CONTINUOUS GROUPS AND SCHWARZ’ LEMMA*

BY

MAX ZORN

INTRODUCTION

The famous lemma of H. A. Schwarz is doubtless one of the basic theorems in the theory of analytic functions. In this paper I propose to study the lemma from a topological point of view. The results have been announced, without proof, in a previous note.† Several changes, corrections, and additions have been made; I use the opportunity to state here my indebtedness to D. W. Hall for his inspiring interest and helpful criticism.

The theory to be presented is a by-product of a more comprehensive treatment of conformal mappings‡ which will be communicated elsewhere.

Like the theories of Kerékjártó§ and Stoilow|| our investigations are made with a direct view to the characterization of conformal mappings. Yet both authors deal with the conformal mappings individually, whereas we aim more at the characterization of the system of all conformal mappings of a Riemann surface $S$ in itself. As an equivalent to this simplification of the problem we attempt to keep the space $S$ general as long as possible, whereas usually $S$ is supposed to be locally plane from the outset.

The theory of Schwarz’ lemma has been separated from the rest because of its independence and also because it seems to be of value for the study of the hardest characterization problem, the problem of Brouwer.

The present paper is divided into three parts. Part I is of a rather general nature and can be read without any topological preparation. For the other parts a certain familiarity with topological notions and theorems is necessary. Parts I and II together lead up to a theorem which is formally identical with the Schwarz lemma. In III, particularly in §8, we show that this formal identity is material identity; in §9 we derive, with the aid of the geometric theorems from II some interesting topological features of the underlying space.

---

* Presented to the Society, February 25, 1939; received by the editors September 9, 1938.

1
Notations. We use only italic letters; consequently, concepts of different logical types are often denoted by letters of the same alphabet: \( d, i, m, n, a \) are indices, \( d, i \) and \( m, n \) natural numbers, \( a \) is arbitrary; \( e, p, q, r, s, x, y, z \) are points (most of them in \( S \)); \( S, A, C, E, K, L, O, U_x, V_y \) denote sets of points, usually contained in \( S \); \( F, F^*, F', G, H, P, R, R_p, R_i, R^{(x)} \) are transformations, usually continuous single-valued mappings of \( S \) in itself; \( N \) is a family of transformations; in general the transformations \( F \), and so on, will belong to \( N \).

If all \( x_i \) are in a set such as \( C \), we call \( x_i \) a sequence from \( C \). A "subsequence" of a sequence, say \( x_i \), is formed by choosing an increasing sequence \( (i_{n+1} > i_n) \) of indices; it is convenient to denote the new sequence by \( x'_i \); a subsequence of the subsequence would be written \( x''_i \).

Theorems and definitions are numbered together; a definition is indicated by brackets, a theorem by parentheses.

Part I

1. Continuous transformations. We make the following definition:

[1.1] \( S \) is a topological space which we assume to be metrizable. The metric of the space does not occur explicitly, but we shall have to use limit relations like \( \lim x_n = x \), functions like the closure \( \overline{A} \), the boundary \( \text{Bd}(A) \), the frontier \( \text{Fr}(A) \), and properties like open, closed, connected, locally connected; the terms compact, limited are defined explicitly for obvious reasons.

In Part I, however, we do not need all the consequences of the metrizability; it is sufficient to assume that \( S \) is an \( L^* \)-space as defined, for example, in Kuratowski's book on topology.†

[1.1.1] \( S \) is an \( L^* \)-space if convergence of sequences is defined and satisfies the following conditions:

I. If \( \lim x_n = x \), then \( \lim x_n' = x \).
II. If \( x_n = x \), then \( \lim x_n = x \).
III. If for every subsequence \( x_n' \) of \( x_n \) a subsequence \( x_n'' \) with \( \lim x_n'' = x \) can be found, then \( \lim x_n = x \).

[1.2] A point \( x \) is a limit point of \( A \) if a sequence \( x_i \) from \( A \) exists such that \( \lim x_i = x \). A point \( x \) is a limit point of a sequence \( x_i \) if a subsequence \( x'_i \) with \( \lim x'_i = x \) exists.

If every sequence \( x_i \) from \( A \) has at least one limit point, then \( A \) is called "limited."

A is "compact" if every sequence from \( A \) has a limit point in \( A \).

