SOME EXISTENCE THEOREMS IN THE CALCULUS OF VARIATIONS

IV. ISOPERIMETRIC PROBLEMS IN NON-PARAMETRIC FORM*

BY

E. J. McSHANE

1. **Heuristic outline of proof.** The preceding notes in this series have already appeared in these Transactions,† and will be cited as I, II, III, respectively.

Here we shall consider isoperimetric problems with one side integral, in the non-parametric form. A heuristic outline of the proof is as follows. We seek to minimize the integral

$$(1.1)\qquad \mathcal{J}[y] = \int_{x_0}^{X} f(x, y, \dot{y})dx$$

in the class K of a.c. curves $y=y(x)$ joining two fixed points (x_0, y_0) and (X, Y) and giving a fixed value γ to the integral

$$(1.2)\qquad \mathcal{G}[y] = \int_{x_0}^{X} g(x, y, \dot{y})dx.$$

Under suitable conditions, we can use as minimizing sequence a sequence of polygons Π_n: $y=y_n(x)$ such that $\mathcal{J}[y_n]$ tends to its lower bound μ on the class K and $\gamma_n=\mathcal{G}[y_n]$ tends to γ. Also we may assume that if s_n is the number of vertices of Π_n, then Π_n is a "best" polygon, in the sense that in the class of polygons of not more than s_n vertices joining (x_0, y_0) to (X, Y) and giving the value γ_n to $\mathcal{G}[y]$, the polygon Π_n minimizes $\mathcal{J}[y]$.

There are known conditions which must be satisfied by a curve $\bar{C}$: $y=\bar{y}(x)$, $(x_0 \leqq x \leqq X)$, which is of class D' and minimizes $\mathcal{J}(C)$ on the class K. For our purposes the statement of these conditions is more conveniently written by passing to the parametric notation, by equation (1.1) of I. That is, we write

---

* Presented to the Society, April 15, 1938; received by the editors October 29, 1937 and, in revised form, February 16, 1938.

† I. *The Dresden corner condition*, these Transactions, vol. 44 (1938), pp. 429–438; II. *Existence theorems for isoperimetric problems in the plane*, ibid., pp. 439–453; III. *Existence theorems for nonregular problems*, these Transactions, vol. 45 (1939), pp. 151–171.

$$F(z, z') = F(z^0, z^1, \cdots, z^q, z^{0\prime}, \cdots, z^{q\prime}) \\ = z^{0\prime} f(z^0, z^1, \cdots, z^q, z^{1\prime}/z^{0\prime}, \cdots, z^{q\prime}/z^{0\prime}) \tag{1.3}$$

for $z^{0\prime} > 0$. Correspondingly we write $\overline{C}$ in the form $z = \bar{z}(t)$, $(0 \leqq t \leqq 1)$. Then it is known that there are constants λ and λ_0, not both zero, such that the DuBois-Reymond relation holds:

$$H_i(\bar{z}(t), \bar{z}'(t)) = \int_0^t H_{z^i}(\bar{z}(t), \bar{z}'(t))dt + c_i, \qquad i = 0, 1, \cdots, q, \tag{1.4}$$

where $H = \lambda_0 F - \lambda G$. If we suppose that $\overline{C}$ is a normal curve, so that $\lambda_0 \neq 0$, we can suppose $\lambda_0 = 1$. Let $\bar{t}$ define a corner of $\overline{C}$, and write $\bar{p} = \bar{z}'(\bar{t} - 0)$, $\bar{r} = \bar{z}'(\bar{t} + 0)$. By the Dresden corner condition*

$$\Omega_H(\bar{z}(\bar{t}), \bar{p}, \bar{r}) \leqq 0. \tag{1.5}$$

Also, from (1.4) we derive the Weierstrass-Erdman corner condition:

$$H_i(\bar{z}(\bar{t}), \bar{p}) = H_i(\bar{z}(\bar{t}), \bar{r}), \qquad i = 0, 1, \cdots, q. \tag{1.6}$$

Of course we cannot expect the polygons Π_n to satisfy these conditions. Nevertheless, each Π_n does minimize $\mathcal{J}(C)$ in a certain class of polygons with $\mathcal{G}(C) = \text{const.}$, and it is therefore not unreasonable to hope that some approximate form of (1.4), (1.5), and (1.6) is satisfied by each Π_n.

Condition (1.6) can be written in a way which does not involve λ_0 and λ. We make the following definition:

(1.7) *A set A of (nonzero) vectors is an approach set at z if the matrix*

$$\Delta(z, p, r) \equiv \begin{pmatrix} F_i(z, p) - F_i(z, r) \\ G_i(z, p) - G_i(z, r) \end{pmatrix}, \qquad i = 0, 1, \cdots, q,$$

has rank less than 2 for each pair p, r of vectors belonging to A.

Then $\bar{p}$ and $\bar{r}$ belong to an approach set at $\bar{z}(\bar{t})$. For by (1.6) a linear combination of the rows (λ_0 times the first row minus λ times the second) vanishes. We shall shortly see that for a fairly large class of functions G each approach set A at z determines a pair of numbers $\lambda_0 = 1$ and λ for which (1.6) holds.

We now proceed to over-simplify the situation by proceeding as though conditions (1.4), (1.5), and (1.6) actually hold on each Π_n. Furthermore, we assume that at each point z each approach set contains at most two unit vectors p, r, and $\Omega_H(z, p, r) \neq 0$. We suppose that the sequence $z_n(t)$ of functions defining the polygons Π_n converges uniformly to a limit function $z_0(t)$;

* I, Theorem 2.

this is not hard to bring about. Consider a particular value t_0 of t and a subsequence $\{\Pi_m\}$ for which $z_m'(t_0)/|z_m'(t_0)|$ tends to a limit p. If $\{t_m\}$ is any sequence of points approaching t, then by (1.4) the difference

$$H_i(z_m(t_m), z_m'(t_m)) - H_i(z_m(t_0), z_m'(t_0))$$

tends to zero as $m \to \infty$, since it is the integral of a bounded function over an interval which is shrinking to zero. Hence as $m \to \infty$ the vector $z_m'(t_m)/|z_m'(t_m)|$, if it converges to any limit, converges to a limit r for which $H_i(z_0(t_0), r) = H_i(z_0(t_0), p)$, $(i = 0, 1, \cdots, q)$; that is, p and r belong to an approach set at $z_0(t_0)$. In other words, if we choose a small arc of Π_m corresponding to t near t_0, each side of Π_m has direction either near p or near r. To be specific, we assume $\Omega_H(z_0(t_0), p, r) < 0$. Then no side of Π_m with direction near r can be followed by one with direction near p. For if this happened, at such a corner $\bar{t}$ we would have $\Omega_H(z_m(\bar{t}), z_m'(\bar{t}-0), z_m'(\bar{t}+0))$ near $\Omega_H(z_0(t_0), r, p)$, which is positive; and this contradicts (1.5). Therefore the arc of Π_m near $z_0(t_0)$ consists of a succession of sides with directions near p followed by a succession of sides with directions near r. That is, the arc consists of two subarcs, each almost linear. This makes it highly plausible that (as would certainly be the case if both subarcs were actually linear) the integrals of F and G along this arc of Π_m tend, respectively, to the integrals of F and G along the arc of $z = z_0(t)$ to which they tend. If this could be applied to a succession of subarcs covering the whole range of t, it would follow that $\mathcal{J}(\Pi_n)$ and $\mathcal{G}(\Pi_n)$ tended, respectively, to $\mathcal{J}(C_0)$ and $\mathcal{G}(C_0)$, and C_0 would be the solution sought.

This outline is, of course, over-simplified, and in order to make the proof rigorous it is necessary to depart from it somewhat and add a mass of analytical details. Nevertheless the generalization (8.1) of the assumption that $\Omega_H(z, p, r) \neq 0$ if p, r satisfy (1.7) is the vital essence of the proof.

Although we are here concerned with integrals in non-parametric form, in the next paper we shall study integrals in parametric form by closely similar methods. Therefore the lemmas of this paper will be so arranged as to apply simultaneously to both forms of the problem. The chief differences are caused by the requirement $z^{0\prime} > 0$ which occurs in studying non-parametric problems. Consequently I shall use the device of enclosing certain statements in square brackets; these are needed if the lemma is understood as applying to non-parametric problems, but should be disregarded for problems in parametric form.

2. **A property of approach sets.** Let us suppose that we are given two integrands $F(z, z')$, $G(z, z')$ and two distinct fixed points z_1, z_2 in the space of points $(z^0, z^1, \cdots, z^q)$. We make the following definition:

(2.1) *K is the class of all rectifiable curves C joining* z_1 *to* z_2, *and* $K[\mathcal{G}=l]$ *is the subclass of K consisting of those curves C of K for which* $\mathcal{G}(C)=l$.

[In case F and G arise from a non-parametric problem, we must restrict our attention to curves representable in the form $z^i=z^i(z^0)$, $(i=1, \cdots, q)$. In this case we define K as follows:

(2.2) *K is the class of all curves having a.c. representations* $z^i=z^i(z^0)$, $(i=1, \cdots, q)$, *and joining two fixed points* z_1, z_2 *with* $z_1^0<z_2^0$. $K[\mathcal{G}=l]$ *is the subclass of K consisting of those curves C for which* $\mathcal{G}(C)=l$.

The points z_1, z_2 will also be denoted by (x_0, y_0), (X, Y), respectively.]

The problem considered is that of minimizing $\mathcal{J}(C)$ on the class $K[\mathcal{G}=l]$.

[We shall now make a hypothesis in the nature of a continuity requirement on F and G.

