SOME EXISTENCE THEOREMS IN THE CALCULUS
OF VARIATIONS

III. EXISTENCE THEOREMS FOR NONREGULAR
PROBLEMS*

BY

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1. Outline of the method of proof. This note is the third of a series, the
first and second of which have already appeared in these Transactions.†

In the present paper we shall establish an existence theorem for single-
integral problems of the calculus of variations without requiring quasi-regu-
larity. In order to help the reader to find his way through the many details
of this proof, we first present a heuristic outline of the method used.

Given an integral \( \mathcal{J}(C) = \int F(z, z') \, dt \), where \( z = (z^1, \cdots, z^s) \), we seek to find
a rectifiable curve (in \( g \)-dimensional space) joining two fixed points \( z_1, z_2 \) for
which this integral is least. It is always possible to find a minimizing sequence
\( \{C_n\} \); that is, a sequence for which \( \mathcal{J}(C_n) \) tends to the greatest lower bound \( \mu \)
of \( \mathcal{J}(C) \) on the class of curves under consideration. Moreover, it is easy to
find hypotheses on \( \mathcal{J}(C) \) which will ensure that a minimizing sequence exist
which has a curve of accumulation \( C_0 \). That is, the curves \( C_n \) have representa-
tions \( z = z_n(t) \), \((0 \leq t \leq 1)\), for which \( z_n(t) \) tends uniformly to a limit function
\( z_0(t) \). If now we have some method of selecting the \( C_n \) so that \( z_n'(t) \) tends al-
most everywhere to \( z_0'(t) \), then \( \mathcal{J}(C_n) \to \mathcal{J}(C_0) \), and the curve \( C_0 \) is the one
sought.

Suppose that a curve \( C: z = z(t) \), \((t_1 \leq t \leq t_2)\), in \( g \)-dimensional space mini-
mizes an integral \( \mathcal{J}(C) \) in the class of all curves of class ‡ \( D' \) joining two fixed
points \( z_1 \) and \( z_2 \). As is well known, the equation

\[
(1.1) \quad F_i(z(t), z'(t)) = \int_{t_1}^{t} F_i(z(t), z'(t)) \, dt + c_i
\]

holds. Also, at each corner \( z(t_0) \) of \( C \) the relations

\[
(1.2) \quad F_i(z(t_0), z'(t_0 - 0)) = F_i(z(t_0), z'(t_0 + 0))
\]

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in revised form, February 16, 1938.
‡ A curve \( C \) is of class \( D' \) if it has a representation \( z = z(t) \), \((t_1 \leq t \leq t_2)\), such that the functions \( z'(t) \)
are continuous and the interval \([t_1, t_2]\) contains a finite number of points between which \( z'(t) \) is non-
vanishing and uniformly continuous. (At these points \( z'(t) \) may be \((0, \cdots, 0)\) or may fail to exist.)
and\(^*\)

\begin{equation}
\Omega(z(t_0), z'(t_0 - 0), z'(t_0 + 0)) \leq 0
\end{equation}

hold, where

\begin{equation}
\Omega(z, p, r) = p^aF_{z^a}(z, r) - r^aF_{z^a}(z, p).
\end{equation}

If now we choose a sequence of polygons \( \Pi_n \) joining \( z_1 \) to \( z_2 \) and such that \( J(\Pi_n) \) tends to the lower bound \( \mu \) of \( J(C) \) on the class of curves under consideration, we can not immediately make any statement similar to (1.1) or (1.3) for \( \Pi_n \). However, if the number of vertices of \( \Pi_n \) is \( s_n \), and \( \Pi_n \) actually minimizes \( J(C) \) in the class of all polygons joining \( z_1 \) and \( z_2 \) and having not more than \( s_n \) vertices, then it is to be expected that relations (1.1), (1.2), and (1.3) hold in an approximate sense for \( \Pi_n \).

We now define an approach set at \( z \) to be an aggregate of vectors \( p \) such that

\begin{equation}
F_i(z, p_i) - F_i(z, p_2) = 0, \quad i = 1, \ldots, q,
\end{equation}

for each pair \( p_1, p_2 \) of vectors of the set. In this terminology, equation (1.2) states that \( z'(t_0 - 0) \) and \( z'(t_0 + 0) \) belong to an approach set at \( z(t_0) \). Suppose then that the polygons \( \Pi_n \) have been chosen as above so that \( \Pi_n \) minimizes \( J(C) \) in the class of polygons which have not more than \( s_n \) vertices and which join \( z_1 \) to \( z_2 \). We suppose that these are represented by functions \( z = z_n(t), \ 0 \leq t \leq 1 \), which are piecewise linear on \([0, 1]\). Also we suppose that we have already chosen a convergent sequence, so that \( z_n(t) \) tends to a limit function \( z_0(t) \) uniformly on \([0, 1]\). If we fix on any number \( t_0 \) in \([0, 1]\), it is possible to choose a subsequence \( \{\Pi_m\} \) of the sequence \( \{\Pi_n\} \) in such a way that the vectors \( z'_m(t_0 + 0) \) tend to a limit \( p_0 \). Whenever \( m \) is large and \( t \) is near \( t_0 \), the point \( z_m(t) \) is near \( z_0(t_0) \) and the vector \( z_m(t_0 + 0) \) is near \( p_0 \). If now equation (1.1) holds, at least approximately, for \( \Pi_m \), the value of \( F_i(z_m(t), z_m'(t)) \) can change only a small amount on a short arc of \( \Pi_m \). For by (1.1) the change in \( F_i(z_m(t), z_m'(t)) \) is equal to the integral of \( F_i \) over a short interval. Hence for all \( t \) near \( t_0 \) the values of the \( F_i(z_m(t), z_m'(t)) \) are nearly equal to the values of \( F_i(z_0(t_0), z_0'(t_0)) \), and these in turn are nearly equal to \( F_i(z_0(t_0), p) \). Thus for all large \( m \) and for all \( t \) near \( t_0 \) the equations

\[ F_i(z_0(t_0), z_0'(t)) = F_i(z_0(t_0), p_0), \quad t = 0, 1, \ldots, q, \]

are almost true, and we may expect that there is an approach set \( A \) such that \( z'_m(t) \) is in or near \( A \) whenever \( m \) is large and \( t - t_0 \) small.

Suppose now that each such approach set contains only a finite number

\(^*\) Cf. I, Theorem 1.
of unit vectors, which can be set in an order \( p_1, p_2, \ldots, p_k \) such that

\[ (1.6) \quad \Omega(z_0(t_0), p_j, p_i) > 0, \quad j > i. \]

Now (1.3) must hold approximately on \( \Pi_m \), and for \( t \) near \( t_0 \) and \( m \) large the directions \( z_m' (t) / |z_m' (t)| \) are each near some \( p_i \); hence by (1.3) and (1.6) we find that a side with direction near \( p_j \) cannot precede one with direction near \( p_i \) if \( j > i \). Thus the arc of \( \Pi_m \) near \( z_0(t_0) \) can be split into subarcs, the first of which consists of sides with directions near \( p_1 \), the second of sides with directions near \( p_2 \), and so on. That is, each such arc is almost a line segment. So is its limit arc; and for line segments \( l_n \) tending to a limit \( l_0 \) it is clear that \( J(l_n) \to J(l_0) \). Thus our point \( t_0 \) is in an interval along which the integrand \( F(z_n, z_n') \) converges, with arbitrarily small error, to \( F(z_0, z_0) \).

This argument, applied to all \( t_0 \), would yield \( J(\Pi_m) \to J(C_0) \). Since \( J(\Pi_m) \to \mu \), we have \( J(C_0) = \mu \), and \( C_0 \) is the curve sought.

In the following pages the argument just suggested will be generalized and made rigorous.

2. Choice of a minimizing sequence. We now suppose that we are given an integral \( J(C) \) in parametric form, and seek the minimum of \( J(C) \) in the class of all rectifiable curves \( C \) joining two given points \( z_1 \) and \( z_2 \). We suppose that \( J(C) \) satisfies the following condition:

\[ (2.1) \quad \text{For every constant } M \text{ there is a number } L_M \text{ such that all curves } C \text{ joining } z_1 \text{ and } z_2 \text{ and giving } J(C) \text{ a value less than or equal to } M \text{ have lengths not greater than } L_M. \]

For example, (2.1) is satisfied if there is a number \( c > 0 \) such that

\[ F(z, z') \geq c |z'| / (1 + |z|). \]

We shall reserve the word “vertex” for points on a polygon at which successive sides join; that is, the initial and final points will not be called vertices.

**Lemma 1.** Let \( J(C) \) satisfy (2.1). Let \( K_s \) be the class of all polygons joining \( z_1 \) to \( z_2 \) and having not more than \( s \) vertices. Then for every \( s \) the class \( K_s \) contains a polygon which minimizes \( J(C) \) on the class \( K_s \).