† C. Kuratowski, Topologie I, Warsaw, 1933, pp. 76-77; cf. also the literature mentioned there.
In the sequel we shall be concerned mostly with a family $N$ of single-valued, continuous transformations $F$, $G$, $H$, $I$, $P$, $R$, $\ldots$. The domain (of definition) is always $S$, the range (of values) $F(S)$ is a subset of $S$. The natural definition of continuity in $L^*$-spaces is the following:

\[\text{1.3] } F \text{ is continuous if } \lim x_n = x \text{ implies } \lim F(x_n) = F(x).\]

Convergent sequences of (continuous) functions $F$ will occur rather often; it seems that the type of convergence which has been introduced as "continuous convergence"\footnote{Cf. C. Carathéodory, *Conformal Representation*, Cambridge Tracts, no. 28.} is the most appropriate one for the abstract theory of conformal mappings.

\[\text{1.4] } \text{A sequence of transformations } F_n \text{ is said to converge towards } F, \text{ that is, } \lim F_n = F, \text{ if } \lim x_n = x \text{ implies } \lim F_n(x_n) = F(x).\]

Obviously this implies $F_n(x) = F(x)$; but the converse is not true.

By virtue of the definition \[\text{1.4] } \text{any set of continuous mappings } F \text{ forms an } L^*-\text{space. We shall have to use the corresponding property III in our proofs; hence we state explicitly the following theorem:}\]

\[\text{1.4.1) If every subsequence } F'_n \text{ contains a subsequence } F''_n \text{ with } \lim F''_n = F, \text{ then } \lim F_n = F.\]

Indeed, let $\lim x_n = x$, and consider the sequence of points $F_n(x_n)$. From every subsequence $F'_n(x'_n)$ we can select $F''_n(x''_n)$ such that $\lim F''_n(x''_n) = F(x)$. Consequently, $\lim F_n(x_n) = F(x)$, which implies $\lim F_n = F$.

\[\text{1.5] } \text{A sequence } F_n \text{ is called "properly divergent" if for no point } x \text{ the sequence } F_n(x) \text{ has a limit point.}\]

We recall the usual notations and conventions about composition of functions:

\[\text{1.6] } \text{The product } H = FG \text{ of } F \text{ and } G \text{ (in this order) is defined by } H(x) = FG(x) = F(G(x)). \text{ The identity is the transformation } I \text{ which leaves all points invariant, } I(x) = x. \text{ A function } G \text{ is the inverse of } F \text{ if } GF = I; \text{ it may not exist, but if it does, then it is unique and satisfies } FG = I. \text{ Powers } F^n \text{ are defined as usual; if the inverse exists, it is always written as } F^{-1}.\]

\[\text{1.7) If } \lim F_n = F \text{ and } \lim G_n = G, \text{ then } \lim F_n G_n = FG.\]

This is an immediate consequence of the "continuity" of the convergence.

Let $\lim x_n = x$. It follows from $\lim G_n(x_n) = G(x)$ that $\lim F_n(G_n(x_n)) = F(\lim G_n(x_n)) = F(G(x)) = FG(x)$.

\[\text{1.8) If } \lim F_n = F \text{ and } \lim F_n^{-1} = G, \text{ then } G = F^{-1}.\]

In other words, if the inverses $F_n^{-1}$ exist and converge towards a limit, then
\[ \lim F_n^{-1} = (\lim F_n)^{-1}. \]

The proof is an algebraic consequence of (1.7), for \( GF = (\lim F_n^{-1})( \lim F_n) \) = \( \lim F_n^{-1}F_n = I. \)

(1.9) \( F \) is called nilpotent if a point \( p \) exists such that \( \lim x_n = x \) implies \( \lim F^n(x_n) = p. \)

[1.10] The transformation which maps every point on the same point \( p \) is called “constant” and denoted by \( P. \)

With this terminology we can say that \( F \) is nilpotent exactly if its powers converge towards a constant.

(1.11) The point \( p \) is a fixed point of \( F, F(p) = p. \) It is also the only fixed point of \( F. \)

Indeed, writing \( p \) in the form \( \lim F^n(p) \), we obtain \( F(p) = F(\lim F^n(p)) = \lim F^{n+1}(p) = \lim F^n(p) = p. \)

If, on the other hand, \( F(q) = q \), the relations \( F^n(q) = q, \lim F^n(q) = q, \) and \( \lim F^n(q) = p \) give \( q = p. \)

2. The family \( N. \) In this section we introduce a group of definitions and assumptions which describe abstractly some features of the analytic mappings of the unit circle in itself.