(2.3) *If, as* $n\to\infty$, *the points* $z_{i,n}$, $\bar{z}_{i,n}$, $\zeta_{i,n}$, *and* $\bar{\zeta}_{i,n}$ *all converge to a common limit* z, $(i=0, 1, \cdots, q)$, *the numbers* $\theta_{i,n}$ *and* $\bar{\theta}_{i,n}$ *tend to* 0, $(i=0, 1, \cdots, q)$, p_n *tends to a vector* p *with* $p^0>0$, *and* r_n *tends to a vector* r *with* $|r|>0$, *and if for each n the matrix*

$$\begin{pmatrix} F_i(\bar{z}_{i,n}, r_n) - F_i(z_{i,n}, p_n) + \theta_{i,n} \\ G_i(\bar{\zeta}_{i,n}, r_n) - G_i(\bar{\zeta}_{i,n}, p_n) + \bar{\theta}_{i,n} \end{pmatrix} \tag{2.4}$$

has rank less than 2, *then* $r^0>0$.

Then the following lemma is immediate:

LEMMA 1. *If* (2.3) *holds and the sequences* $z_{i,n}$, $\bar{z}_{i,n}$, $\zeta_{i,n}$, $\bar{\zeta}_{i,n}$, $\theta_{i,n}$, $\bar{\theta}_{i,n}$, p_n, r_n *satisfy the conditions of* (2.3), *then* $\Delta(z, p, r)$ (*of* (1.7)) *has rank less than* 2.

For then all the functions in the matrix (2.4) are continuous at (z, p, r), and the elements in the matrix tend, respectively, to the elements of the matrix $\Delta(z, p, r)$.]

For problems in parametric form the analogue of Lemma 1 is trivial. It is as follows:

LEMMA 1′. *If as* $n\to\infty$ *the points* $z_{i,n}$, $\bar{z}_{i,n}$, $\zeta_{i,n}$, *and* $\bar{\zeta}_{i,n}$ *all converge to a common limit* z, $(i=0, 1, \cdots, q)$, *the numbers* $\theta_{i,n}$ *and* $\bar{\theta}_{i,n}$ *tend to* 0, $(i=0, 1, \cdots, q)$, *and* p_n *and* r_n *tend to nonzero limit vectors* p, r, *and if for each n the matrix* (2.4) *has rank less than two, then the matrix* $\Delta(z, p, r)$ *has rank less than two.*

This is an immediate consequence of the continuity of all the elements of the matrix (2.4).

As is customary in isoperimetric problems, we are often interested in linear combinations of the integrands F and G. In order to have a notation for such combinations we adopt the definition: *For each number* λ,

$$H(z, z'; \lambda) \equiv F(z, z') - \lambda G(z, z'). \tag{2.5}$$

When there is no danger of misunderstanding, we write $H(z, z')$ in place of $H(z, z'; \lambda)$.

Let us make the definition:

(2.6) *$G(z, r)$ is positive quasi-regular normal at z_0 if $\mathcal{E}_G(z_0, p, r) > 0$ whenever $|p| > 0$ and $r \neq kp$, $(k \geqq 0)$ [and p^0 and r^0 are positive]; $G(z, r)$ is negative quasi-regular normal at z_0 if $\mathcal{E}_G(z_0, p, r) < 0$ whenever $|p| > 0$ and $r \neq kp$, $(k \geqq 0)$ [and p^0 and r^0 are positive]; $G(z, r)$ is quasi-regular normal at z_0 if it is positive or negative quasi-regular normal at z_0.*

Then we state the following interesting useful property of approach sets:

LEMMA 2. *If $G(z, r)$ is quasi-regular normal at z, and A is an approach set at z, there is a number $\lambda(z, A)$ such that the function $H(z, r) \equiv H(z, r; \lambda(z, A)) = F(z, r) - \lambda(z, A) G(z, r)$ satisfies the equations*

$$H_i(z, p) = H_i(z, r), \qquad i = 0, 1, \cdots, q, \tag{2.7}$$

for all p, r in A.

Since $H(z, r)$ is positively homogeneous of degree zero in r, it is enough to establish (2.7) for unit vectors in A. Moreover, it is enough to consider the case in which $G(z, r)$ is positive quasi-regular normal at z; if it is negative quasi-regular normal at z, we replace it by $-G(z, r)$.

Let p_1, p_2, p_3 be any three distinct unit vectors in A. The relationship

$$\begin{aligned} a_1[G_i(z, p_3) - G_i(z, p_2)] + a_2[G_i(z, p_1) - G_i(z, p_3)] \\ + a_3[G_i(z, p_2) - G_i(z, p_1)] = 0, \qquad i = 0, 1, \cdots, q, \end{aligned} \tag{2.8}$$

obviously holds if $a_1 = a_2 = a_3$. We shall show that it holds for no other a_1, a_2, a_3. Suppose (2.8) holds. By a cyclic interchange of the subscripts 1, 2, 3, followed, if necessary, by a change of sign of all three, we can bring about the relationship $a_1 \leqq a_2 \leqq a_3$. If we subtract a_2 from each of the numbers a_i, (2.8) remains valid and assumes the form

$$\begin{aligned} (a_1 - a_2)[G_i(z, p_3) - G_i(z, p_2)] \\ + (a_3 - a_2)[G_i(z, p_2) - G_i(z, p_1)] = 0, \qquad i = 0, \cdots, q. \end{aligned} \tag{2.9}$$

Now we multiply by $p_2{}^i$ and sum over $i = 0, \cdots, q$; this yields

$$-(a_1 - a_2)\mathcal{E}_G(z, p_3, p_2) + (a_3 - a_2)\mathcal{E}_G(z, p_1, p_2) = 0. \tag{2.10}$$

But $-(a_1 - a_2)$ and $(a_3 - a_2)$ are nonnegative, and their coefficients are positive; so this is only possible if $a_1 - a_2 = a_3 - a_2 = 0$.

Given any two distinct unit vectors p, r of A, the matrix $\Delta(z, p, r)$ is (by definition of A) of rank 0 or 1. Since its second row is not 0, its first row is a number $\lambda(p, r)$ times the second. From the form of Δ it is clear that $\lambda(p, r) = \lambda(r, p)$. We must show that $\lambda(p, r)$ has the same value for all p, r in A. If A contains only two distinct unit vectors p, r, this is at once evident from the equation $\lambda(p, r) = \lambda(r, p)$. It remains to establish the formula in case A contains more than two distinct unit vectors. To do this it is sufficient to show (with above notation) that $\lambda(p_1, p_3) = \lambda(p_1, p_2)$. For then if we fix p_1 and $p_2 \neq p_1$, $\lambda(p_1, p_3)$ has the same value for all unit vectors p_3 in A. That is, $\lambda(p_1, r)$ is independent of its second argument. Since $\lambda(p, r) = \lambda(r, p)$, it is also independent of its first argument, and has only one value.

Suppose then that p_1, p_2, and p_3 are distinct unit vectors in A. By definition of $\lambda(p, r)$ we have

$$
\begin{aligned}
F_i(z, p_3) - F_i(z, p_2) &= \lambda(p_3, p_2)[G_i(z, p_3) - G_i(z, p_2)],\\
F_i(z, p_2) - F_i(z, p_1) &= \lambda(p_1, p_2)[G_i(z, p_2) - G_i(z, p_1)],\\
F_i(z, p_1) - F_i(z, p_3) &= \lambda(p_1, p_3)[G_i(z, p_1) - G_i(z, p_3)].
\end{aligned}
$$

Adding, we obtain (2.8) with $a_1 = \lambda(p_3, p_2)$, $a_2 = \lambda(p_1, p_3)$, $a_3 = \lambda(p_1, p_2)$. Hence by the preceding proof we know that $\lambda(p_1, p_3) = \lambda(p_1, p_2)$; so $\lambda(p, r)$ has but one value, which we denote by $\lambda(z, A)$.

Thus (2.7) holds if p, r are distinct unit vectors in A. Obviously it holds if $p = r$; and the proof is complete.

Under the hypotheses of Lemma 2, the set A is an approach set for $H(z, r)$ at the point z as defined in III, for equations (2.7) are exactly the equations (1.5) used there to define an approach set. Therefore we have available a geometric interpretation of approach sets. When $G(z, r)$ is quasi-regular normal at z and A is an approach set at z, then there is a single hyperplane $u = l_\alpha r^\alpha$ tangent to the hypersurface $u = F(z, r) - \lambda(z, A)G(z, r)$ at all points (u, r) with r in A. Conversely, let λ be a number, and let A be the set of (nonzero) vectors r such that a given hyperplane $u = l_\alpha r^\alpha$ is tangent to $u = H(z, r; \lambda)$ whenever r is in A. Then as in §7 of III, A is an approach set for $H(z, r; \lambda)$, and equations (2.7) hold. But equations (2.7) imply that the rank of $\Delta(z, p, r)$ is less than two; so A is an approach set at z as defined in (1.7).

This interpretation shows that any approach set A containing two or more distinct unit vectors has a unique maximal extension, provided that $G(z, r)$ is quasi-regular normal at z. For then any two unit vectors of A determine $\lambda(z, A)$, and hence determine $H(z, r)$. Let $u = l_\alpha r^\alpha$ be the hyperplane tangent to $u = H(z, r)$ whenever r is in A. Let A^* be the set of all nonzero vectors r for which $u=l_\alpha r^\alpha$ is tangent to $u=H(z, r)$. Then A^* is an approach set and contains A, and is the largest approach set which contains A.