Let \( \mu_s \) be the greatest lower bound of \( J(C) \) on the class \( K_s \). We establish a correspondence between the polygons of \( K_s \) and the points \( (\xi^1, \ldots, \xi^{s+1}) \) of \( s \)-dimensional space by writing the coordinates of the first vertex, then those of the second, and so on.* By (2.1), there is a number \( R \) such that if any coordinate of the point \( \xi \) representing a polygon \( \Pi \) exceeds \( R \) in absolute value, then \( J(\Pi) > \mu + 1 \). So the minimum of \( J(\Pi) \) on the class \( K_s \) is to be

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* We may suppose that every polygon of \( K_s \) has \( s \) vertices, because if it has less we can insert points of division in the sides and increase the number of vertices up to \( s \).
sought among the images of the set $|\xi| \leq R_{sq}$. This is bounded and closed, and on it $\mathcal{J}(\Pi)$ is continuous; hence the minimum is assumed.

Now we introduce the notation:

(2.2) $K$ is the class of all rectifiable curves joining the two distinct fixed points $z_1$ and $z_2$, and $\mu$ is the greatest lower bound of $\mathcal{J}(C)$ on the class $K$.

Then we have the following lemma:

**Lemma 2.** If $\mathcal{J}(C)$ satisfies (2.1), there exists a sequence of polygons $\Pi_n$: $z = z_n(t)$, ($0 \leq t \leq 1$), in the class $K$ with the properties:

(a) $\mathcal{J}(\Pi_n) \to \mu$.

(b) $s_n$ being the number of vertices of $\Pi_n$, and $K_{s_n}$ being the subclass of $K$ consisting of polygons of not more than $s_n$ vertices, $\Pi_n$ minimizes $\mathcal{J}(C)$ on the class $K_{s_n}$.

(c) There is a constant $L$ such that $|\dot{z}_n(t)| \leq L$ for all $n$ and all $t$ in $[0, 1]$.

(d) $z_n(t)$ converges uniformly to a limit function $z_0(t)$, ($0 \leq t \leq 1$).

Let $\{C_n\}$ be a sequence of curves of $K$ for which $\mathcal{J}(C_n) \to \mu$. For each fixed $n$ we inscribe in $C_n$ a sequence of polygons $\{\Pi_k^*\}$ with sides tending to 0. It is well known that under these conditions $\mathcal{J}(\Pi_k^*)$ tends to $\mathcal{J}(C_n)$. Hence we can find a particular $k$ for which $\mathcal{J}(\Pi_k^*) < \mathcal{J}(C_n) + 1/n$.

Let $s_k^*$ be the number of vertices of $\Pi_k^*$. By Lemma 1, there is a polygon $\Pi_n$ of $s_n$ vertices, where $s_n \leq s_k^*$, which minimizes $\mathcal{J}(C)$ on the class $K_{s_k^*}$. Therefore $\mathcal{J}(\Pi_n) \leq \mathcal{J}(\Pi_k^*) < \mathcal{J}(C) + 1/n$. Since $\Pi_n$ minimizes $\mathcal{J}(C)$ in the class $K_{s_k^*}$ and is in the class $K_{s_k^*} \subset K_{s_n}$, it is clear that $\Pi_n$ minimizes $\mathcal{J}(C)$ on $K_{s_k^*}$.

Also,

$$\lim \sup_{n \to \infty} \mathcal{J}(\Pi_n) \leq \lim_{n \to \infty} \mathcal{J}(C_n) = \mu.$$ 

But $\Pi_n$ is in $K$; so $\mathcal{J}(\Pi_n) \geq \mu$, and $\lim \inf_{n \to \infty} \mathcal{J}(\Pi_n) \geq \mu$. Thus (a) and (b) are established.

There is no loss of generality in supposing $\mathcal{J}(\Pi_n) < \mu + 1$ for all $n$. Hence, by hypothesis (2.1), there is an $L$ such that each $\Pi_n$ has length $L(\Pi_n)$ less than $L$. On $\Pi_n$ we introduce the parameter $t = s/L(\Pi_n)$, where $s$ is arc length. Then $\Pi_n$ has a representation $z = z_n(t)$, ($0 \leq t \leq 1$), with $|\dot{z}_n(t)| \leq L(C_n) < L$, and (c) is satisfied.

Finally, the $z_n(t)$ all satisfy a Lipschitz condition with constant $L$; so by Ascoli's theorem there is a subsequence (which we may without loss of generality suppose to be the whole sequence) for which $z_n(t)$ converges uniformly to a limit function $z_0(t)$. This completes the proof of the lemma.

**Remark.** If a sequence $\{\Pi_n\}$ satisfies (a), (b), (c), and (d) of Lemma 2, so does every subsequence of $\{\Pi_n\}$.

Before stating the next lemma we introduce a definition.
(2.3) If \([\alpha, \beta]\) is an interval having points in common with \([0, 1]\) and \(t_{n,1}, t_{n,2}, \ldots, t_{n,k}\) are (in that order) the values of \(t\) in \([\alpha, \beta]\) which define vertices of \(\Pi_n\), then
\[
\theta(n, \alpha, \beta) = t_{n,1}, \quad \Theta(n, \alpha, \beta) = t_{n,k}.
\]

If no value of \(t\) in \([\alpha, \beta]\) defines a vertex of \(\Pi_n\), we define
\[
(2.4) \quad \theta(n, \alpha, \beta) = \Theta(n, \alpha, \beta) = (\alpha + \beta)/2.
\]

Then we have the following lemma:

**Lemma 3.** If \(\{\Pi_n\}\) is the sequence of Lemma 2, then for every \(\varepsilon > 0\) and every \(t_0\) in \([0, 1]\) there is a \(\delta > 0\) such that
\[
(2.5) \quad |F_i(z(t_0), z'(t')) - F_i(z(t_0), z'(t''))| \leq \varepsilon, \quad i = 1, \ldots, q,
\]
if \(t_0, t',\) and \(t''\) are all in the (open) interval \((\theta(n, t_0 - \delta, t_0 + \delta), \Theta(n, t_0 - \delta, t_0 + \delta))\) and \(z(t')\) and \(z(t'')\) are not vertices\(^\dagger\) of \(\Pi_n\).

We shall suppose, to be specific, that \(t' < t''\). Let \(t_1, t_2, \ldots, t_k\) define successive vertices of \(\Pi_n\) such that \(t_1 < t' < t_2 \) and \(t_{k-1} < t'' < t_k\). By the definition of \(\theta\) and \(\Theta\) we know that
\[
(2.6) \quad t_0 - \delta < t_1 < t_k < t_0 + \delta.
\]

From \(\Pi_n\) we now form the polygon \(\Pi_n(t')\) by displacing each of the vertices \(z(t_1), \ldots, z(t_{k-1})\) by the amount \((0, \ldots, \tau, 0, \ldots, 0)\), the \(\tau\) being in the \(j\)th place. Because of the minimizing property of \(\Pi_n\) we have
\[
(2.7) \quad \left. \frac{d}{dt} \mathcal{J}(\Pi_n(t)) \right|_{t=0} = 0.
\]

But if we apply formula (2.8) of I to the successive sides of \(\Pi_n(t')\) defined by \(t_1 \leq t \leq t_2, \ldots, t_{k-1} \leq t \leq t_k\), and then add, we obtain
\[
(2.8) \quad \left. \frac{d}{dt} \mathcal{J}(\Pi_n(t)) \right|_{t=0} = F_j(z(t), z'(t)) - F_j(z(t), z'(t))
\]
\[
+ \int_{t_1}^{t_2} F_s(z(t), z'(t)) \frac{t - t_1}{t_2 - t_1} \, dt + \int_{t_k}^{t_{k-1}} F_s(z(t), z'(t)) \, dt
\]
\[
+ \int_{t_{k-1}}^{t_k} F_s(z(t), z'(t)) \frac{t_k - t}{t_k - t_{k-1}} \, dt,
\]
\(^\dagger\) It is easy to see that this restriction is essentially no restriction, for if \(t'\) defines a vertex, we first write (2.5) with \(t'\) replaced by \(t' + h, h > 0\), and then let \(h \rightarrow 0\). For small \(h\) the value of \(s'(t' + h)\) is constantly equal to \(s'(t' + 0)\); whence we find (2.5) with \(t'\) replaced by \(t' + 0\). Similarly, we could replace \(t'\) by \(t' - 0\), and analogously for \(t''\).
where \( t_1 < t < t_2 \) and \( t_{k-1} < t < t_k \). All the curves \( \Pi_n \) lie in a bounded closed subset of the space, and all \( |\dot{z}_n| \) are less than \( L \). Hence the functions \( F_{ij}(z_n, \dot{z}_n) \) are bounded, say less than \( N \) in absolute value. Then the sum of the three integrals on the right is at most \( N(t_k - t_1) \) in absolute value, so that

\[
0 = F_i(z_n(t), z_n'(t)) - F_i(z_n(t), z_n'(t)) + \theta_i,
\]

where \( |\theta_i| \leq N(t_k - t_{k-1}) \leq 2N\delta \).