[2.1] The family \( N \) is a set of transformations with the following properties:

I. Continuity. The elements of \( N \) are single-valued continuous transformations of \( \mathbb{S} \) into itself, \( F(S) \subset S. \)

II. Composition, identity. The identity \( I \) is in \( N, \) and if \( F \) and \( G \) are in \( N, \) then their product \( FG \) is in \( N. \)

III. Cancellation. If \( F, G, H \) are in \( N, \) and if \( F \) is not constant, then the equality \( GF = HF \) implies \( G = H. \)

IV. Normality. Every sequence \( F_n \) from \( N \) contains a subsequence \( F'_n \) which is either properly divergent or else converges towards an element \( F \) of \( N. \)

If we want \( S \) to be the unit circle of the complex number plane and \( N \) the set of all analytic mappings \( F, F(S) \subset S, \) we speak of “the classical case.”

In the classical case, I–IV are fulfilled; I–III are elementary, whereas IV has perhaps a more advanced character and belongs to the theory of normal families.†

From these assumptions alone we shall derive a topological version of the Schwarz lemma. In Part II a geometrical formulation will be established on

the basis of further restrictions on $S$ and $N$. Finally we show how the abstract theorem yields the ordinary Schwarz lemma in the classical case.

The geometry in $S$ will be provided by those elements of $N$ which have an inverse in $N$. In particular, the analogue of the ordinary rotations is of use.

[2.2] A transformation $R$ is called a rotation if

(a) $R$ is in $N$;

(b) the inverse $R^{-1}$ exists;

(c) $R^{-1}$ is in $N$;

(d) $R$ has a fixed point $p$.

We shall also say that $R$ is a rotation "about $p"$ or "with center $p"$; the fixed point will often be indicated by the subscript $p$: $R_p(p) = p$.

The point $p$, unless stated otherwise, may be considered fixed in advance. In particular, it will be fixed for the following definitions of "rotatory," "invariant," "circumference."

(2.3) The rotations about $p$ form a group.

That means that $R_1 R_2$ is a rotation, $(R_1 R_2) R_3 = R_1 (R_2 R_3)$, $I$ is a rotation, and $R^{-1}$ is a rotation satisfying $R^{-1} R_1 = R_1 R^{-1} = I$. (The proof is omitted.)

[2.4] If the set $A$ contains its image $R_p(A)$ under every rotation, it is called invariant. Since $R_p^{-1}$ is also a rotation, we might have said $R_p(A) = A$.

[2.5] A set $A$ is "rotatory" if for any two points $q$, $r$ in $A$, there exists a rotation $R_p$ such that $R_p(q) = r$.

[2.6] A set which contains a point $q$ is called a circumference $L_q$ if it is invariant and rotatory.

If necessary, we say "circumference through $q$ with center $p."$

This definition is justified by the fact that $L_q$ consists of all points of the form $R_p(q)$.

The definitions and assumptions set forward in these two introductory paragraphs enable us now to formulate and prove the first (topological) version of Schwarz' lemma. The rotations will hereby play a quite important role; we shall establish first some of their properties.

3. Rotations and circumferences. We make the following assertion:

(3.1) Let $\{F_a\}$ be a subset of $N$, the index $a$ ranging over an arbitrary set of symbols. If for one single point $p$ the set $\{F_a(p)\}$ is limited, then for every point $x$ the set $\{F_a(x)\}$ is limited.

We derive this from the normality property IV in the following more general form:
(3.1.1) If \( \{F_\alpha(p)\} \) is limited, then every sequence \( F_\alpha_i \) contains a subsequence \( F_\alpha_i' \) which converges towards an element of \( N \).

Indeed, we only have to select a subsequence \( F_\alpha_i' \) which is either convergent or properly divergent. The second possibility cannot arise, since \( F_\alpha_i'(p) \) has at least one limit point. Consequently, any sequence \( F_\alpha_i(x) \) has at least the limit point \( \lim F_\alpha_i(x) \). From the theorem (3.1) we shall generally use the following special case:

(3.1.2) If \( F_i(p) = p \), then \( F_i \) has a convergent subsequence \( F_i' \).

Two other consequences are the following:

(3.1.3) The circumferences \( L_q \) are limited.

(3.1.4) If the transformations \( F_i \) are in \( N \), and if the sequence \( F_i \) converges "pointwise" towards \( F \), that is, for every \( x \) \( \lim F_i(x) = F(x) \), then \( F \) is in \( N \), and the convergence

\[ \lim F_i = F \]

is continuous.

The theorem (3.1.3) is obvious since \( L_q \) consists of all points \( R_p(q) \), and \( R_p(p) = p \). We shall afterwards show that \( L_q \) is even compact. The second statement is based on (1.4.1), and we prove it in a more general form. We do not need the generalization; it is inserted merely as the abstract background of the theorems of Stieltjes-Porter-Vitali-Blaschke.†

(3.1.5) Let \( A \) be such that \( F(x) = G(x) \) for all \( x \) in \( A \) implies that \( F = G \) in \( S \), in case \( F \) and \( G \) are in \( N \). Suppose that for all \( x \) in \( A \) \( \lim F_i(x) \) exists. Then \( F_i(x) \) converges for all \( x \) in \( S \), towards say \( F(x) \); \( F(x) \) is in \( N \), and we have \( \lim F_i = F \).