In view of this lemma it might possibly be more desirable, when G is not quasi-regular normal at z to define an approach set at z to be a set A such that there exist numbers λ_0, λ, not both zero, for which (if we write $H(z, r)=\lambda_0 F(z, r)-\lambda G(z, r)$)

$$H_i(z, p) = H_i(z, r), \qquad i = 0, 1, \cdots, q,$$

for all p, r in A. The interest in this remark is rather slight, since the methods of proof which are the central feature of this paper seem not to apply unless $G(z, r)$ is quasi-regular normal.

3. **Existence of minimizing polygons.** We now add some further hypotheses concerning the integrands. These added requirements are much more stringent for problems in non-parametric form than for those in parametric form.

[(3.1) *For every bounded set* S_0 *there exist positive numbers* δ *and* a *and a number* $b \geqq 0$ *such that*

$$| F_{z^i}(z, z') | \leqq a | z' | + bF(z_0, z')$$

wherever z_0 *is in* S_0 *and* $|z-z_0| < \delta$, *and* $z^{0\prime} > 0$; *and* G *also satisfies this condition.*

(3.2) *On every bounded set* S_0 *the relations*

$$\lim_{|y'|\to\infty} f(x, y, y')/| y' | = \infty, \qquad \lim_{|y'|\to\infty} g(x, y, y')/f(x, y, y') = 0$$

hold uniformly in (x, y).

(3.3) *The points* $z_1=(x_0, y_0)$ *and* $z_2=(X, Y)$ *and the integrals* $\mathcal{J}$, $\mathcal{G}$ *have the property that there are constants* $a_0 \geqq 0$ *and* a_1 *such that for every number* H *there is a bounded set* S_H *containing all the a.c. curves* C: $y=y(x)$ *joining* z_1 *and* z_2 *and having* $a_0\mathcal{J}(C)+a_1\mathcal{G}(C) < H$.]

For problems in parametric form the role of these hypotheses is filled by the following assumption:

(3.4) *The points* z_1, z_2 *and the integrals* $\mathcal{J}$, $\mathcal{G}$ *have the property that if* K *is a class of rectifiable curves joining* z_1 *and* z_2 *for which* $\mathcal{J}(C)$ *is bounded above and* $\mathcal{G}(C)$ *is bounded, then the curves of* K *have uniformly bounded lengths.*

Then we have the following lemma:

LEMMA 3.† *Let* ν *be an integer and* l *a number. If the subclass of* $K[\mathcal{G}=l]$, *consisting of polygons of not more than* ν *vertices, is not empty, it contains a polygon which minimizes* $\mathcal{J}(C)$ *on that class of polygons.*

† For problems in parametric form this is a trivial extension of Lemma 1 of III.

[Let $y=y_n(x)$, $(x_0\leqq x\leqq X)$, be a sequence of polygons of not more than ν vertices such that $\mathcal{G}[y_n]=l$ and $\mathcal{J}[y_n]$ tends to its lower bound μ. By Lemma 1 of II, all the curves $y=y_n(x)$ are in a circle Q. By Lemma 4 of II, the functions $y_n(x)$ are equi-absolutely continuous; so by Ascoli's theorem there is a subsequence (which we may suppose to be the original sequence) converging uniformly to a limit function $y=y_0(x)$. We see readily that this limit curve is also a polygon of not more than ν vertices, and that $y_n'(x)\rightarrow y_0'(x)$ except at the vertices of $y=y_0(x)$. So by Lemma 5 of II we have $\mathcal{G}[y_0]=l$ and $\mathcal{J}[y_0]\leqq\lim \mathcal{J}[y_n]=\mu$. But by the definition of μ we cannot have $\mathcal{J}[y_0]<\mu$; so $\mathcal{J}[y_0]=\mu$, and $y=y_0(x)$ is the polygon sought.]

4. **Choice of a particular minimizing sequence.** Continuing with the hypotheses of the preceding section, we assume that l is a number such that $K[\mathcal{G}=l]$ is not empty, and we denote by μ the greatest lower bound of $\mathcal{J}(C)$ on the class $K[\mathcal{G}=l]$. There is [by Lemma 2 of II] a sequence of polygons $\overline{\Pi}_n$ [of the form $y=\bar{y}_n(x)$] joining (x_0, y_0) to (X, Y) and such that $\lim \mathcal{J}(\overline{\Pi}_n)=\mu$, $\lim \mathcal{G}(\overline{\Pi}_n)=l$. Let ν_n be the number of vertices of $\overline{\Pi}_n$. By Lemma 3 of the preceding section, there is, for each n, a polygon Π_n [of the form $y-y_n(x)$] in K such that

$$(4.1)\qquad \mathcal{G}(\Pi_n) = \mathcal{G}(\overline{\Pi}_n)\rightarrow l,$$

while Π_n minimizes $\mathcal{J}(C)$ on the class of polygons in K having not more than ν_n vertices and satisfying the equation $\mathcal{G}(C)=\mathcal{G}(\overline{\Pi}_n)$. Hence

$$(4.2)\qquad \mathcal{J}(\Pi_n) \leqq \mathcal{J}(\overline{\Pi}_n).$$

From this sequence we now select a subsequence $\overline{\Pi}_{n_i}$ such that

$$(4.3)\qquad \lim_{i\rightarrow\infty} \mathcal{J}(\Pi_{n_i}) = \liminf_{n\rightarrow\infty} \mathcal{J}(\Pi_n) \leqq \lim_{n\rightarrow\infty} \mathcal{J}(\overline{\Pi}_n) = \mu.$$

We may assume that this subsequence is the whole sequence, so that

$$(4.4)\qquad \lim \mathcal{J}(\Pi_n) \text{ exists},\quad \lim \mathcal{J}(\Pi_n) \leqq \mu.$$

By Lemma* 1 of II the polygons Π_n are all in a circle Q. [So by Lemma 4 of II the functions $y_n(x)$ are equi-absolutely continuous, and we can select a subsequence (which we suppose to be the whole sequence) converging uniformly to a limit function $y_0(x)$:

$$(4.5)\qquad \lim_{n\rightarrow\infty} y_n(x) = y_0(x) \quad \text{uniformly for} \quad x_0 \leqq x \leqq X.$$

This selection of a subsequence leaves (4.4) still valid.

* Although this lemma was stated for non-parametric problems, its extension to problems in parametric form is trivial.

Hypothesis (3.2) implies that $f(x, y, y')$ is bounded below for (x, y) in Q, say $f(x, y, y') \geqq \kappa$. Since addition of a constant to f merely changes $\mathcal{J}[y]$ by the same number for all curves $y=y(x)$ of K and leaves all minima unaffected, we may assume without loss of generality that for all (x, y) in Q and all y'

$$f(x, y, y') \geqq 0. \tag{4.6}$$

Now we return to the parametric notation.] We have a sequence of polygons $\{\Pi_n\}$ [converging to a limit curve C_0] in K, and

$$\lim \mathcal{G}(\Pi_n) = l, \qquad \bar{\mu} = \lim \mathcal{J}(\Pi_n) \leqq \mu. \tag{4.7}$$

Also, the lengths of the Π_n are bounded (by (3.4) if the problem is in parametric form, by Lemma 4 of II otherwise), and are not less than $|z_1-z_2|$:

$$|z_1 - z_2| \leqq \mathcal{L}(\Pi_n) \leqq L. \tag{4.8}$$

On each Π_n we now introduce the parameter $t=s/\mathcal{L}(\Pi_n)$, $(0\leqq s\leqq\mathcal{L}(\Pi_n))$. Thus Π_n has the representation

$$\Pi_n: \ z = z_n(t), \qquad 0 \leqq t \leqq 1,$$

with

$$0 < |z_1 - z_2| \leqq |\dot{z}_n(t)| \leqq \mathcal{L}(\Pi_n) \leqq L \tag{4.9}$$

except at vertices of Π_n.

Since the functions z_n all satisfy the same Lipschitz condition, there is a subsequence (we suppose it the whole sequence) which converges uniformly to a limit function $z_0(t)$ on $[0, 1]$:

$$\lim_{n\to\infty} z_n(t) = z_0(t) \quad \text{uniformly on} \quad [0, 1]. \tag{4.10}$$

The curve $z=z_0(t)$ is therefore a limit curve* of the Π_n. [But these have the unique limit C_0; so $z=z_0(t)$, $(0\leqq t\leqq 1)$, is a representation of C_0.]

Now we define

$$\phi_n(t) = \int_0^t F(z_n(t), \dot{z}_n(t))dt, \qquad n = 0, 1, \cdots. \tag{4.11}$$

[By (4.6), these functions are monotonic increasing; and they are bounded, for

$$0 = \phi_n(0) \leqq \phi_n(t) \leqq \phi_n(1) = \int_0^1 F(z_n, z_n')dt \to \bar{\mu} \leqq \mu,$$

* If the reader is omitting bracketed statements, we here define C_0 to be the curve $z=z_0(t)$, $(0\leqq t\leqq 1)$.

by (4.7).] So [by Helly's theorem] we can select a subsequence of $\{\phi_n\}$ (we suppose it the whole sequence) which converges* for all t in $[0, 1]$ to a limit function $\phi(t)$:

$$\lim_{n\to\infty} \phi_n(t) = \phi(t), \qquad 0 \leqq t \leqq 1. \tag{4.12}$$

For this subsequence, (4.7) and (4.10) remain valid. By Lemma 3 of II, the functions

$$\gamma_n(t) \equiv \int_0^t G(z_n(t), \dot z_n(t))dt, \qquad 0 \leqq t \leqq 1, \ n = 0, 1, \cdots, \tag{4.13}$$

are equi-absolutely continuous. Hence we can select a subsequence (we suppose it to be the whole sequence) which converges uniformly on $[0, 1]$ to a limit function $\gamma(t)$:

$$\lim_{n\to\infty} \gamma_n(t) = \gamma(t) \quad \text{uniformly on} \quad [0, 1]. \tag{4.14}$$

For this subsequence (4.7), (4.10), and (4.12) remain valid.