On the interval \((t_1, t_2)\) the derivative \( z_n'(t) \) is constantly equal to \( z_n'(t') \), and on \((t_{k-1}, t_k)\) it is constantly equal to \( z_n'(t'') \). Hence from (2.9) we obtain

\[
|F_i(z_n(t'), z_n'(t')) - F_i(z_n(t''), z_n'(t''))| \leq |F_i(z_n(t'), z_n'(t')) - F_i(z_n(t), z_n'(t'))| + |F_i(z_n(t), z_n'(t'')) - F_i(z_n(t'), z_n'(t''))| + |\theta_i|.
\]

Because of the continuity of \( F_i \), each of the first two terms on the right is less than \( \epsilon/3 \) if \( |z_n(t) - z_n(t')| \) and \( |z_n(t) - z_n(t'')| \) are less than a certain \( \gamma > 0 \). This is surely the case if \( |t - t'| \) and \( |t - t''| \) are less than \( \gamma/L \), which again is surely true if \( 2\delta < \gamma/L \). Also, if \( \delta < \epsilon/6N \), the term \( |\theta_i| \) is at most \( 2N\delta < \epsilon/3 \). Hence, if \( \delta \) is any number less than the smaller of \( \gamma/2L \) and \( \epsilon/6N \), inequality (2.10) implies (2.5).

**Remark.** From the last sentence of the proof it is obvious that \( \delta \) can be chosen as near to 0 as desired.

### 3. Directions of the sides of the minimizing sequence

The next lemma will explain the name “approach set” given to the sets satisfying (1.5).

**Lemma 4.** Let \( \{\Pi_n\} \) be a sequence of polygons satisfying (a), (b), (c), and (d) of Lemma 2, and let \( z_1 \neq z_2 \). Let \( t_0 \) be interior to \((0, 1)\). Then there is a subsequence \( \{\Pi_m\} \) of \( \{\Pi_n\} \) and an approach set \( A \) at \( z_0(t_0) \) such that for every \( \epsilon > 0 \) there is a \( \delta > 0 \) and an \( m_0 \) such that \( z_m(t) \) is in the \( \epsilon \)-neighborhood of \( A \) whenever \( z_m(t) \) is not a vertex, \( m > m_0 \), and

\[
\theta(m, t_0 - \delta, t_0 + \delta) * < t < \Theta(m, t_0 - \delta, t_0 + \delta).
\]

If there is a \( \delta > 0 \) and a subsequence \( \{\Pi_m\} \) such that not more than one \( t \) in \((t_0 - \delta, t_0 + \delta)\) defines a vertex of \( \Pi_m \), then the conclusion holds vacuously, for then \( \theta(m, t_0 - \delta, t_0 + \delta) = \Theta(m, t_0 - \delta, t_0 + \delta) \), and (3.1) is not satisfied by any \( t \).

Otherwise, for each integer \( m \) there are infinitely many \( n_m \) such that the interval

\[
(\theta(n_m, t_0 - m^{-1}, t_0 + m^{-1}), \Theta(n_m, t_0 - m^{-1}, t_0 + m^{-1}))
\]

has length greater than zero. We can suppose \( n_m \) so chosen that \( n_1 < n_2 < \cdots \).
In each interval (3.2) we choose a number $t^*_{m}$ which does not define a vertex of $\Pi_{n_m}$. The vectors $z'_{m}(t^*_{m})$ all have lengths less than or equal to $L$; hence they have an accumulation vector $p_0$. We select a subsequence $\{m\}$ of the sequence $\{n_m\}$ such that $z'_{m}(t^*_{m}) \rightarrow p_0$.

Now let $A$ be the approach set at $z_0(t_0)$ which contains $p_0$, and let $U$ be the set of all vectors $u$ whose distance from $A$ is less than $\varepsilon$. Then $U$ is clearly open. We are to show that $z'_{m}(t)$ is in $U$ if $t$ and $m$ satisfy the conditions of our lemma. If $V$ is the set of all vectors $v$ of length $|v| \leq L$ which are not in $U$, then $V$ is bounded and closed and has no point in common with $A$. Hence the function

$$(3.3) \quad \sum_{i=1}^{q} |F_i(z_0(t_0), p_0) - F_i(z_0(t_0), v)|$$

is positive for all $v$ in $V$. Being continuous† on $V$, it is greater than a positive number $3\gamma$.

Now

$$\sum_{i=1}^{q} |F_i(z_0(t_0), z'_{m}(t)) - F_i(z_0(t_0), p_0)|$$

$$\leq \sum_{i=1}^{q} |F_i(z_0(t_0), z'_{m}(t)) - F_i(z_0(t_0), z''_{m}(t))|$$

$$+ \sum_{i=1}^{q} |F_i(z_0(t_0), z''_{m}(t)) - F_i(z_0(t_0), z'_{m}(t))|$$

$$(3.4) + \sum_{i=1}^{q} |F_i(z_0(t_0), z'_{m}(t)) - F_i(z_0(t_0), p_0)|$$

Here, as in Lemma 3, the parameter $t$ is $s/L_{m}$, so that $|z'_{m}(t)| = L_{m}\geq |z_2 - z_1| > 0$. Thus the arguments in the first term on the right have a bounded closed range and $z''_{m}(t)$ is bounded from 0. Hence the first term is less than $\gamma$ if $|z_0(t_0) - z_m(t)|$ is smaller than a certain number $\kappa$. Since $z_m(t)$ tends uniformly to $z_0(t)$ and $z_0(t)$ is continuous, this is true if $\delta$ is small enough and $m$ is greater than a certain $m_0$.

In the last term on the right, $z'_{m}(t)$ tends to $p_0$ and $z''_{m}(t)$ to $z_0(t_0)$; so this term is less than $\gamma$ if $m$ is greater than a certain $m_2$.

If $\delta$ is small enough and $m$ is greater than a certain $m_3$, then by Lemma 3 the third term is less than $\gamma$ whenever $t$ does not define a vertex of $\Pi_m$ and (3.1) holds. Hence if $\delta$ is small enough and $m > m_0 = \max (m_1, m_2, m_3)$, then for all $t$ satisfying the requirements of our lemma we have

† The only point at which $F_i(z_0(t_0), v)$ is discontinuous is $v = 0$, which is a limit point of $A$ and therefore not in $V$. 

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\[ \sum_{i=1}^{q} \left| F_i(z_0(t_0), z_m'(t)) - F_i(z_0(t_0), p_0) \right| < 3\gamma. \]

But if \( z_m'(t) \) were in \( V \), this expression (see (3.3)) would exceed \( 3\gamma \). So \( z_m'(t) \) is not in \( V \). Also, it satisfies the condition \( |z_m'(t)| \leq L \). Hence it must belong to \( U \), and the lemma is established.

**Remark.** Again it is clear that the \( \delta \) can be chosen as small as desired.

4. **Proof of the principal theorem.** We now impose a new hypothesis on the integral \( J(C) \). We shall make the following supposition:

\[ (4.1) \text{For every } z, \text{ each approach set } A \text{ at } z \text{ is the sum of a finite number of convex sets } A_1, \ldots, A_k, \text{ which can be so ordered that} \]

\[ \Omega(z, p_i, p_j) < 0 \]

if \( p_i \) is in \( A_i \) and \( p_j \) in \( A_j \), \((i < j)\).

Our principal theorem is the following:

**Theorem 1.** If \( F(z, z') \) satisfies conditions (2.1) and (4.1), then in the class \( K \) of all rectifiable curves joining two distinct points \( z_1 \) and \( z_2 \) there is a curve which minimizes the integral

\[ J(C) = \int_C F(z, \dot{z})dt. \]

We select a sequence of polygons \( II_n \) and a curve \( C_0 \) satisfying the conclusions of Lemma 2, and we define

\[ (4.2) \phi_n(t) = \int_{0}^{t} F(z_n, \dot{z}_n)dt, \quad 0 \leq t \leq 1; n = 0, 1, 2, \ldots. \]

Since \( |\dot{z}_n(t)| \leq L \), the integrands are bounded, and the \( \phi_n \) all satisfy the same Lipschitz condition. Hence by Ascoli's theorem we can select a subsequence (we suppose it to be the whole sequence) such that \( \phi_n(t) \) tends uniformly to a limit function \( \phi(t) \). By (a) of Lemma 2,

\[ (4.3) \phi(1) = \lim_{n \to \infty} \phi_n(1) = \lim_{n \to \infty} J(II_n) = \mu. \]

Since

\[ (4.4) \phi_0(1) = J(C_0), \]

we must show then that

\[ (4.5) \phi_0(1) = \phi(1) = \mu. \]

We shall in fact show that

\[ (4.6) \phi_0(t) = \phi(t), \quad 0 \leq t \leq 1. \]
Let \( t_0 \) be a point interior to \((0, 1)\) at which \( \phi'(t) \) and \( \phi_0'(t) \) are both defined. We shall first show

\[
(4.7) \quad \phi'(t_0) = \phi_0'(t_0).
\]

This will be established if we can prove the following statement:

\[
(4.8) \text{For every positive number } \gamma \text{ there is a } t^* > t_0 \text{ such that if } t_0 < \rho < \sigma < t^*,
\]

then

\[
(4.9) \quad |[\phi(\sigma) - \phi(\rho)] - [\phi_0(\sigma) - \phi_0(\rho)]| \leq \gamma (\sigma - \rho).
\]

For if (4.8) holds, then by continuity (4.9) holds with \( \rho \) replaced by \( t_0 \). Dividing by \( \sigma - t_0 \) and letting \( \sigma \) tend to \( t_0 \) we obtain

\[
|\phi'(t_0) - \phi_0'(t_0)| \leq \gamma.
\]

Since \( \gamma \) is an arbitrary positive number, this can be true only if (4.7) holds. We therefore take \( \gamma \) to be a positive number and proceed to establish (4.8).

