $T$ be a perspective mapping of $L(\theta, c(p))$ on $L(\theta, c')$, and let $c_{p+1} = T(c_r)$ for $r = 1, \cdots, 2^p$. Then $(\lambda)_{p+1}, (\mu)_{p+1},$ and $(\nu)_{p+1}$ will be satisfied. Since we can define $\tilde{c}_1 = c$ to satisfy $(\lambda)_0, (\mu)_0,$ and $(\nu)_0$, it follows that we can define, by induction on $p$, an infinite sequence $c_n, (n = 1, 2, \cdots),$ satisfying $(\lambda)_p, (\mu)_p,$ and $(\nu)_p,$ for all $p = 0, 1, \cdots$. Then we have $\tilde{c}_1, \tilde{c}_2, \cdots$ independent over $\theta$ and $c_n \sim \tilde{c}_{n+1}$ (by Theorem 5.1 since $c_n \sim c$, $c \sim c_{n+1}$); hence $\tilde{c}_1 = \theta$ by Lemma 4.3. Since $c = \tilde{c}_1$ we have $c = \theta$. This proves the lemma.

**Lemma 6.12.** Without the condition $c \leq a_1$, the remaining hypotheses of Lemma 6.10 imply $c = \theta$.

**Proof.** $\theta \leq c_1, c \sim a_1$, imply, by Lemma 6.6, the existence of a $c_1$ with $\theta \leq c_1 \leq a_1$, and $c \sim c_1$. By Lemma 6.5 (III), $c_1$ and $a_1 \geq a_2 \geq \cdots$ satisfy the
hypotheses of Lemma 6.10; hence Lemma 6.11 implies $c_1 = \theta$. Since $c \sim c_1 = \theta$ and $\theta \leq c$, Lemma 4.4 gives $c = \theta$. This proves the lemma.

**Lemma 6.13.** If $\theta$ is defined and $c \prec (a \oplus b)$, there exists a decomposition $c = c_1 \oplus c_2$ with $c_1 \prec a$, $c_2 \prec b$.

**Proof.** By Lemma 6.6, $c \sim u$, $\theta \leq u \leq (a \oplus b)$. Let $u_1 = au$, define $u'_1 = [u - u_1]$, and let $u_2 = (a + u'_1) b$; then $u = u_1 \oplus u'_1$, and $u_2 \sim u'_1$ with axis $a$, for

$$u'_1 + a = (u'_1 + a)(b + a) = (a + u'_1)b + a = u_2 + a,$$

hence (i) is satisfied; and

$$u'_1 a = u'_1 ua = u'_1 u_1 = \theta = u_2 ba = u_2 a,$$

hence (ii) is satisfied.

Now let $c_1 = T^{-1}(u_1)$ and $c_2 = T^{-1}(u'_1)$; then $c = c_1 \oplus c_2$, $c_1 \sim u_1 \leq a$, and $c_2 \sim u_2 \leq b$ (by Theorem 5.1, since $c_2 \sim u'_1$ and $u'_1 \sim u_2$). This proves the lemma.

**Lemma 6.14.** If $a_1 \geq a_2 \geq \cdots$ and $c \prec a_n$ for $n = 1, 2, \cdots$, then $c \prec \prod_n a_n$.

**Proof.** Let $\mathfrak{a} = \prod_n a_n$, $\mathfrak{b} = c\mathfrak{a}$. Apply Lemma 5.1 to $\mathfrak{a}$ and $a_1 \geq a_2 \geq \cdots$ to obtain $a_n = a_n' \oplus b$ with $a'_1 \geq a'_2 \geq \cdots$. Then $\prod_n a_n' = (\prod_n a_n') a_n' = \theta$.

Let $c_0 = c$ and $\mathfrak{a}_0 = \mathfrak{a}$. Suppose $c_r, c'_r, a_r, a'_r$, have been defined for $1 \leq r < p$ and for some $p = 1, 2, \cdots$ in such a way that the following conditions are satisfied:

$$(\alpha)_p c_{r-1} = c_r \oplus c'_r, \quad a_{r-1} = a_r \oplus a'_r, \quad c'_r \sim a'_r,$$

$$(\beta)_p c_{p-1} = a_{p-1} (p > 1), \quad c_{p-1} \prec (a'_n + a_{p-1}), \text{ for } n = 1, 2, \cdots.$$

Then, since $c_{p-1} \prec (a'_n + a_p)$, we can define $c_p, c'_p, a'_p$, by Lemma 6.13, so that

$$c_{p-1} = c_p \oplus c'_p, \quad c_p \prec a'_p, \quad c'_p \sim a'_p \leq a_{p-1}.$$

Now define $\mathfrak{a}_p = [\mathfrak{a}_{p-1} - \mathfrak{a}_p]$. Then, by the use of Lemma 6.8,

$$\mathfrak{a}_{p-1} = \mathfrak{a}_p \oplus \mathfrak{a}_p', \quad c_p \prec (a'_n \oplus \mathfrak{a}_p), \text{ for } n = 1, 2, \cdots.$$

Thus $(\alpha)_{p+1}, (\beta)_{p+1}$ are satisfied. Since $(\alpha)_1$ and $(\beta)_1$ are satisfied by $c_0, \mathfrak{a}_0$, it follows that we can define by induction $c_r, c'_r, a_r, a'_r$, for $r = 1, 2, \cdots$, to satisfy $(\alpha)_p$ and $(\beta)_p$ for all $p = 1, 2, \cdots$.