Our task is now to prove

$$\gamma(1) = \gamma_0(1), \qquad \phi(1) \geqq \phi_0(1). \tag{4.15}$$

For these are the same as

$$\lim \mathcal{G}(\Pi_n) = \mathcal{G}(C_0), \qquad \lim \mathcal{J}(\Pi_n) \geqq \mathcal{J}(C_0). \tag{4.16}$$

If we can establish them, then the first equation shows that C_0 is in $K[\mathcal{G}=l]$, by (4.1). Hence $\mathcal{J}(C_0) \geqq \mu$. But the second part of (4.16), with (4.7), gives $\mathcal{J}(C_0) \leqq \mu$; hence $\mathcal{J}(C_0) = \mu$, and $y = y_0(x)$ is the curve sought.

It is worth noticing that all the properties of the sequence which have been established here are also possessed by every subsequence of this sequence.

5. **Convergence of directions at one point.** We observe that, $\phi(t)$, $\phi_0(t)$, [being monotonic increasing, and] $\gamma(t)$, $\gamma_0(t)$, being a.c., there is a set E of measure 1 contained in the open interval $0 < t < 1$ and having the following property:

(5.1) *The derivatives $\phi'(t)$, $\phi_0'(t)$, $\gamma'(t)$, and $\gamma_0'(t)$ exist and are finite for all t in E.*

We now wish to prove the following lemma:

---

* For problems in parametric form we note that the integrands in (4.11) are uniformly bounded. So the $\phi_n(t)$ all satisfy the same Lipschitz condition, and a subsequence can be selected which converges uniformly to a limit $\phi_0(t)$. A like remark applies to (4.14).

LEMMA 4. *If t_1 belongs to E, there is a subsequence $\{\Pi_m\}$, $(m=n_1, n_2, \cdots)$, of $\{\Pi_n\}$ and a sequence $\{t_m\}$ of points of $[0, 1]$ such that $\lim_{m\to\infty} t_m = t_1$, and $z_m'(t_m)$ converges* to a limit p [and $p^0>0$].*

[Let ϵ_i be a sequence of positive numbers less than $1-t_1$. The numbers $[\phi(t_1+\epsilon_i)-\phi(t_1)]/\epsilon_i$ tend to $\phi'(t_1)$, which is finite. Hence there is an h such that

$$|\phi(t_1+\epsilon_i)-\phi(t_1)|/\epsilon_i < h.$$

Therefore, by (4.14), the inequality

$$(5.2)\qquad |\phi_n(t_1+\epsilon_i)-\phi_n(t_1)|/\epsilon_i < h+1$$

holds for all n greater than a certain $n(i)$. We choose a sequence $\{n_i\}$ such that $n_1<n_2<\cdots$ and $n_i>n(i)$. Since ϕ_n is a.c. and (by 4.6) monotonic increasing, from (5.2) we see that there is a set of positive measure in $[t_1, t_1+\epsilon_i]$ on which $|\phi_{n_i}'(t)|<h+1$. In this set we choose a t_{n_i} which does not define a vertex of Π_{n_i}; and we use the letter m to replace n_i. Then $z_m'(t_m)$ is defined, and

$$F(z_m(t_m), z_m'(t_m)) = \phi_m'(t_m) < h+1.$$

Therefore by (4.9)

$$(5.3)\quad F(z_m(t_m), z_m'(t_m)/|z_m'(t_m)|) < (h+1)/|z_m'(t_m)| \leqq (h+1)/|z_1-z_2|.$$

Hypothesis (3.2), in its parametric form, states that there is a $\delta>0$ such that

$$(5.4)\qquad F(z, z') > \frac{h+1}{|z_1-z_2|}, \qquad z \text{ in } Q, \quad |z'|=1,\ z^{0\prime}<\delta.$$

Therefore from (5.3) we conclude

$$(5.5)\qquad z_m^{0\prime}(t_m) \geqq \delta.$$

From the bounded sequence $\{z_m'(t_m)\}$ we now choose a subsequence converging to a limit p; we may suppose this subsequence to be the whole sequence. Then, by (5.5), $p^0 \geqq \delta$, and the lemma is established.]

6. **The directions of the sides of the polygons.** In our sketch of the proof, in the introduction, the two essentials were first to show that the directions of the sides of the Π_n were, on short arcs, near approach sets, and second to establish an order relation between the sides. The next lemma treats the first of these two needs.

---

* For problems in parametric form this is an immediate consequence of the Bolzano-Weierstrass theorem, since the lengths $|z_m'(t_m)|$ are bounded.

LEMMA 5. *Let* $\{\Pi_n\}$ *be the sequence of polygons specified in* §4, *and let* t_0 *be a point of* E. *Then either*

(i) *there is a subsequence* $\{\Pi_m\}$ *of* $\{\Pi_n\}$ *and a sequence of numbers* $t_m \to t_0$ *such that each* t_m *is contained in an interval* $h_m \leqq t_m \leqq k_m$ *on which* $z_m(t)$ *is linear, and* $k_m - h_m$ *does not tend to* 0, *or*

(ii) *there is an approach set* A *at* $z_0(t_0)$ *and a subsequence* $\{\Pi_m\}$ *such that if* $t_m \to t_0$ *and* $z_m(t_m)$ *is not a vertex* of* Π_m, *then all accumulation points of the* $z_m'(t_m)$ *belong to* A.

Suppose that (i) does not hold. Let $\{t_n\}$ be a sequence tending to t_0, and let h_n, k_n define the initial and final points of the side of Π_n to which t_n belongs. Then $h_n \to t_0$ and $k_n \to t_0$. For otherwise either $t_n - h_n$ or $k_n - t_n$ would not tend to 0, and $z_n(t)$ would be linear on the intervals (h_n, t_n) and (t_n, k_n).

By Lemma 4, there is a subsequence $\{\Pi_h\}$ of $\{\Pi_n\}$ and a sequence of numbers t_h^1 such that $t_h^1 \to t_0$ and $z_h'(t_h^1)$ tends to a limit p_1 [with $p_1^0 > 0$]. If for every sequence $t_h \to t_0$ the relation $z_h'(t_h) \to p_1$ holds, then case (ii) holds. Otherwise, for some sequence $t_h^2 \to t_0$ the vectors $z_h'(t_h^2)$ have an accumulation point $p_2 \neq p_1$. For a subsequence $\{t_w^2\}$ we have $z_w'(t_w^2) \to p_2$, while it is still true that $z_w'(t_w^1) \to p_1$.

To be specific, we suppose $t_w^1 < t_w^2$ for all w. Let $v_0, v_1, \cdots, v_{s+1}$ be values of t which define successive vertices of Π_w and satisfy $v_0 < t_w^1 < v_1 < \cdots < v_s < t_w^2 < v_{s+1}$. By the first paragraph, v_0 and v_{s+1} tend to t_0 as $w \to \infty$. If we displace the vertices $z_w(v_1), \cdots, z_w(v_s)$ all by the same amount $(0, 0, \cdots, 0, \tau, 0, \cdots, 0)$, where τ is in the $(j+1)$st place, we obtain a polygon $\Pi_w^j(\tau)$. [For small τ this can be represented in the form $z^i = z^i(z^0)$, $(i = 1, \cdots, n)$.] From formula (2.8) of I we have

$$
\begin{aligned}
\frac{d}{d\tau} \mathcal{J}(\Pi_w^j(\tau)) \Big|_{\tau=0} &= F_j(z_w(t), z_w'(t)) - F_j(z_w(\bar{t}), z_w'(\bar{t})) \\
&\quad + \int_{v_0}^{v_1} \frac{t - v_0}{v_1 - v_0} F_{z^j}(z_w, z_w') dt \\
&\quad + \int_{v_s}^{v_{s+1}} \frac{t - v_{s+1}}{v_s - v_{s+1}} F_{z^j}(z_w, z_w') dt \\
&\quad + \int_{v_1}^{v_s} F_{z^j}(z_w, z_w') dt,
\end{aligned} \tag{6.1}
$$

where $v_0 < \bar{t} < v_1$ and $v_s < \bar{\bar{t}} < v_{s+1}$.