Let us first dispose of the relatively simple case in which there is a subsequence \( \{\Pi_k\} \) and a \( \delta > 0 \) such that either (a) \( \lim_{k \to \infty} \theta(h, t_0 - \delta, t_0 + \delta) > t_0 \) or (b) \( \lim_{k \to \infty} \Theta(h, t_0 - \delta, t_0 + \delta) \leq t_0 \). If (a) holds, we denote \( \lim \theta \) by \( t^* \); if (b) holds, we let \( t^* \) equal \( t_0 + \delta \). Let \( \rho, \sigma \) be numbers such that \( t_0 < \rho < \sigma < t^* \). Then if \( h \) is large enough, the interval \([\rho, \sigma]\) is contained in case (a) in \([t_0 - \delta, \theta(h, t_0 - \delta, t_0 + \delta)]\), and in case (b) in \([\Theta(h, t_0 - \delta, t_0 + \delta), t_0 + \delta]\). In either case, \( z_h(t) \) is linear on \([\rho, \sigma]\) for all large \( h \). The same is then true of its limit \( z_0(t) \), and it follows at once that \( z_h'(t) \) tends to \( z_0'(t) \) uniformly on \([\rho, \sigma]\). Therefore

\[
\phi'(t_0) - \phi_0'(t_0) = \lim_{h \to \infty} \{\phi_h(t_0) - \phi_h(\rho)\} = \lim_{h \to \infty} \int_{\rho}^{\sigma} F(z_h, z_h') dt
\]

\[
= \int_{\rho}^{\sigma} F(z_0, z_0') dt = \phi_0(\sigma) - \phi_0(\rho);
\]

and (4.8) is established for all positive \( \gamma \) at once.

Now we return to the general case. By Lemma 4, there is an approach set \( A \) at \( z_0(t_0) \) with the properties there specified. If \( p_0 \) is any fixed vector in \( A \), then for every vector \( \rho \) in \( A \) the equations

\[
F_i(z_0(t_0), \rho) = F_i(z_0(t_0), p_0), \quad i = 1, \ldots, q,
\]

hold, by definition. If we denote the right-hand members of these equations by \( l_1, \ldots, l_q \), respectively, then on multiplying by \( \rho^i \) and summing we find, for all \( \rho \) in \( A \), that

\[
F(z_0(t_0), \rho) = l_\rho^\rho.
\]
The set of vectors $p$ in $A$ with $|p| \leq L$ is bounded and closed, and the function $F$ is continuous. Therefore there is a $\lambda > 0$ such that

$$|F(z, p) - l_0 p^a| < \gamma/2$$

if $|z - z_0(t_0)| < \lambda$ and $p$ is in the $\lambda$-neighborhood $U_\lambda$ of the intersection of $A$ with the sphere $|p| \leq L$.

The set $A$ is the sum of convex sets $A_1, \ldots, A_k$. Let us denote by $U_{j, e}$ the $e$-neighborhood of the intersection of $A_j$ with the sphere $|p| \leq L$. Then $U_e = U_{1, e} + \cdots + U_{k, e}$. By hypothesis, the $A_j$ have the property that

$$\Omega(z_0(t_0), p_i, p_j) < 0$$

if $p_i$ is in $A_i$, $p_j$ is in $A_j$, and $i < j$. Therefore on the bounded closed set of arguments $(z, p_i, p_j)$ which satisfy the conditions $z = z_0(t_0)$, $p_i$ in $A_i$, $p_j$ in $A_j$, $(i < j)$, $|z_2 - z_1| \leq |p_i| \leq L$, $|z_2 - z_1| \leq |p_j| \leq L$ this function has a negative upper bound $-2\kappa$. Since it is continuous, we find that there is an $\eta > 0$ such that

$$\Omega(z, p_i, p_j) < -\kappa$$

whenever $p_i$ is in $U_{i, e}$, $p_j$ is in $U_{j, e}$, $(i < j)$, $|z_2 - z_1| \leq |p_i| \leq L$, $|z_2 - z_1| \leq |p_j| \leq L$, and $|z - z_0(t_0)| < 3\eta$.

Now define

$$\epsilon = \min (\eta, \lambda).$$

We use this for the $\epsilon$ in Lemma 4, and choose (see the remark after the lemma) a value for $\delta$ which is less than $\epsilon/2(L + 1)$. Furthermore, from the sequence $\{ \Pi_m \}$ of that lemma we discard those polygons (finite in number) for which the inequality

$$|z_m(t) - z_0(t)| < \epsilon/2,$$

fails to hold. Then, by Lemma 4, together with (4.10) and (4.13), we find:

(4.15) **For all functions $z_m(t)$ and all $t$ such that**

$$\theta(m, t_0 - \delta, t_0 + \delta) < t < \Theta(m, t_0 - \delta, t_0 + \delta),$$

**the inequality**

$$|F(z_m(t), \dot{z}_m(t)) - l_0 \dot{z}_m^a(t)| < \gamma/2$$

**holds, and $z_m'(t)$ is in one of the sets $U_{1, e}, \ldots, U_{k, e}$ if $z_m'(t)$ is defined.**

Let $t_1, t_2, t_3$ define three consecutive vertices of $\Pi_m$, and let the inequalities

$$\theta(m, t_0 - \delta, t_0 + \delta) \leq t_1 < t_2 < t_3 \leq \Theta(m, t_0 - \delta, t_0 + \delta)$$

**hold. The sides of $\Pi_m$ corresponding to** $[t_1, t_2]$ and $[t_2, t_3]$ have directions
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By (4.15) these belong respectively to neighborhoods $U_{i,e}, U_{j,e}$; and the inequalities

$$|z_2 - z_1| \leq |z_m'(t)| \leq L$$

always hold. We now show that $i$ is less than $j$. Suppose the contrary. If we interchange these two sides of $\Pi_m$, we obtain a new polygon $\Pi_m^*$, and by formula (3.19) of $I$

$$\mathcal{J}(\Pi_m^*) - \mathcal{J}(\Pi_m) = - \left| z_m(t_1) - z_m(t_2) \right| - \left| z_m(t_2) - z_m(t_1) \right| \cdot \Omega(\bar{z}, z_m'(t_1 + 0), z_m'(t_2 + 0)),$$

where $\bar{z}$ is in the parallelogram determined by the two sides. However, $|t_1 - t_0| < \delta < \epsilon/2L$, so $|z_m(t_1) - z_m(t_0)| < L \cdot \epsilon/2L = \epsilon/2$, while by (4.14), $|z_m(t_0) - z_m(t_0)| < \epsilon/2$. Hence $|z_m(t_1) - z_m(t_0)| < \epsilon$. The sides of this parallelogram have lengths less than $|t_1 - t_0| L < 2\delta L < \epsilon$. Thus the whole parallelogram lies in the $3\epsilon$-neighborhood of $z_0(t_0)$, and in particular $|\bar{z} - z_0(t_0)| < 3\epsilon \leq 3\eta$, by (4.13). Therefore by (4.12) and (4.13) the factor $\Omega$ in (4.16) is positive,† and $\mathcal{J}(\Pi_m^*) < \mathcal{J}(\Pi_m)$. This contradicts the minimizing property of $\Pi_m$, and establishes the inequality $i < j$.

We have therefore established, as a sort of addition to (4.15), that for $t$ between $\theta$ and $\Theta$ no side of $\Pi_m$ whose direction is in a $U_{i,e}$ is followed by one whose direction is in a $U_{j,e}$ with $i < j$. Let us now define

$$t_{m,0} = \theta(m, t_0 - \delta, t_0 + \delta), \quad t_{m,k} = \Theta(m, t_0 - \delta, t_0 + \delta).$$

By the previous remark, there are numbers $t_{m,i}$ defining vertices of $\Pi_m$ such that

$$t_{m,0} \leq t_{m,1} \leq \cdots \leq t_{m,k},$$

and if $t_{m,i-1} < t < t_{m,i}$ and $z_m'(t)$ is defined, then $z_m'(t)$ is in $U_{j,e}$. We now select a subsequence $\{\Pi_h\}$ of $\{\Pi_m\}$ such that for each $j = 0, 1, \cdots, k$ the limit

$$\tau_j = \lim_{h \to \infty} t_{h,j}$$

exists. We may suppose $\tau_0 \leq t_0 < \tau_k$, for otherwise we are back to the simpler case first considered.