By Lemma 2.11

$$c = \sum_{n=1}^{\infty} (\oplus c'_n) \oplus \prod_{n=1}^{\infty} c_n, \quad \mathfrak{a} = \sum_{n=1}^{\infty} (\oplus \mathfrak{a}'_n) \oplus \prod_{n=1}^{\infty} \mathfrak{a}_n.$$

Since

$$\prod_{n=1}^{\infty} a'_n = \theta \leq \prod_{n=1}^{\infty} c_n \leq c_r \prec a'_r \text{ for } r = 1, 2, \cdots,$$
Lemma 6.12 implies $\prod_{n=1}^{\infty} c_n = \theta$. Hence $c = \sum_{n=1}^{\infty} (\oplus c'_n)$. Since
\[
c\left(\sum_{n=1}^{\infty} (\oplus a'_n)\right) = c\bar{a}\left(\sum_{n=1}^{\infty} (\oplus a'_n)\right) = \theta,
\]
Lemma 2.10 implies $c'_n$, $a'_n$, $(n=1, 2, \cdots)$, are independent. Lemmas 2.3 and 4.2 now give
\[
c = \sum_{n=1}^{\infty} (\oplus c'_n) \sim \sum_{n=1}^{\infty} (\oplus a_n) \leq \bar{a} = \prod_{n=1}^{\infty} a_n.
\]
Hence $c \propto \prod_n a_n$, which proves the lemma.

**Lemma 6.15.** If $\theta$ is defined, and if
\[
a \oplus a' = b \oplus b',
\]
then $b \propto a$ implies $a' \propto b'$.

**Proof.** Suppose $b \sim a_1 \leq a$, and define $a'_1 = [a - a_1]$. Then $a_1 \oplus (a'_1 \oplus a') = b \oplus b'$, and Lemma 6.2 implies $(a'_1 \oplus a') \sim b'$; hence by Lemma 6.5 (II) $a' \propto b'$.

**Lemma 6.16.** If $a_1 \leq a_2 \leq \cdots$ and if $a_n \propto c$ for $n = 1, 2, \cdots$, then $\sum_n a_n \propto c$.

**Proof.** Let $a_1 c = \theta$, and define $u_1 = a_1$, $u_n = [a_n - a_{n-1}]$ for $n = 2, 3, \cdots$; then $a_n = \sum_{r=1}^{n} u_r$, $(n = 1, 2, \cdots)$, and $u_1, u_2, \cdots$ are independent over $\theta$ by Lemma 2.6, since $u_{n+1}(u_1 + \cdots + u_n) = [a_{n+1} - a_n] a_n = \theta$ for $n = 1, 2, \cdots$. Set $b_n = \sum_{r=n+1}^{\infty} (\oplus u_r)$ for $n = 0, 1, \cdots$; then $b_0 \geq b_1 \geq \cdots$, and
\[
b_0 = \sum_{r=1}^{\infty} (\oplus u_r) = a_n \oplus b_n, \quad n = 1, 2, \cdots.
\]
Hence $b_0 \geq \sum_{n=1}^{\infty} a_n$. Since $u_n \leq a_n$ we also have $b_0 = \sum_{n=1}^{\infty} u_n \geq \sum_{n=1}^{\infty} a_n$; thus $b_0 = \sum_{n=1}^{\infty} a_n$. Now define $c' = [(b_0 + c) - c]$ and $b'_0 = [(b_0 + c) - b_0]$. Then
\[
c \oplus c' = b_0 \oplus b'_0 = a_n \oplus (b_n \oplus b'_n)
\]
for $n = 1, 2, \cdots$, and Lemma 6.15 implies $c' \propto (b_n \oplus b'_n)$ for $n = 1, 2, \cdots$. Applying Lemma 6.14 to $c'$ and $(b_1 + b'_1) \geq (b_2 + b'_2) \geq \cdots$, we obtain
\[
c' \propto \prod_n (b_n + b'_n) = \left(\prod_n b_n\right) + b'_0 = \theta + b'_0 = b'_0.
\]
Lemma 6.15 now implies $\sum_n a_n = b_0 \propto c$. This proves the lemma.