Let us denote the sum of the last three terms in (5.6) by θ_j. [By (3.1) and (4.9) the inequality

* This requirement is not essential.

$$(6.2)\qquad \begin{aligned} |\theta_j| &\leqq \int_{v_0}^{v_{s+1}} \{a|\dot{z}_w| + bF(z_w, \dot{z}_w)\}dt \\ &\leqq aL|v_{s+1} - v_0| + b\{\phi_w(v_{s+1}) - \phi_w(v_0)\} \end{aligned}$$

holds.*] Also, on the intervals (v_0, v_1) and (v_s, v_{s+1}) the derivative $z_w'(t)$ is constant. Hence (5.1) can be written

$$(6.3)\qquad D_j \equiv \frac{d}{dt}\mathcal{J}(\overset{j}{\Pi}_w(\tau))\Big|_{\tau=0} = F_j(z_w(\bar{\bar{t}}), z_w'(t_w^2)) - F_j(z_w(\bar{t}), z_w'(t_w^1)) + \theta_j.$$

A similar result holds for $\mathcal{G}(C)$:

$$(6.4)\qquad \begin{aligned} \bar{D}_j \equiv \frac{d}{d\tau}\mathcal{G}(\overset{j}{\Pi}_w(\tau))\Big|_{\tau=0} &= G_j(z_w(\bar{\bar{t}}), z_w'(t_w^2)) - G_j(z_w(\bar{t}), z_w'(t_w^1)) + \bar{\theta}_j, \\ & v_0 < \bar{t} < v_1,\ v_s < \bar{\bar{t}} < v_{s+1}, \end{aligned}$$

where for non-parametric problems

$$(6.5)\qquad |\bar{\theta}_j| \leqq \bar{a}L(v_{s+1} - v_0) + \bar{b}\{\gamma_w(v_{s+1}) - \gamma_w(v_0)\},$$

while for parametric problems

$$(6.5')\qquad |\bar{\theta}_j| \leqq \overline{M}(v_{s+1} - v_0).$$

Now we show that the matrix

$$(6.6)\qquad \begin{pmatrix} D_0 & D_1 \cdots & D_q \\ \bar{D}_0 & \bar{D}_1 \cdots & \bar{D}_q \end{pmatrix}$$

is of rank less than 2. Otherwise, suppose, say, that the determinant

$$(6.7)\qquad \begin{vmatrix} D_j & D_k \\ \bar{D}_j & \bar{D}_k \end{vmatrix}$$

is not 0. We form the polygon $\Pi_w(\tau, \sigma)$ by displacing each of the vertices $z_w(v_1), \cdots, z_w(v_s)$ by the amount $(0, 0, \cdots, \tau, 0, \cdots, 0, \sigma, \cdots, 0)$, where τ is in the $(j+1)$st place and σ in the $(k+1)$st. Then the jacobian of the functions $\mathcal{J}(\Pi_w(\tau, \sigma))$, $\mathcal{G}(\Pi_w(\tau, \sigma))$ at $\tau=\sigma=0$ is exactly the determinant (6.7). The equations

$$(6.8)\qquad \begin{aligned} \mathcal{J}(\Pi_w(\tau, \sigma)) + u - \mathcal{J}(\Pi_w) &= 0, \\ \mathcal{G}(\Pi_w(\tau, \sigma)) - \mathcal{G}(\Pi_w) &= 0 \end{aligned}$$

have the initial solutions $\tau=\sigma=u=0$. By the implicit functions theorem, we can solve for τ, σ as functions of u, defined for all u near 0. However, if $u>0$

* For parametric problems this is replaced by the inequality (6.2′) $|\theta_j| \leqq M(v_{s+1} - v_0)$, where M is an upper bound for $|F_{z^j}(z, z')|$ for z near C_0 and $|z'| \leqq L$.

these functions would define values of τ, σ for which (6.8) held; that is,

$$\begin{aligned} \mathcal{G}(\Pi_w(\tau, \sigma)) &= \mathcal{G}(\Pi_w), \\ \mathcal{J}(\Pi_w(\tau, \sigma)) &< \mathcal{J}(\Pi_w). \end{aligned} \tag{6.9}$$

This is incompatible with the minimizing property of the polygons Π_w. Hence the matrix (6.6) is of rank less than two.

We now write (6.6) in the form

$$\begin{pmatrix} F_j(z_w(\tilde t_w), z_w'(t_w^2)) - F_j(z_w(\bar t_w), z_w'(t_w^1)) + \theta_j \\ G_j(z_w(\tilde{\bar t}_w), z_w'(t_w^2)) - G_j(z_w(\bar{\bar t}_w), z_w'(t_w^1)) + \bar\theta_j \end{pmatrix}. \tag{6.10}$$

For parametric problems inequalities (6.2′) and (6.5′) make it evident that θ_j and $\bar\theta_j$ approach 0 with $v_{s+1}-v_0$. But v_{s+1} and v_0 approach t_0 as $w\to\infty$; so θ_j and $\bar\theta_j$ approach 0 as $w\to\infty$. [For problems in non-parametric form we use (6.2) and (6.5). Since $\phi'(t_0)$ exists, and $\phi(t)$ is monotonic increasing, for every positive ϵ there is a $\delta>0$ such that

$$\phi(t_0) - \epsilon < \phi(t_0 - \delta) \leqq \phi(t_0) \leqq \phi(t_0 + \delta) < \phi(t_0) + \epsilon; \tag{6.11}$$

and if w is large enough,

$$|\phi_w(t_0 - \delta) - \phi(t_0 - \delta)| < \epsilon, \qquad |\phi_w(t_0 + \delta) - \phi(t_0 + \delta)| < \epsilon.$$

Hence if

$$t_0 - \delta < t < t_0 + \delta, \tag{6.12}$$

then

$$\phi(t_0) - 2\epsilon < \phi_w(t_0 - \delta) \leqq \phi_w(t) \leqq \phi_w(t_0 + \delta) < \phi(t_0) + 2\epsilon. \tag{6.13}$$

For all large w the points v_0 and v_{s+1} both satisfy (6.12), since they both approach t_0 as $w\to\infty$; so by (6.13) we have

$$|\phi_w(v_{s+1}) - \phi_w(v_0)| < 4\epsilon. \tag{6.14}$$

Thus $\phi_w(v_{s+1})-\phi_w(v_0)$ approaches 0 as $w\to\infty$. The other term $aL(v_{s+1}-v_0)$ also tends to 0, as previously observed; so θ_j tends to zero with $1/w$. The proof that $\bar\theta_j\to 0$ is easier. The functions $\gamma_w(t)$ are equi-absolutely continuous; so when $w\to\infty$ (and consequently $v_{s+1}-v_0\to 0$) the term $\gamma_w(v_{s+1})-\gamma_w(v_0)$ tends to 0. The other term in (5.10) also tends to 0; hence $\bar\theta_j$ approaches 0 as $w\to\infty$.]

Since $z_w(t)$ tends uniformly to $z_0(t)$, and the numbers $\tilde t_w$, $\tilde{\bar t}_w$, $\bar t_w$, and $\bar{\bar t}_w$ all tend to t_0, we have

$$\lim z_w(\tilde t_w) = \lim z_w(\tilde{\bar t}_w) = \lim z_w(\bar t_w) = \lim z_w(\bar{\bar t}_w) = z_0(t_0). \tag{6.15}$$

Finally, we already know that $\lim z_w'(t_w^1) = p_1$ [where $p_1^0>0$], and that

$\lim z_w'(t_w^2) = p_2$. Therefore [from (2.4) it follows that $p_2^0 > 0$, and by Lemma 1] (for parametric problems by Lemma 1′) the vectors p_1 and p_2 belong to an approach set at $z_0(t_0)$, and the matrix

$$(6.16) \qquad \begin{pmatrix} F_j(z_0(t_0), p_h) & F_j(z_0(t_0), p_k) \\ G_j(z_0(t_0), p_h) & G_j(z_0(t_0), p_k) \end{pmatrix}$$

has rank less than two, where $h=1$ and $k=2$.

Now we can define the approach set A to be the maximal approach set at z containing p_1 and p_2 (see remark at end of §2).

Suppose now that there is a third sequence $t_w^3 \to t_0$ such that $z_w'(t_w^3)$ has an accumulation point p_3 different from p_1 and p_2. We select a subsequence $\{l\}$ of the integers $\{w\}$ such that $z_l'(t_l^3) \to p_3$. By the preceding argument, the matrix (6.16) has rank less than two if $h=1$ and $k=3$ or if $h=2$ and $k=3$. So p_3 belongs to an approach set at z containing p_1 and p_2, and therefore belongs to the maximal set A. The lemma is thus established.

7. **A lemma on the order of sides.** The next lemma is the one by which the Ω-function is made useful to us.

LEMMA 6. *Let p^*, r^* be distinct unit vectors [with $p^{*0} > 0$ and $r^{*0} > 0$] belonging to an approach set A at z^*. Let the inequality*

$$\Omega_H(z^*, p^*, r^*) < 0$$

hold, where $H(z, z') = H(z, z'; \lambda(z^*, A)) = F(z, z') - \lambda(z^*, A) G(z, z')$. Let $G(z, r)$ be quasi-regular normal (either positive or negative) at z^*. Then there is a $\delta > 0$ such that if ABC is a polygon of two sides lying entirely in the δ-neighborhood of z^*, the side AB having direction r such that $|r - r^*| < \delta$ and the side BC having direction p such that $|p - p^*| < \delta$, then ABC does not *minimize $\mathcal{J}(C)$ in the class of all two-sided polygons ADC joining A and C and having $\mathcal{G}(ADC) = \mathcal{G}(ABC)$.*

To be specific we assume that $G(z, r)$ is positive quasi-regular normal at z^*. This involves no loss of generality. For if $G(z, r)$ is negative quasi-regular normal at z^* we have only to replace it by $-G(z, r)$. This causes $\lambda(z^*, A)$ to be replaced by $-\lambda(z^*, A)$, and $H(z, r)$ is unchanged.

In order to avoid repetition we assume that every positive constant introduced in this proof is less than one, and we let M be an upper bound for the sums

$$2 \sum_{i=0}^{q} \{ | H_{z^i}(z, r) | + | G_{z^i}(z, r) | \}$$

for all (z, r) such that $|z - z^*| \leqq 5$ and $|r| = 1$ [and $r^0 \geqq \frac{1}{2} \min (p^{*0}, r^{*0})$]. We shall connote that a vector is a unit vector by giving it a subscript u.