There is a first $j$ such that $\tau_j > t_0$; hence $\tau_{j-1} \leq t_0 < \tau_j$. Let $\rho, \sigma$ be numbers such that $t_0 < \rho < \sigma < \tau_j$. For all large $h$ the inequalities $\tau_{h,j-1} < \rho < \tau_{h,j}$ hold, so that $z_h(t)$ is in $U_{j,e}$ if $\rho \leq t \leq \sigma$. The set $U_{j,e}$ is the $\epsilon$-neighborhood of the

† Recall that the interchange of $p_i$ and $p_j$ in (4.12) changes the sign of $\Omega$. 

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intersection of two convex sets, \( A_i \) and the sphere \(|\rho| \leq L\); so \( U_i, \epsilon \) is convex. By Jensen’s inequality (in geometric form), if \( \rho \leq t < t' \leq \sigma \), the vector 
\[
\frac{1}{t' - t} \int_0^{t'} z_h(t') dt = \frac{\dot{z}_h(t') - \dot{z}_h(t)}{t' - t}
\]
is in the closure \( \overline{U}_{i, \epsilon} \). Letting \( h \to \infty \), we find that the limit
\[
\frac{z_h(t') - z_h(t)}{t' - t}
\]
is also in \( \overline{U}_{i, \epsilon} \). Now let \( t' \to t \); we find that the vector \( z_0'(t) \) is in \( \overline{U}_{i, \epsilon} \), if it is defined. Since the zero vector is in \( U_{i, \epsilon} \), we see that in any case \( \dot{z}_0(t) \) is in \( \overline{U}_{i, \epsilon} \) for \( \rho \leq t < \sigma \).

Turning now to (4.10), we therefore obtain
\[
|F(z_0(t), \dot{z}_0(t)) - l_\alpha \dot{z}_0^\alpha(t)| \leq \gamma/2
\]
if \( \rho \leq t < \sigma \). Hence
\[
(4.18) \quad |\phi_0(\sigma) - \phi_0(\rho) - l_\alpha \{z_0^\alpha(\sigma) - z_0^\alpha(\rho)\}| = \left| \int_\rho^\sigma \{F(z_0, \dot{z}_0) - l_\alpha \dot{z}_0^\alpha\} dt \right| 
\leq \gamma(\sigma - \rho)/2.
\]
On the other hand, by (4.15)
\[
(4.19) \quad |\phi_\alpha(\sigma) - \phi_\alpha(\rho) - l_\alpha \{z_\alpha(\sigma) - z_\alpha(\rho)\}| = \left| \int_\rho^\sigma \{F(z_\alpha, \dot{z}_\alpha) - l_\alpha \dot{z}_\alpha^\alpha\} dt \right| 
< \gamma(\sigma - \rho)/2.
\]
If we here let \( h \to \infty \), we obtain
\[
(4.20) \quad |\phi(\sigma) - \phi(\rho) - l_\alpha \{z_\alpha(\sigma) - z_\alpha(\rho)\}| \leq \gamma(\sigma - \rho)/2.
\]
From (4.18) and (4.20) we at once obtain (4.9). Thus statement (4.8), and with it (4.7), is established.

Since \( \phi(0) = \phi_0(0) = 0 \) and (4.7) holds for almost all \( t \), and since the functions \( \phi \) and \( \phi_0 \) are both absolutely continuous, it follows that \( \phi(t) = \phi_0(t) \), so that (4.5) is established. Therefore the curve \( z = z_0(t) \) is the minimizing curve sought.

5. Example. If \( F(z, z') \) satisfies (2.1) and is quasi-regular† (see (6.2)), every approach set \( A \) at every point \( z_0 \) consists of a single convex set; so (4.1) is trivially satisfied. Hence for such integrands there is always a minimizing

† Necessarily positive quasi-regular, for if it is negative quasi-regular and not linear, (2.1) cannot hold.
curve for $\mathcal{F}(C)$ in the class $K$ of all curves joining two given points $z_1$ and $z_2$. This is, of course, a special case of a known theorem.

An example of an essentially different type is $\int F dt$ where

$$F(x, y, x', y') = (x'^2 + y'^2)^{1/2} + y^2\left[4(x'^2 + y'^2)^{1/2} - (x'^2 + 8y'^2)^{1/2}\right].$$

This integrand is not positive quasi-regular, for if $|y| > 1/2$, the $E$-function can be negative. Given a set $(x, y, p, q)$ with $p^2 + q^2 = 1$, we seek to determine all $(\bar{p}, \bar{q})$ with $\bar{p}^2 + \bar{q}^2 = 1$ such that

$$(5.1) \quad F_+(x, y, \bar{p}, \bar{q}) = F_+(x, y, p, q), \quad F_+(x, y, \bar{p}, \bar{q}) = F_+(x, y, p, q).$$

We can write (5.1) in the form

$$\begin{align*}
\rho [1 + 4y^2 - \rho^2(8 - 7\rho^2)^{-1/2}] &= \bar{\rho} [1 + 4\rho^2 - \rho^2(8 - 7\rho^2)^{-1/2}], \\
q [1 + 4y^2 - 8y^2(1 + 7q^2)^{-1/2}] &= \bar{q} [1 + 4y^2 - 8y^2(1 + 7q^2)^{-1/2}].
\end{align*}$$

The function in the left-hand member of the first of these equations is easily seen to have a positive derivative with respect to $p$ for $|p| \leq 1$; so the only solution of the first equation is $\bar{p} = p$. Since $p^2 + q^2 = \bar{p}^2 + \bar{q}^2 = 1$, this implies $\bar{q} = \pm q$. Hence we substitute $-q$ for $\bar{q}$ in the second equation of (5.2). If it is not satisfied, then the entire approach set at $(x, y)$ containing $(p, q)$ is the set of multiples $(kp, kq)$, $(k > 0)$. If it is satisfied, then

$$1 + 4y^2 - 8y^2(1 + 7q^2)^{-1/2} = 0$$

or

$$(5.3) \quad q = \pm \left[\frac{1}{7} \left\{\left(\frac{8y^2}{1 + 4y^2}\right)^2 - 1\right\}\right]^{1/2}.$$

If $q$ has the value (5.3), then $A$ consists of the positive multiples of $(p, q)$ and of $(p, -q)$, which are distinct if $q \neq 0$. Equation (5.3) can hold with $q \neq 0$ only for $|y| > 1/2$, and it is easy to see that $|q| < (3/7)^{1/2}$.

In any case, the approach set at $(x, y)$ containing $(p, q)$ consists either of a half-line or of two half-lines; and a half-line is a convex set. If the set consists of a single half-line, (4.1) clearly holds. Otherwise, let $A$ consist of the multiples of $(p, q)$ and of $(p, -q)$. If $y > 0$, we define $A_1$ to be the multiples of $(p, -|q|)$ and $A_2$ to be the multiples of $(p, |q|)$; if $y < 0$, we interchange the definitions of $A_1$ and $A_2$. Then each vector in $A_1$ has the form $(p_1, q_1) = (k_1p, -k_1\text{ sgn } y|q|)$, $(k_1 > 0)$, and each vector in $A_2$ has the form $(p_2, q_2) = (k_2p, k_2\text{ sgn } y|q|)$, $(k_2 > 0)$. Hence

$$\Omega_F(x, y; p_1, q_1; p_2, q_2) = 2y\left\{q_1\left[4(p_1^2 + q_1^2)^{1/2} - (p_1^2 + 8q_1^2)^{1/2}\right] - q_2\left[4(p_2^2 + q_2^2)^{1/2} + (p_2^2 + 8q_2^2)^{1/2}\right]\right\}.$$
The factors in square brackets are positive, and \( yq_1 \) and \(-yq_2\) are negative (\( q_1 \) and \( q_2 \) are not zero, since by hypothesis \( A \) contains more than one unit vector). Hence the left member is negative, and (4.1) holds.

6. **Generalization to sets \( S \) with boundary points.** A slight generalization of Theorem 1 can be established at once. We need not assume that \( z_2 \neq z_1 \). The only use made of this hypothesis was to ensure that \( |z'_n(t)| \) had a positive lower bound. If \( z_2 = z_1 \), we distinguish two cases.

**Case I.** The \( |z'_n(t)| \) have a positive lower bound. In this case the preceding proof applies without change.

**Case II.** The \( \liminf |z'_n(t)| = 0 \). Since \( |z'_n(t)| = \mathcal{L}(\Pi_n) \) for almost all \( t \), this implies that for a subsequence \( \{\Pi_m\} \) of the minimizing sequence \( \{\Pi_n\} \) we have \( \mathcal{L}(\Pi_m) = 0 \). Then \( \mu = \lim \mathcal{J}(\Pi_n) = 0 \); so the degenerate curve consisting of the single point \( z_1 \) is the curve sought.