**Definition 6.2.** If $a_1, a_2, \cdots$ is an infinite sequence, we define
\[
\limsup a_n = \prod_p \left(\sum_{n=p}^{\infty} a_n\right), \quad \liminf a_n = \sum_p \left(\prod_{n=p}^{\infty} a_n\right).
\]
The sequence is called convergent if \( \lim \sup a_n = \lim \inf a_n \), and for a convergent sequence we define \( \lim a_n = \lim \sup a_n = \lim \inf a_n \).

**Lemma 6.17.** If \( a_1 \leq a_2 \leq \cdots \) \( (a_1 \geq a_2 \geq \cdots) \) then \( \lim a_n \) is defined and is equal to \( \sum_n a_n (\prod_n a_n) \).

**Proof.**

\[
\lim \sup a_n = \prod_{p=1}^{\infty} \left( \sum_{n=p}^{\infty} a_n \right) = \prod_{p=1}^{\infty} \left( \sum_{n=1}^{\infty} a_n \right) = \sum_{n=1}^{\infty} a_n = \prod_{p=1}^{\infty} (a_p) = \prod_n a_n;
\]

\[
\lim \inf a_n = \sum_{p=1}^{\infty} \left( \prod_{n=p}^{\infty} a_n \right) = \sum_{p=1}^{\infty} (a_p) = \sum_{n=1}^{\infty} a_n = \sum_{p=1}^{\infty} \left( \prod_{n=1}^{\infty} a_n \right) = \prod_{n=1}^{\infty} a_n.
\]

Hence \( \lim \sup a_n = \lim \inf a_n = \sum_n a_n (\prod_n a_n) \) and the lemma follows from Definition 6.2.

**Theorem 6.1.** Continuity of perspectivity. If \( a_1, a_2, \cdots \) and \( b_1, b_2, \cdots \) are convergent sequences with \( \lim a_n = \bar{a} \) and \( \lim b_n = \bar{b} \), then \( a_n \sim b_n \) for \( n = 1, 2, \cdots \) implies \( \bar{a} \sim \bar{b} \).

**Proof.** For every fixed \( p = 1, 2, \cdots \) we have

\[
\left( \prod_{n=p}^{\infty} a_n \right) \leq a_r \sim b_r \leq \left( \sum_{n=r}^{\infty} b_n \right),
\]

and Lemma 6.14 implies \( (\prod_{n=p}^{\infty} a_n) \times (\prod_{n=p}^{\infty} b_n) = \bar{b} \). Lemma 6.16 gives \( \bar{a} = \sum_{p=1}^{\infty} (\prod_{n=p}^{\infty} a_n) \), that is \( \bar{a} \prec \bar{b} \). Similarly \( \bar{b} \prec \bar{a} \). Then, by Lemma 6.15(IV), \( \bar{a} \sim \bar{b} \), which proves the theorem.

**Corollary.** If \( a_1 \leq a_2 \leq \cdots \) and \( b_1 \leq b_2 \leq \cdots \) \( (a_1 \geq a_2 \geq \cdots \) and \( b_1 \geq b_2 \geq \cdots \) then \( a_n \sim b_n \) for \( n = 1, 2, \cdots \) implies \( \sum_n a_n \sim \sum_n b_n (\prod_n a_n \sim \prod_n b_n) \).

**Proof.** By Lemma 6.17 this is a special case of Theorem 6.1.

**Theorem 6.2.** Additivity of perspectivity. If \( \theta \) is defined and if \( a_n, 1 \leq n < p \), and \( b_n, 1 \leq n < p \), are each independent over \( \theta \), where \( p \) is finite or infinite, then \( a_n \sim b_n \) for \( 1 \leq n < p \) implies

\[
\sum_{n=1}^{p} (\oplus a_n) \sim \sum_{n=1}^{p} (\oplus b_n).
\]
Proof. By Lemma 6.4 $\sum_{n=1}^{r} (\oplus a_n) \sim \sum_{n=1}^{r} (\oplus b_n)$ for all $r < p$. If $p$ is finite this proves the theorem, and if $p$ is infinite we have, using the corollary to Theorem 6.1,

$$\sum_{n=1}^{\infty} (\oplus a_n) = \sum_{r} \left( \sum_{n=1}^{r} (\oplus a_n) \right) \sim \sum_{r} \left( \sum_{n=1}^{r} (\oplus b_n) \right) = \sum_{n=1}^{\infty} (\oplus b_n).$$

This proves the theorem.*

* For the special case of an irreducible geometry (finite dimensional or continuous), all the lemmas and theorems of §6 are easy consequences of the existence of a dimension function, and conversely, some of them are useful in establishing the existence of the dimension function (see C.G. part 1, chaps. 6 and 7). The notion of a convergent sequence is given in an equivalent form by von Neumann, Proceedings of the National Academy of Sciences, vol. 22 (1936), p. 107 (see the definition of lim** given there).

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