By hypothesis, $\Omega_H(z^*, p^*, r^*) < -3m < 0$. Thus by continuity there are positive numbers ζ, δ (we take ζ, δ, m all less than 1) such that

$$(7.1)\qquad \Omega_H(z, p_u, r_u) < -2m, \quad [p_u^0 > \tfrac{1}{2}\min(p^{*0}, r^{*0}),\ r_u^0 > \tfrac{1}{2}\min(p^{*0}, r^{*0})]$$

if $|z-z^*| < 5\zeta$, $|p_u-p^*| < 3\delta$, and $|r_u-r^*| < 3\delta$.

Again by hypothesis, the numbers $\mathcal{E}_G(z^*, p^*, r^*)$ and $\mathcal{E}_G(z^*, r^*, p^*)$ are positive. Let $3e$, $(0 < e < 1)$, be smaller than the smaller of them. Then if ζ and δ are small enough,

$$(7.2)\qquad G(z, r_u) - r_u^\alpha G_\alpha(\bar{z}, p_u) > 2e, \quad G(z, p_u) - p_u^\alpha G_\alpha(\bar{z}, r_u) > 2e$$

if z and $\bar{z}$ are in the 5ζ-neighborhood of z^* and $|p_u-p^*| < 3\delta$, $|r_u-r^*| < 3\delta$. By definition of M,

$$(7.3)\qquad |\Omega_G(z, p_u, r_u)| < M \text{ if } |z-z^*| < 5\zeta,\ |p_u-p^*| < 3\delta,\ |r_u-r^*| < 3\delta.$$

By Lemma 2, $H_i(z^*, p^*) = H_i(z^*, r^*)$, $(i = 0, 1, \cdots, q)$. If we multiply by p^{*i} or by r^{*i} and sum, we find

$$(7.4)\qquad \mathcal{E}_H(z^*, p^*, r^*) = \mathcal{E}_H(z^*, r^*, p^*) = 0.$$

Then if ζ and δ are small enough, we find, as in establishing (7.2), that

$$(7.5)\qquad \begin{aligned} |H(z, r_u) - r_u^\alpha H_\alpha(\bar{z}, p_u)| &< em/M, \\ |H(z, p_u) - p_u^\alpha H_\alpha(\bar{z}, r_u)| &< em/M \end{aligned}$$

if $|z-z^*| < 5\zeta$, $|\bar{z}-z^*| < 5\zeta$, $|p_u-p^*| < 3\delta$, and $|r_u-r^*| < 3\delta$.

Let ABC be a two-sided polygon lying in the ζ-neighborhood of z^*. We denote A, B, C by z_1, z_2, z_3, respectively, and for their directions and lengths we use the symbols

$$(7.6)\qquad \begin{aligned} p_1 &= (z_2 - z_1)/|z_2 - z_1|, & p_2 &= (z_3 - z_2)/|z_3 - z_2|, \\ l_1 &= |z_2 - z_1|, & l_2 &= |z_3 - z_2|; \end{aligned}$$

and we assume

$$(7.7)\qquad |p_1 - r^*| < \delta, \qquad |p_2 - p^*| < \delta.$$

We now distinguish the two cases $l_1 \geqq l_2$, $l_1 < l_2$. In the former case we write

$$(7.8)\qquad z_\tau = z_1 + (1 + \tau)(z_3 - z_2);$$

in the latter case,

$$(7.9)\qquad z_\tau = z_3 + (1 + \tau)(z_1 - z_2).$$

The two cases differ only trivially; so we suppose, to be specific, that $l_1 \geqq l_2$; the alterations needed to cover the other case are obvious. If $|\tau| < 1$, then $|z_\tau - z_1| < 2|z_3 - z_2| = 2l_2 < 4\zeta$; so z_τ is in the 5ζ-neighborhood of z^*. Therefore for $|\tau|<1$ the quadrilateral $z_1z_2z_3z_\tau$ lies entirely in the sphere $|z-z^*|<5\zeta$.

By an elementary trigonometric computation we find that if $|\tau|<\delta l_1/l_2$ (and therefore if $|\tau|<\delta$, since $l_1\geqq l_2$), then the unit vector

$$(7.10)\qquad p(\tau)\equiv(z_3-z_\tau)/|z_3-z_\tau|$$

satisfies the inequality $|p(\tau)-p_1|<2\delta$. Hence, by (7.7),

$$(7.11)\qquad |p(\tau)-r^*|<3\delta \quad \text{if} \quad |\tau|<\delta.$$

Also, if $|\tau|<1$, then (by (7.8))

$$(7.12)\qquad |z_\tau-z_3|=|z_1-z_2+\tau(z_3-z_2)|\leqq|z_1-z_2|+|z_3-z_2|\leqq 2l_1.$$

We henceforth assume

$$(7.13)\qquad |\tau|<\delta.$$

The point z_τ will also be denoted by D_τ. The figure $ABCD_0$ is a parallelogram, by (7.8). Hence by formula (3.19) of I

$$(7.14)\qquad \mathcal{H}(AD_0C)-\mathcal{H}(ABC)=-l_1l_2\Omega_H(\bar{z},p_1,p_2),$$

$$(7.15)\qquad \mathcal{G}(AD_0C)-\mathcal{G}(ABC)=-l_1l_2\Omega_G(\tilde{z},p_1,p_2),$$

where $\bar{z}$ and $\tilde{z}$ are in the parallelogram $ABCD_0$. If we recall that $\Omega_H(z,p,r)=-\Omega_H(z,r,p)$, these inequalities, with (7.7), (7.1), and (7.3), yield

$$(7.16)\qquad \mathcal{H}(AD_0C)-\mathcal{H}(ABC)<-2l_1l_2m,$$

$$(7.17)\qquad |\mathcal{G}(AD_0C)-\mathcal{G}(ABC)|<l_1l_2M.$$

Now we compute the derivative of $\mathcal{G}(AD_\tau C)$ with respect to τ. The side AD is defined by the equation $z=z_1+t(z_3-z_2)$, $(0\leqq t\leqq 1+\tau)$. So

$$(7.18)\qquad \begin{aligned}\frac{d}{d\tau}\mathcal{G}(AD_\tau)&=\frac{d}{d\tau}\int_0^{1+\tau}G(z_1+t(z_3-z_2),z_3-z_2)dt\\&=G(z_\tau,z_3-z_2)=l_2G(z_\tau,p_2).\end{aligned}$$

The derivative of $G(D_\tau C)$ we compute by formula (2.8) of I, taking $\pi_1=z_3-z_2$, $\pi_2=0$. The side $D_\tau C$ is defined by the equation $z=(1-t)z_\tau+tz_3$, $(0\leqq t\leqq 1)$; so

$$(7.19)\qquad \begin{aligned}\frac{d}{d\tau}\mathcal{G}(D_\tau C)&=-G_\alpha(\bar{z},z_3-z_2)[z_3^\alpha-z_2^\alpha]\\&\quad+\int_0^1 G_{z^\alpha}((1-t)z_\tau+tz_3,z_3-z_\tau)[z_3^\alpha-z_2^\alpha](1-t)dt\\&=-l_2G_\alpha(\bar{z},p_2)p_2^\alpha\\&\quad+\int_0^1 G_{z^\alpha}((1-t)z_\tau+tz_3,p(\tau))p_2^\alpha l_2|z_3-z_\tau|(1-t)dt,\end{aligned}$$

where $\bar{z}$ is on the line segment $D_\tau C$. By the definition of M, together with (7.11) and (7.12), the integrand in (7.19) does not exceed $2Ml_1l_2(1-t)$ in absolute value. Hence the absolute value of the integral is at most Ml_1l_2, and if we add (7.18) and (7.19) we find

$$(7.20)\quad \frac{d}{d\tau}\mathcal{G}(AD_\tau C) = l_2[G_\alpha(z_\tau, p_2) - p_2^\alpha G_\alpha(\bar{z}, p(\tau))] + \theta M l_1 l_2, \qquad |\theta| \leqq 1.$$

The points z_τ and $\bar{z}$ are in the 5ζ-neighborhood of z^*, and (7.11) and (7.7) hold; so by (7.2),

$$(7.21)\qquad \frac{d}{d\tau}\mathcal{G}(AD_\tau C) > 2l_2 e + \theta M l_1 l_2, \qquad |\theta| \leqq 1,$$

if $|\tau| \leqq \delta$.

The conditions thus far imposed on ζ and δ merely require that they be sufficiently small. We now require

$$(7.22)\qquad \zeta < em\delta/2M^2.$$

Then

$$(7.23)\qquad l_1 < 2\zeta < e\delta m/M^2 < e/M,$$

and (7.21) yields

$$(7.24)\qquad \frac{d}{d\tau}\mathcal{G}(AD_\tau C) > l_2 e \quad \text{if} \quad |\tau| < \delta.$$

A formula similar to (7.20) holds for $\mathcal{H}$ also. Here, however, we can use (7.5) and (7.23) to obtain

$$(7.25)\qquad \begin{aligned} \left|\frac{d}{d\tau}\mathcal{H}(AD_\tau C)\right| &< l_2[em/M] + \theta M l_1 l_2, \qquad |\theta| \leqq 1, \\ &\leq 2lem/M. \end{aligned}$$

As τ traverses the interval $[-\delta, \delta]$ inequality (7.24) remains valid; so by the theorem of mean value

$$(7.26)\qquad \begin{aligned} \mathcal{G}(AD_\delta C) - \mathcal{G}(AD_0 C) &> \delta l_2 e, \\ \mathcal{G}(AD_0 C) - \mathcal{G}(AD_{-\delta} C) &> \delta l_2 e. \end{aligned}$$