Less trivial is the generalization which allows us to study integrands not defined for all \( z \). Before stating this theorem it is desirable to introduce some definitions and establish a lemma.

\[(6.1) \text{ The function } F(z, z') \text{ is positive (negative) regular at the point } z_0 \text{ iff } u^\alpha F_{\alpha\beta}(z_0, p)u^\beta > 0 (-0) \text{ for all pairs } u, p \text{ of orthogonal unit vectors.}\]

\[(6.2) \text{ The function } F(z, z') \text{ is positive (negative) quasi-regular at } z_0 \text{ if } u^\alpha F_{\alpha\beta}(z_0, p)u^\beta = 0 (^0) \text{ for all pairs } u, p \text{ of orthogonal unit vectors.}\]

We use the abbreviations p.q.r., n.q.r. for positive quasi-regular, negative quasi-regular, respectively. It is well known that \( F(z, z') \) is p.q.r. (n.q.r.) at \( z_0 \) if and only if \( \mathcal{E}(z_0, p, r) \geq 0 (\leq 0) \) for all \( p \neq 0 \) and all \( r \). Although we do not use the concept in this note, we shall also make the following definition:

\[(6.3) \text{ The function } F(z, z') \text{ is p.q.r. normal (n.q.r. normal) at } z_0 \text{ if } \mathcal{E}(z_0, p, r) > 0 (<0) \text{ whenever } p \neq 0 \text{ and } r \neq kp, k \geq 0.\]

With this terminology we state the following lemma:

**Lemma 5.** If \( \phi_0'(t_0) \) and \( \phi'(t_0) \) are defined, and \( F(z, z') \) is p.q.r. at \( z_0(t_0) \), then \( \phi'(t_0) \leq \phi'_0(t_0) \).

Let \( \epsilon \) be an arbitrary positive number, and let \( F'(z, z') = F(z, z') + \epsilon |z'| \). By an elementary computation, we find that if \( u \) and \( p \) are orthogonal unit vectors, then

\[(6.4) u^\alpha F_{\alpha\beta}(z, p)u^\beta = u^\alpha F_{\alpha\beta}(z, p)u^\beta + \epsilon.\]

The first term on the right is nonnegative at \( z = z_0(t_0) \), by hypothesis. Hence the left-hand member is positive for all \( z \) of the set \( S \) in a neighborhood \( U \).
of $z_0(t_0)$. If $h$ is a sufficiently small positive number, less than $1 - t_0$, then for all large $n$ the arcs $z = z_n(t)$ and $z = z_0(t)$, $(t_0 \leq t \leq t_0 + h)$, are in $U$, and by a known semicontinuity theorem

$$ \lim \inf \int_{t_0}^{t_0+h} F^e(z_n, \dot{z}_n) \, dt \geq \int_{t_0}^{t_0+h} F^e(z_0, \dot{z}_0) \, dt. $$

Recalling the definition of $F^e$ and the inequality $|\dot{z}_n| \leq L$, we deduce that

$$ \lim \inf \left[ \phi_n(t_0 + h) - \phi_n(t_0) \right] + \epsilon h L \geq \phi_0(t_0 + h) - \phi_0(t_0). $$

That is,

$$ \phi(t_0 + h) - \phi(t_0) + \epsilon h L \geq \phi_0(t_0 + h) - \phi_0(t_0). $$

If we divide by $h$ and let $h$ tend to 0, we obtain

$$ \phi'(t_0) + \epsilon L \geq \phi_0'(t_0). $$

But $\epsilon$ is an arbitrary positive number; so (6.8) implies

$$ \phi'(t_0) \geq \phi_0'(t_0), $$

which was to be proved.

Our next theorem is the following:

**Theorem 2.** Let $S$ be a closed point set in $q$-dimensional space. If $F(z, \dot{z})$ satisfies condition (2.1), and condition (4.1) holds at every interior point of $S$, while $F(z, \dot{z})$ is positive quasi-regular at each boundary point of $S$, then in the class $K$ there is a curve which minimizes $\mathcal{J}(C)$.

Choose first a minimizing sequence $C_n^*$ of curves $z = z_n^*(t)$, $(0 \leq t \leq 1)$. There is a set of values of $t$ open relative to $[0, 1]$ for which $z_n^*(t)$ has distance greater than $1/n$ from the boundary of $S$. This open set consists of a finite or denumerable number of intervals, open relative to $[0, 1]$. Only a finite number $h_n$ of these define arcs $C_{n,j}$, $(j = 1, \ldots, h_n)$, of $C_n^*$ of length greater than $1/n$. These arcs we replace by polygonal arcs $\Pi_{n,j}^*$ having the same end points and such that $|\mathcal{J}(\Pi_{n,j}^*) - \mathcal{J}(C_{n,j})| < 1/n h_n$, $(j = 1, \ldots, h_n)$. If we denote by $C_n'$ the curve obtained by replacing the arcs $C_{n,j}$ by the polygonal arcs $\Pi_{n,j}^*$, then $|\mathcal{J}(C_n') - \mathcal{J}(C_n^*)| < 1/n$; so lim $\mathcal{J}(C_n') = \mu$. By a minor modification of Lemma 1, each arc $\Pi_{n,j}^*$ can be replaced by an arc $\Pi_{n,i}$ having a number $s_{n,i}$ of vertices which is not greater than the number of vertices of $\Pi_{n,j}^*$ and which minimizes $\mathcal{J}(C)$ in the class of polygons joining the ends of $\Pi_{n,j}^*$ and having not more than $s_{n,i}$ vertices. Let $C_n$ be the curve obtained by replacing the arcs $\Pi_{n,j}^*$ by the arcs $\Pi_{n,i}$. Then lim $\mathcal{J}(C_n) = \mu$.

† If $t_0 = 1$, we take $h$ negative; modifications are obvious.
Now let \( t_0 \) be a point of \( E \). If \( z_0(t_0) \) is interior to \( S \), there is a neighborhood of \( z_0(t_0) \) on which the \( C_n \) are polygonal if \( n \) is large. All the discussion leading up to equation (4.7) remains valid, for Lemmas 3 and 4 are purely local. So in this case (6.9) is valid. If \( z_0(t_0) \) is a boundary point of \( S \), then by Lemma 5 inequality (6.9) holds. Integrating, we obtain

\[
\mu = \phi(1) \geq \phi_0(1) = \mathcal{J}(C_0).
\]

But \( C_0 \) is in \( K \); so \( \mathcal{J}(C_0) \geq \mu \). This, with (6.10), proves \( \mathcal{J}(C_0) = \mu \), and the theorem is established.

7. Geometric interpretation of approach sets. For fixed \( z \), let us construct the graph in \( (u, \ p) \)-space of the function \( u = F(z, \ p) \). As is well known, this is a conical hypersurface with vertex at the origin. Let \( A \) be an approach set at \( z \). If \( p_1 \) and \( p_2 \) are both in \( A \), then

\[
(F_i(z, p_i) = F_i(z, p_2)), \quad i = 1, \ldots, q.
\]

But \( u = p \omega F_\omega(z, p) \) is the equation of the hyperplane tangent to \( u = F(z, p) \) at \( p_j \), \( (j = 1, 2) \). Therefore equation (7.1) shows that this same hyperplane is tangent to the surface \( u = F(z, p) \) at all points of \( A \).

Conversely, let \( u = l_ap^a \) be a hyperplane tangent to the hypersurface \( u = F(z, p) \) for all \( p \) in a set \( A \). Then if \( p \) is in \( A \), the partial derivatives \( F_i(z, p) \) must be the same as the partial derivatives \( l_i \) of the osculating function \( l_ap^a \). Therefore \( F_i(z, p) = l_i \) for all \( p \) in \( A \), and \( A \) is an approach set at \( z \).

If \( F(z, p) > 0 \) for \( |p| > 0 \), this interpretation can be formulated somewhat differently. In the preceding paragraphs set \( u = 1 \). The hyperplane \( u = 1 \) intersects \( u = F(z, p) \) in a \( (q-1) \)-dimensional hypersurface, and the hyperplane \( u = l_ap^a \) is tangent to \( F(z, p) = 1 \) if and only if \( 1 = l_ap^a \) is tangent to \( F(z, p) = 1 \).

Thus if \( q = 2 \) and \( F(z, p) > 0 \) for \( |p| > 0 \), we can find all approach sets at \( z \) by constructing the curve \( F(z, p) = 1 \) and finding all the points \( (p^1, p^2) \) at which an arbitrary line \( a_1p^1 + a_2p^2 + a_3 = 0 \) is tangent to the curve.

This interpretation suggests that it might be more descriptive to replace the name “approach set” by “isotangential set.”

8. \( \mathcal{E} \)-admissibility of approach sets; a more general existence theorem.

The geometric interpretation in §7 suggests the following line of reasoning. If \( z = z(t) \) minimizes \( \mathcal{J}(C) \), then for each \( t_0 \) the surface \( u = F(z(t_0), p) \) never sinks below the plane tangent to it at \( p = z'(t_0) \). The analytical statement is \( \mathcal{E}(z(t_0), z'(t_0), p) \geq 0 \) for all \( p \). The definition of approach sets \( A \) was suggested by the Weierstrass-Erdman corner condition. Might it not be true that only those approach sets \( A \) are of importance which have the property that the hypersurface \( u = F(z, p) \) never sinks below the hyperplane tangent to it at
the points \( p \) of \( A \)? We shall give this property of the set \( A \) the name "\( \mathcal{E} \)-admissibility." Stated analytically the definition is as follows:

(8.1) If \( A \) is an approach set at \( z \), it is \( \mathcal{E} \)-admissible if \( \mathcal{E}(z, p_0, p) \geq 0 \) for all \( p_0 \) in \( A \) and all \( p \).

It would make no difference here if we replaced the words "all \( p_0 \) in \( A \)" by the words "some one \( p_0 \) in \( A \)." For if \( p_0, p_1 \) are any two vectors in \( A \), then for all \( p \) the equation

\[
\mathcal{E}(z, p_0, p) = F(z, p) - p^a F_a(z, p_0) = F(z, p) - p^a F_a(z, p_1) = \mathcal{E}(z, p_1, p)
\]

holds.

The conjecture that approach sets which are not \( \mathcal{E} \)-admissible can be disregarded is indeed true, and we shall establish it in Theorem 3 below. However, the proof is rather complicated. It does not seem possible to rest upon the minimizing property of each \( \Pi_n \), as we have done before. For in any direct analogue of the proof of the Weierstrass condition polygons are introduced which have new vertices, and \( \Pi_n \) does not necessarily minimize \( \mathcal{J}(C) \) in the class of polygons with not more than \( s_n + 1 \) vertices. The proof which we shall give is therefore based on the property of the sequence as a whole that

\[
\lim_{m \to \infty} \mathcal{J}(\Pi_m) = \mu.
\]

**Theorem 3.** Theorems 1 and 2 remain valid if in hypothesis (4.1) the words "each approach set" are replaced by "each \( \mathcal{E} \)-admissible approach set."

In the proof of Theorems 1 and 2 the approach set \( A \) entered by way of Lemma 4, and then only in case there was no subsequence \( \{\Pi_k\} \) and \( \delta > 0 \) for which either (a) \( \lim_{h \to \infty} \Theta(h, t_0 - \delta, t_0 + \delta) > t_0 \) or (b) \( \lim_{h \to \infty} \Theta(h, t_0 - \delta, t_0 + \delta) \leq t_0 \). For if either (a) or (b) held, the proof was relatively simple and did not involve any approach set \( A \). We may suppose then that there is no subsequence \( \{\Pi_k\} \) and no \( \delta > 0 \) for which either (a) or (b) holds. If we can then show that the approach set \( A \) in Lemma 4 is necessarily \( \mathcal{E} \)-admissible, our proof is complete.

In Lemma 4 there is no loss of generality in assuming that the common value of the partial derivatives \( F_i(z_0(t_0), p) \) for all \( p \) in \( A \) is zero. For let us replace \( F(z, p) \) by \( F(z, p) - p^a F_a(z_0(t_0), p_0) \), where \( p_0 \) is in \( A \). All the curves \( C \) of the family \( K \) join \( z_1 \) to \( z_2 \); so \( \mathcal{J}(C) \) changes by \( (z_2^a - z_1^a) F_a(z_0(t_0), p_0) \), independent of \( C \). Therefore the minimizing properties of the \( \Pi_n \) are unchanged by the alteration of \( F(z, p) \). The statement that \( A \) is not \( \mathcal{E} \)-admissible assumes the form that there is a \( p_1 \) such that \( F(z_0(t_0), p_1) < 0 \). Because of the homogeneity of \( F \), we may suppose that \( p_1 \) is a unit vector, and we write

\[
F(z_0(t_0), p_1) = -\gamma < 0.
\]
By the continuity of $F$, there is an $\epsilon > 0$ such that

$$F(z, p_1) < -6\gamma$$

if $|z - z_0(t_0)| < 4\epsilon$. Furthermore, $F_i(z_0(t_0), p) = 0$ if $p$ is in $A$; hence we may suppose that $\epsilon$ has been chosen small enough so that the following statement holds:

(8.4) The vector $(F_1(z, p), \ldots, F_q(z, p))$ has length less than $\gamma$ if $|z - z_0(t_0)| < 4\epsilon$, $|z_2 - z_1| \leq |p| \leq 2L + 1$, and $p$ is in the $3\epsilon$-neighborhood of $A$.

Clearly we may suppose $\epsilon < \min \{1, |z_2 - z_1|\}$.

On the bounded closed set $|z - z_0(t_0)| \leq 1$, $|p| \leq 2L + 1$, the functions $F_i(z, p)$ are continuous; hence we have the following statement:

(8.5) There is an $M > 0$ such that the vector $(F_1(z, p), \ldots, F_q(z, p))$ has length less than $M$ if $|z - z_0(t_0)| \leq 1$ and $|p| \leq 2L + 1$.

Now from the subsequence $\{\Pi_m\}$ of Lemma 4 we choose a subsequence $\{\Pi_h\}$ with the following properties:

(8.6) $|z_h(t) - z_0(t)| < \epsilon$ for $0 \leq t \leq 1$ and all $h$.

(8.7) The limits $t_1 = \lim_{h \to \infty} \Theta(h, t_0 - \delta, t_0 + \delta)$ and $t_2 = \lim_{h \to \infty} \Theta(h, t_0 - \delta, t_0 + \delta)$ exist, where $\delta$ is prescribed by Lemma 4.

(As previously remarked, we need consider only the case $t_1 \leq t_0 < t_2$.)

(8.8) $\delta$ is chosen less than $\epsilon/L$.

(8.9) For all $h$ the following inequality holds:

$$\Theta(h, t_0 - \delta, t_0 + \delta) - \Theta(h, t_0 - \delta, t_0 + \delta) > (t_2 - t_1)/2.$$

Each of the conditions (8.6), (8.7), (8.9) is readily satisfied by appropriate choice of a subsequence; for (8.6) and (8.9) we need only reject a finite number of terms.

Let $d_{h,1}, \ldots, d_{h,n(h)}$ be a sequence of points with the following properties:

(8.10) $d_{h,1} < d_{h,2} < \cdots$.

(8.11) $|d_{h,i+1} - d_{h,i}| < \gamma/M$ for $j = 1, \cdots, n(h) - 1$.

(8.12) $d_{h,1} = \Theta(h, t_0 - \delta, t_0 + \delta)$, $d_{h,n(h)} = \Theta(h, t_0 - \delta, t_0 + \delta)$.

(8.13) Every $t$ between $\Theta(h, t_0 - \delta, t_0 + \delta)$ and $\Theta(h, t_0 - \delta, t_0 + \delta)$ which defines a vertex of $\Pi_h$ is included among the $d_{h,i}$.

(Thus $z_h(t)$ is linear on $d_{h,i} \leq t \leq d_{h,i+1}$.)

Let $m_{h,i}$ be the mid-point of the interval $[d_{h,i}, d_{h,i+1}]$.

On each interval $[d_{h,i}, d_{h,i+1}]$ we shall replace the line segment $z = z_h(t)$ belonging to $\Pi_h$ by a polygon of two sides having the same beginning and the same end as the line segment. To keep the notation from becoming too complicated, we temporarily replace the symbols $d_{h,i}, m_{h,i}, d_{h,i+1}, z_h$ by $d_1, m, d_2, z$, 

respectively. For \(0 \leq \tau \leq \epsilon\) we define \(\Pi(\tau)\) to be the two-sided polygon \(z = z(t, \tau)\), where

\[
\begin{align*}
z(t, \tau) &= z(d_1) + (t - d_1)\tau p_1, & d_1 \leq t < m, \\
z(t, \tau) &= z(2t - d_2) + (d_2 - t)\tau p_1, & m \leq t \leq d_2.
\end{align*}
\]

We readily verify that \(z(t, \tau)\) is continuous on \([d_1, d_2]\) and that

\[
z(d_1, \tau) = z(d_1), \quad z(d_2, \tau) = z(d_2).
\]

Also it is easy to show that

\[
\int_{d_1}^{d_2} F(z(t, 0), z'(t, 0)) dt = \int_{d_1}^{d_2} F(z(t), z'(t)) dt,
\]

where the prime denotes the derivative with respect to \(t\).