By (7.17) and (7.23)

$$(7.27)\qquad |\mathcal{G}(AD_0 C) - \mathcal{G}(ABC)| < e\delta l_2;$$

so by (7.26) the number $\mathcal{G}(ABC)$ lies between $\mathcal{G}(AD_{-\delta}C)$ and $\mathcal{G}(AD_\delta C)$. But

$G(AD_\tau C)$ is a continuous function of τ; so there is a value of τ, we call it σ, such that

$$\mathcal{G}(AD_\sigma C) = \mathcal{G}(ABC). \tag{7.28}$$

Using another form of the theorem of mean value, we obtain

$$\frac{\mathcal{K}(AD_\sigma C) - \mathcal{K}(AD_0 C)}{\mathcal{G}(AD_\sigma C) - \mathcal{G}(AD_0 C)} = \frac{\dfrac{d}{d\tau}\mathcal{K}(AD_\tau C)}{\dfrac{d}{d\tau}\mathcal{G}(AD_\tau C)}, \tag{7.29}$$

the right-hand member being calculated for some τ between 0 and σ. By (7.21) and (7.22) the right-hand member of (7.29) has an absolute value less than $2m/M$; so, using (7.29), (7.28), (7.17), and (7.16), we obtain

$$\begin{aligned} \mathcal{K}(AD_\sigma C) - \mathcal{K}(AD_0 C) &< (2m/M)\,|\,\mathcal{G}(AD_\sigma C) - \mathcal{G}(AD_0 C)\,| \\ &= (2m/M)\,|\,\mathcal{G}(ABC) - \mathcal{G}(AD_0 C)\,| \\ &< 2ml_1 l_2 < \mathcal{K}(ABC) - \mathcal{K}(AD_0 C). \end{aligned} \tag{7.30}$$

That is,

$$\mathcal{K}(AD_\sigma C) < \mathcal{K}(ABC). \tag{7.31}$$

It follows from (7.28) and (7.31) that

$$\begin{aligned} \mathcal{J}(AD_\sigma C) &= \mathcal{K}(AD_\sigma C) + \lambda(z^*, A)\,\mathcal{G}(AD_\sigma C) \\ &< \mathcal{K}(ABC) + \lambda(z^*, A)\,\mathcal{G}(ABC) \\ &= \mathcal{J}(ABC). \end{aligned} \tag{7.32}$$

We have thus shown that if ABC is in the ζ-neighborhood of z^* and inequalities (7.7) hold, then ABC fails to minimize $\mathcal{G}(ADC)$ in the class of two-sided polygons ADC such that $\mathcal{G}(ADC) = \mathcal{G}(ABC)$. For (7.28) and (7.32) show that $AD_\sigma C$ is in the given class and gives a smaller value to $\mathcal{J}(C)$. If we let δ_1 be the smaller of ζ and δ, it then serves as the δ of the conclusion of the lemma. The proof is therefore complete.

8. **The existence theorem.** We now make another assumption concerning $\mathcal{J}$ and $\mathcal{G}$:

(8.1) *For each z_0, every approach set A at z_0 consists of the positive multiples of a finite number of unit vectors $p_1, \cdots, p_k$; and these can be so ordered that*† $\Omega_H(z_0, p_i, p_j) < 0$ *if* $i < j$.

We wish to prove the following lemma:

† As usual, $H(z, r) = F(z, r) - \lambda(z_0, A)G(z, r)$.

LEMMA 7. *If t_0 is in E, then*

$$\phi_0'(t_0) = \phi'(t_0) \tag{8.2}$$

and

$$\gamma_0'(t_0) = \gamma'(t_0), \tag{8.3}$$

where $\phi, \phi_0, \gamma, \gamma_0$ are the functions defined in §4.

At t_0, either case (i) or case (ii) of Lemma 5 holds. Suppose case (i). Then there is a sequence $\{\Pi_m\}$ and a sequence $t_m \to t_0$, each t_m being contained in an interval $h_m < t_m < k_m$ on which $z_m(t)$ is linear, and $k_m - h_m$ does not tend to zero. We now select a subsequence $\{\Pi_l\}$ for which $k_l - h_l$ tends to a limit greater than zero, and further select a subsequence $\{\Pi_p\}$ of $\{\Pi_l\}$ such that h_p and k_p tend to limits h, k, respectively. Then $k > h$. Also, since $h_p < t_p < k_p$ and $t_p \to t_0$, it is clear that $h \leqq t_0 \leqq k$. Suppose, to be specific, that $h \leqq t_0 < k$.

We now choose numbers l, m such that $h < l < m < k$. On the interval $[l, m]$ all functions $z_p(t)$ except a finite number are linear. Since $z_p(t)$ tends to $z_0(t)$ and is linear on $[l, m]$, it is also true that $z_p'(t)$ tends to $z_0'(t)$, and in fact uniformly, on $[l, m]$. So

$$\phi(m) - \phi(l) = \lim_{p\to\infty} (\phi_p(m) - \phi_p(l)) = \lim_{p\to\infty} \int_l^m F(z_p, z_p')dt$$

$$= \int_l^m F(z_0, z_0')dt = \phi_0(m) - \phi_0(l).$$

Since ϕ and ϕ_0 are continuous at t_0, we let $l \to t_0$ and obtain

$$\phi(m) - \phi(t_0) = \phi_0(m) - \phi_0(t_0).$$

Dividing by $m - t_0$ and letting $m \to t_0$ gives

$$\phi'(t_0) = \phi_0'(t_0),$$

establishing (8.2). In a like manner we establish (8.3).

Now suppose that case (ii) of Lemma 5 holds. Let A be an approach set at $z_0(t_0)$ consisting of the positive multiples of the unit vectors $p_1, \cdots, p_k$ numbered as in (8.1), and let $\{\Pi_m\}$ be a subsequence of $\{\Pi_n\}$ such that for every sequence $t_m \to t_0$ all accumulation points of the sequence $z_m'(t_m)$ are in A. Then if ϵ is a positive number, there is a $\delta_0 > 0$ and an m_0 such that $z_m'(t)$ is in the ϵ-neighborhood of A if $|t - t_0| < \delta_0$ and $m > m_0$. Otherwise for some $\epsilon > 0$ we could select a subsequence $\{\Pi_p\}$ and a sequence $t_p \to t_0$ for which $z_p'(t_p)$ has distance greater than or equal to ϵ from A, so that no accumulation point of this sequence would be in A.

By Lemma 6, for each pair of numbers i and $j > i$ there is a positive number δ_{ij} with the properties specified in Lemma 6. Let δ be the least of these. We suppose that the ϵ of the preceding paragraph is so small that

$$(8.4) \qquad \left| \, z_m'(t)/\left| z_m'(t) \right| - p_i \, \right| < \delta$$

for some i if $z_m'(t)$ is in the ϵ-neighborhood of A; this is possible because of (4.9). Moreover, we suppose m_0 so large and δ_0 so small that

$$(8.5) \qquad \left| z_m(t) - z_0(t_0) \right| < \delta \quad \text{if} \quad m > m_0, \quad \left| t - t_0 \right| < \delta_0.$$

Now denote by t_m^1 the least t in the interval $(t_0-\delta_0,\, t_0+\delta_0)$ which defines a vertex of Π_m, and denote by T_m the greatest such t. Each side of Π_m between $z_m(t_m^1)$ and $z_m(T_m)$ has a direction $z_m'(t)/\left| z_m'(t) \right|$ which differs from one of the p_i by less than ϵ. By Lemma 6 and the minimizing property of each Π_n, no side with direction near p_j is immediately followed by one with direction near p_i, $(i<j)$. Hence the arc of Π_m between $z_m(t_m^1)$ and $z_m(T_m)$ consists of a subarc (possibly empty) along which $z_m'/\left| z_m' \right|$ differs by less than ϵ from p_1, followed by an arc along which $z_m'/\left| z_m' \right|$ differs by less than ϵ from p_2, followed by $\cdots$, followed by an arc along which $z_m'/\left| z_m' \right|$ differs from p_k by less than ϵ. The values of t which mark the ends of these arcs we denote by $t_m^2, t_m^3, \cdots, t_m^{k+1} = T^m$. Thus if $t_m^j < t < t_m^{j+1}$ and t does not define a vertex of Π_m, the inequality

$$(8.6) \qquad \left| \, z_m'(t)/\left| z_m'(t) \right| - p_j \, \right| < \epsilon$$

holds. We now select a subsequence $\{\Pi_h\}$ such that t_h^j converges to a limit t^j as $h\to\infty$, $(j=1, \cdots, k+1)$. Then $t^1 < t_0 < t^{k+1}$, for otherwise we would be back in case (i) of Lemma 5.

If the interval (t^j, t^{j+1}), $(j=1, \cdots, k)$, is not empty, we choose an interval (l, m) such that $t^j < l < m < t^{j+1}$. Then for all large h the inequality

$$(8.7) \qquad \left| \, z_h'(t)/\left| z_h'(t) \right| - p_j \, \right| < \epsilon$$

holds if $l \leqq t \leqq m$ and $z_h(t)$ is not a vertex. From (8.7) and (4.9),

$$(8.8) \qquad \left| z_h'(t) - \mathcal{L}(\Pi_h)p_j \right| = \mathcal{L}(\Pi_h)\cdot \left| \, z_h'(t)/\left| z_h'(t) \right| - p_j \, \right| < \epsilon L.$$

If t and $t+\tau$ are both in (l, m), this yields

$$(8.9) \qquad \begin{aligned} &\left| z_h(t+\tau) - z_h(t) - \mathcal{L}(\Pi_h)\tau p_j \right| \\ &\qquad = \left| \int_t^{t+\tau} \{ \dot z_h(t) - \mathcal{L}(\Pi_h)p_j \} dt \right| \leqq \epsilon L \left| \tau \right|. \end{aligned}$$

We may without loss of generality suppose that the sequence $\{\Pi_h\}$ was so chosen that the lengths $\mathcal{L}(\Pi_h)$ approach a limit L_0. Then, letting $h\to\infty$, we have

$$(8.10)\qquad |z_0(t+\tau)-z_0(t)-L_0\tau p_j|\leqq \epsilon L|\tau|.$$

Now dividing by τ and letting $\tau\to 0$, we obtain

$$(8.11)\qquad |z_0'(t)-L_0p_j|\leqq \epsilon L,$$

provided only that $z_0'(t)$ exists. For all large h we have $|\mathcal{L}(\Pi_h)-L_0|<\epsilon$, and (8.8) holds; so if $l\leqq t\leqq m$ and $z_0'(t)$ exists, then

$$(8.12)\qquad \begin{aligned}|z_h'(t)-z_0'(t)| &\leqq |z_h'(t)-\mathcal{L}(\Pi_h)p_j|+|\mathcal{L}(\Pi_h)p_j-L_0p_j|\\ &\quad+|L_0p_j-z_0'(t)|<\epsilon(2L+1).\end{aligned}$$

Since F and G are continuous at the point $(z_0(t_0), Lp_j)$, for every $\rho>0$ we can find a neighborhood U of this point set such that

$$(8.13)\qquad |F(z,z')-F(z_1,z_1')|<\rho$$

and

$$(8.14)\qquad |G(z,z')-G(z_1,z_1')|<\rho$$

if (z, z') and (z_1, z_1') are both in U. We now suppose that δ and ϵ are so small and h_0 so large that we have

$$(8.15)\qquad (z_h(t), z_h'(t)) \text{ and } (z_0(t), z_0'(t)) \text{ in } U$$

if $l\leqq t\leqq m$ and $h>h_0$ and the derivatives exist (see (8.8) and (8.12)). Then by (8.13), for all $h>h_0$ we have

$$(8.16)\qquad \begin{aligned}|[\phi_h(m)-\phi_h(l)]-[\phi_0(m)-\phi_0(l)]| &= \left|\int_l^m [F(z_h,\dot z_h)-F(z_0,\dot z_0)]dt\right|\\ &\leqq \rho(m-l).\end{aligned}$$

Letting $h\to\infty$, we get

$$(8.17)\qquad |[\phi(m)-\phi(l)]-[\phi_0(m)-\phi_0(l)]|\leqq \rho(m-l).$$

Now ϕ is continuous on $[t_0-\delta, t_0+\delta]$, for $[z_h^{0\prime}$ is bounded away from 0 by (8.6) (if ϵ is smaller than p_j^0/L), so that$]$ $F(z_h, \dot z_h)$ is uniformly bounded. Also, ϕ_0 is a.c. by its definition. So we can let $m\to t^{j+1}$ and $l\to t^j$, obtaining

$$(8.18)\qquad |\{\phi(t^{j+1})-\phi(t^j)\}-\{\phi_0(t^{j+1})-\phi_0(t^j)\}|\leqq \rho(t^{j+1}-t^j).$$

Although (8.18) was obtained under the assumption $t^{j+1}>t^j$, it clearly holds if $t^{j+1}=t^j$.

Now we add the relations (8.18) for $j=0, 1, \cdots, k$. We find

$$(8.19)\qquad |\{\phi(t^{k+1})-\phi(t^1)\}-\{\phi_0(t^{k+1})-\phi_0(t^1)\}|\leqq \rho(t^{k+1}-t^1).$$

We can divide by $t^{k+1}-t^1$, since we have seen that $t^1<t_0<t^{k+1}$. But the num-

ber δ_0 could be chosen as small as desired, and $t_0-\delta \leqq t^1 < t^{k+1} \leqq t_0+\delta_0$. So if we let δ_0 tend to 0, the difference $t^{k+1}-t^1$ must also approach 0, and (8.19) yields

$$(8.20) \qquad \left| \phi'(t_0) - \phi_0'(t_0) \right| \leqq \rho.$$

But here ρ is an arbitrary positive number; so (8.20) implies

$$(8.21) \qquad \phi'(t_0) = \phi_0'(t_0).$$

Likewise we can establish (8.3).

This completes the proof of the theorem. For $\gamma(t)$ and $\gamma_0(t)$ are a.c., and their derivatives are equal for almost all t, and $\gamma(0)=\gamma_0(0)=0$; so $\gamma(1)=\gamma_0(1)$. Also, $\phi(t)$ is monotonic increasing and $\phi_0(t)$ is a.c., and $\phi'(t)=\phi_0'(t)$ for almost all t; hence

$$\phi(1) \geqq \int_0^1 \dot{\phi}(t)dt = \int_0^1 \dot{\phi}_0(t)dt = \phi_0(1).$$

Thus we have established (4.15), and, as we saw in §4, this implies that C_0 is the minimizing curve sought.

Collecting the hypotheses introduced at various stages of the proof, we obtain the following theorem:

THEOREM. *Let the functions $f(x, y, \dot{y})$ and $g(x, y, \dot{y})$ be defined and continuous with all their first-order partial derivatives for all $(x, y, \dot{y})$. Let $F(z, z')$ and $G(z, z')$ be the parametric integrands associated with f, g, respectively. Assume that hypotheses* (2.4), (3.1), (3.2), (3.3), *and* (8.1) *are satisfied. Let $\mathcal{G}(C)$ be quasi-regular normal, and let (x_0, y_0) and (X, Y) be two points such that $x_0 < X$.*

Then for every l the class of a.c. curves $y=y(x)$, $(x_0 \leqq x \leqq X)$, joining (x_0, y_0) to (X, Y) and such that $\mathcal{G}(y)=l$ either is empty or contains a minimizing curve for $\mathcal{J}(C)$.

9. **Example.** An example satisfying our conditions is

$$f(x, y, y') = \phi(y)(y'^2 + 1), \qquad g(x, y, y') = (1 + y'^2)^{1/2},$$

where $\phi(y) > 0$ and $\phi'(y) > 0$. Transforming to parametric form, we have

$$F(x, y, x', y') = \phi(y)\left(\frac{y'^2}{x'} + x'\right), \qquad G(x, y, x', y') = (x'^2 + y'^2)^{1/2}.$$

To determine the approach sets at a fixed point (x, y), we recall that, by Lemma 2, if $(1, q)$ and $(1, \bar{q})$ form an approach set, then for some number λ these same sets form an approach set for $f-\lambda g$. Hence if we consider the graph of $u=f(x, y, r)-\lambda g(x, y, r)$ as a function of r, the tangent at $r=q$ and the tangent at $r=\bar{q}$ coincide. (Cf. the geometric interpretation of Lemma 2.) Therefore the graph has at least two flex points between q and $\bar{q}$. But

$$f_{y'y'}(x, y, r) - \lambda g_{y'y'}(x, y, r) = 2\phi(y) - \lambda(1 + r^2)^{-3/2}.$$

This can have at most two zeros, of opposite sign. So q and $\bar{q}$ are of opposite sign, and no approach set can contain more than two members of the form $(1, q_1)$ and $(1, q_2)$. We easily see that $(1, q)$ and $(1, -q)$ form an approach set. Hence given any (x, y, q), we see that the entire approach set at (x, y) containing $(1, q)$ consists of the positive multiples of $(1, q)$ and $(1, -q)$.

We readily calculate that, independently of λ,

$$\Omega_H(x, y; 1, q; 1, -q) = 2q\phi'(y)(q^2 + 1),$$

which is positive if $q>0$. So (8.1) is satisfied. It remains only for us to verify (2.4). In (2.4) we may assume, if we wish, that $|p_n| = |r_n| = 1$, since $F_i(z, r)$ and $G_i(z, r)$ are positively homogeneous of degree 0 in r. Then their limits p, r are also unit vectors. With the assumption, the matrix in (2.4) is, for our example,

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},$$

where

$$\begin{aligned} A_{11} &= \phi(\bar{y}_{1,n})\{1 - (r_n^1/r_n^0)^2\} - \phi(y_{1,n})\{1 - (p_n^1/p_n^0)^2\} + \theta_{1,n}, \\ A_{12} &= 2\phi(\bar{y}_{2,n})r_n^1/r_n^0 - 2\phi(y_{2,n})p_n^1/p_n^0 + \theta_{2,n}, \\ A_{21} &= r_n^0 - p_n^0 + \bar{\theta}_{1,n}, \quad A_{22} = r_n^1 - p_n^1 + \bar{\theta}_{2,n}. \end{aligned}$$

We must show $\lim r_n^0 = r^0 > 0$. Suppose the contrary. Then $\lim |r_n^1| = 1$, while $\lim p_n^1 = p^1 \neq 1$ and $\lim \bar{\theta}_{2,n} = 0$. Therefore $\lim A_{22} \neq 0$. The last two terms in A_{11} and the last two in A_{12} have finite limits. If our matrix has rank less than two, then

$$A_{11}A_{22} - A_{12}A_{21} = 0, \qquad n = 1, 2, \cdots.$$

But $|r_n^1/r_n^0|$ tends to ∞, and in the determinant its square occurs with a coefficient which approaches a limit $\phi(y)(r^1 - p^1)$ which is not zero, while its first power has a bounded coefficient. Thus as $n \to \infty$ the absolute value of the determinant will also tend to ∞, contradicting the assumption that it is zero for all n. So (2.4) is established, and all the hypotheses of our theorem are satisfied.

UNIVERSITY OF VIRGINIA,
CHARLOTTESVILLE, VA.