The polygon \(\Pi(\tau)\) may be regarded as arising from \(\Pi(0)\) by displacing the vertex \(z(m, 0)\) by the amount \((d_2 - d_1)\tau p_1/2\). Hence we may calculate \(\mathcal{J}'(\Pi(\tau))\) by applying (2.8) of I to the sides \([d_1, m]\), \([m, d_2]\) and adding. We obtain

\[
\mathcal{J}(\Pi(\tau)) = \frac{1}{2}(d_2 - d_1)F_a(z(\bar{t}, \tau), p_1)p_1^\alpha
\]

\[
- \frac{1}{2}(d_2 - d_1)F_a(z(\bar{t}, \tau), z'(\bar{t}, \tau))p_1^\alpha
\]

\[
+ \int_{d_1}^{d_2} F(z(t, \tau), z'(t, \tau)) dt,
\]

where \(d_1 < \bar{t} < m\) and \(m < \bar{t} < d_2\). Now we must prove the following statements:

\(8.18\) If \(d_1 \leq t \leq d_2\) and \(0 \leq \tau \leq \epsilon\), then \(|z(t, \tau) - z_0(t_0)| < 3\epsilon\).

\(8.19\) If \(m < t < d_2\) and \(0 \leq \tau \leq \epsilon\), then \(z'(t, \tau)\) is in the \(\epsilon\)-neighborhood of \(A\), and \(|z_2 - z_1| \leq \|z'(t, \tau)\| < 2L + 1\).

We establish (8.19) first. By (8.14), we have

\[
z'(t, \tau) = 2z'(2t - d_2) - \tau p_1.
\]

But by our choice of parameter \(t\), we have \(|z_2 - z_1| \leq z'(2t - d_2) \leq L\), while \(|\tau p_1| \leq \epsilon < \min \{|z_2 - z_1|, 1\}\). Hence \(|z_2 - z_1| < \|z'(t, \tau)\| < 2L + 1\). Also, by Lemma 4 the vector \(z'(2t - d_2)\) is in the \(\epsilon\)-neighborhood of \(A\); whence \(z'(t, \tau)\) is in the \(3\epsilon\)-neighborhood of \(A\). Thus (8.19) is established.

By (c) of Lemma 2, \(|z'(t)| \leq L\). Hence if \(t_0 - \delta \leq t \leq t_0 + \delta\), then (recalling (8.8)) we have

\[
|z(t) - z(t_0)| \leq L\delta < \epsilon.
\]

By (8.6), \(|z(t_0) - z_0(t_0)| < \epsilon\). Combining these inequalities, we find that both
\( z(d_1) \) and \( z(d_2) \) are in the \( 2\varepsilon \)-neighborhood of \( z_0(t_0) \). The distance from 
\( z(d_1) = z(d_1, \tau) \) to \( z(m, \tau) \) is \( |(m - d_1)\tau_1| < \varepsilon \). So all three vertices \( z(d_1, \tau), 
\( z(m, \tau) \), and \( z(d_2, \tau) \) are in the \( 3\varepsilon \)-neighborhood of \( z_0(t_0) \), and (8.18) is proved.

Thus the arguments in (8.17) satisfy the requirements laid down in (8.3),
(8.4), and (8.5), and we have from (8.17) and (8.11)
\[
\frac{d}{d\tau} \mathcal{F}(\Pi(\tau)) \leq -3\mathcal{J}(d_2 - d_1) + \frac{3}{2}\mathcal{J}(d_2 - d_1) + \int_{d_1}^{d_2} M |m - t| \, dt
\]
(8.20)
\[
< -2\mathcal{J}(d_2 - d_1) + M(d_2 - d_1)^2 < -\gamma(d_2 - d_1).
\]

If we integrate from \( \tau = 0 \) to \( \tau = \varepsilon \), we obtain (using (8.16))
\[
\mathcal{F}(\Pi(\varepsilon)) < \mathcal{F}(\Pi(0)) - \gamma\epsilon(d_2 - d_1) = \mathcal{F}(\Pi) - \gamma\epsilon(d_2 - d_1).
\]
(8.21)

In the foregoing paragraphs \( [d_1, d_2] \) was any one of the intervals
\( [d_{h,i}, d_{h,i+1}] \). Let the construction be carried out on each of these intervals.
The arc \( z = z_h(t), d_{h,i} \leq t \leq d_{h,n(h)} \) is thereby replaced by a polygonal arc
\( z = z_h(t, \varepsilon) \) having the same ends and satisfying the relation
\[
\int_{h,1}^{d_{h,n(h)}} F(z_h(t, \varepsilon), \hat{z}_h(t, \varepsilon)) \, dt < \int_{h,1}^{d_{h,n(h)}} F(z_h, \hat{z}_h) \, dt - \sum_{j=1}^{n(h)-1} \gamma\epsilon(d_{h,j+1} - d_{h,j}).
\]

If we extend the range of definition of \( z_h(t, \varepsilon) \) by setting it equal to \( z_h(t) \) for
\( 0 \leq t < d_{h,1} \) and for \( d_{h,n(h)} < t \leq 1 \), we obtain a polygon \( \Pi_h(\varepsilon) \) joining \( z_1 \) to \( z_2 \) and
satisfying (by (8.21) and (8.9))
\[
\mathcal{F}(\Pi_h(\varepsilon)) < \mathcal{F}(\Pi_h) - \sum_{j=1}^{n(h)-1} \gamma\epsilon(d_{h,j+1} - d_{h,j}) = \mathcal{F}(\Pi_h) - \gamma\epsilon(d_{h,n(h)} - d_{h,1})
\]
\[
< \mathcal{F}(\Pi_h) - \gamma\epsilon(t_2 - t_1)/2.
\]

Therefore
\[
\lim \sup \mathcal{F}(\Pi_h(\varepsilon)) \leq \lim \mathcal{F}(\Pi_h) - \gamma\epsilon(t_2 - t_1)/2 = \mu - \frac{1}{2}\gamma\epsilon(t_2 - t_1) < \mu.
\]

This is impossible by the definition of \( \mu \), and our theorem is established.

It is evident that the restriction \( z_2 \neq z_1 \) can be removed here just as it was
in \( \S 6 \). We thus arrive at the following theorem, which includes the three pre-
ceding ones as special cases:

**Theorem 4.** Let \( F(z, z') \) be defined and satisfy the usual continuity and
homogeneity conditions* for all \( z \) in a closed set \( S \) and all \( z' \). Let \( F(z, z') \) satisfy
the following conditions:
(i) At every boundary point \( z \) of \( S \), \( F(z, z') \) is positive quasi-regular.
(ii) At every interior point \( z \) of \( S \), every \( \varepsilon \)-admissible approach set \( A \) is

* As specified in I.
the sum of a finite number of convex sets $A_1, \ldots, A_k$ such that $\Omega(z, p_i, p_j) < 0$ if $p_i$ is in $A_i$, $p_j$ in $A_j$, $i < j$.

Let condition (2.1) be satisfied. Then for every two points $z_1, z_2$ of $S$ the class $K$ of all rectifiable curves lying in $S$ and joining $z_2$ to $z_1$ either is empty or contains a minimizing curve for $\mathcal{J}(C)$.

9. Geometric interpretation of $\mathcal{E}$-admissibility. The geometric interpretation of the property of $\mathcal{E}$-admissibility has already been given at the beginning of §8. If $A$ is an approach set at $z$, it is $\mathcal{E}$-admissible if and only if the surface $u = F(z, p)$ never sinks below the hyperplane tangent to it at the points $(u, p)$ with $p$ in $A$. The alternative geometric interpretation of approach sets also carries with it an interpretation of $\mathcal{E}$-admissibility. Let the hyperplane (in $p$-space) $a_x p + b = 0$ be tangent to the hypersurface $F(z, p) = 1$ at all the points $p$ of the surface which belong to $A$. This hyperplane divides $p$-space into two half-spaces. The half-space which contains the origin necessarily contains other points of $F(z, p) = 1$. The set $A$ is then $\mathcal{E}$-admissible if and only if the entire hypersurface $F = 1$ lies in this same half-space.

It follows that if we form the least convex body $Q$ containing $F(z, p) = 1$, then the points $p$ of the hypersurface which are not on the boundary of $Q$ cannot belong to any $\mathcal{E}$-admissible approach set. In particular, if $q = 2$, the points $p_0$ of $F(z, p) = 1$ on the boundary of $Q$ either lie in a line segment belonging to the boundary of $Q$ or they do not. In the second case the approach set $A$ containing $p_0$ consists of the half-line from the origin through $p_0$. In the first case, let $B$ be the set of all points $p$ at which the line segment touches $F(z, p) = 1$. The (maximal) approach set at $z$ containing $p_0$ then consists of all the half-lines from the origin through the points of $B$.

Consider, for example, the function $F(x, y, x', y')$ studied in §5. If $|y| \leq 1/2$, the curve $F = 1$ is convex. If $|y| > 1/2$, the curve is dumbbell-shaped, curved inwards near its intersections with the $x'$-axis. It is then evident from the graph that there are just two approach sets which consist of more than one half-line, and these are determined by the points of contact with the two lines $x' = \text{const.}$ tangent to the curve at the points at which $x'$ is respectively greatest and least. In §5 we showed rather more than this. We showed that these are the only approach sets other than half-lines; that besides these there are no other approach sets other than half-lines, whether $\mathcal{E}$-admissible or not. But Theorem 3 shows that the simpler conclusion reached here is sufficient to show the existence of minimizing curves for $\mathcal{J}(C)$.

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