Indeed, for every subsequence \( F'_n \) there exists a convergent subsequence \( F''_n \), with the limit \( F'' \) contained in \( N \) but formally dependent on the subsequence \( F'_n \). Yet all these possible limit functions are identical on \( A \); consequently, they are identical throughout. That is sufficient (cf. (1.4.1)) for the relation \( \lim F_i = F \).

The foregoing theorems are now applied in the case of rotations (about \( p \)).

(3.2) Every sequence \( R_n \) of rotations contains a convergent subsequence; the limit mapping is in \( N \).

This is again a special case of (3.1.2). But we can make the following stronger statement:

(3.3) The limit of a sequence \( R_i \) of rotations is again a rotation \( R \).

Anticipating the result, we write $\lim R_i = R$. Since $R_i(p) = p$ implies $R(p) = p$, and $R$ is (cf. (3.2)) in $N$, we have only to prove that $R$ has an inverse in $N$.

Consider the sequence of rotations $R_i^{-1}$. There will be at least one convergent subsequence $R_i^{-1}$, $\lim R_i^{-1} = G$, where $G$ is $N$. Since the limit of the corresponding sequence $R_i$ is $R$, (1.8) yields that $G$ is the inverse $R^{-1}$ of $R$.

(3.4) If $\lim R_i = R$, then $\lim R_i^{-1} = R^{-1}$.

Take any subsequence $R_i^{-1}$ of the sequence $R_i^{-1}$. The proof of the foregoing statement shows that we can select a convergent subsequence $R_i^{-1}$ which converges towards $R^{-1}$. On account of (1.4.1) this implies $\lim R_i^{-1} = R^{-1}$.

These theorems may be condensed into the statement that the rotations about $p$, under the continuous convergence, form a compact $L^*$-group.

(3.5) The circumferences $L_q$ are compact.

Let $q_i$ be a sequence from $L_q$; then by definition $q_i = R_i(q)$. Selecting a convergent subsequence $R_i$ with the limit $R'$ we see that the corresponding subsequence $q_i = R_i(q)$ has the limit point $R'(q)$, which is in $L_q$.

(3.6) If $S$ has more than one point, then a rotation is not nilpotent.

4. Topological version of the lemma. The theorem (4.1) is, in the classical case, one of the numerous consequences of Schwarz’ lemma. It expresses as far as possible the tendency of a mapping $F$ which has a fixed point $p$ but is not a rotation, to move the points of $S$ “nearer” $p$. Why we call it a topological version of the classical lemma will be evident afterwards, when the application to the classical case is made.

(4.1) A transformation $F$ in $N$ with the fixed point $p$ is either a rotation (about $p$) or is nilpotent.

The proof is made in two steps, (4.2) and (4.3). We show first that $F$ is already nilpotent if only one subsequence $F^{n_i}$ of the sequence $F^n$ converges towards the constant $P$. If then $F$ is not nilpotent, there must be a convergent sequence $F^{n_i}$ with a nonconstant limit $F^*$ in $N$. It is shown in (4.3) that in this case $F$ has an inverse $F^{-1}$ in $N$, which is more or less explicitly constructed as a limit of a sequence of powers of $F$.

All transformations occurring in this paragraph will be in $N$, either by assumption (as in the case of $F$) or because they are limits of mappings in $N$.

(4.2) If $F(p) = p$ and if a sequence $F^{n_i}$, where $n_{i+1} > n_i$, tends towards the constant $P$, then $F$ is nilpotent and $\lim F^n = P$.

It is sufficient to show that every sequence $F^{m_i}$, $m_{i+1} > m_i$, contains a convergent subsequence $F^{m_{i_{*}}}$ with the limit $P$. 

License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use
We select two subsequences \( m'_i, n'_i \) such that
(a) \( d_i = n'_i - m'_i \) is increasing, and
(b) the sequence \( F^{d_i} \) is convergent with \( \lim F^{d_i} = F' \).

Such a sequence exists; the first condition can be fulfilled because \( n_i \) and \( m_i \) are strictly increasing; the second condition, because \( F(p) = p, F^n(p) = p \).

The relations \( \lim F^{n_i} = \lim F^{n'_i} = p, \lim F^{d_i} = F' \) imply (cf. (1.7)) that \( \lim F^{n''_i} = \lim F^{n'_i} F^{d_i} = P F' = P \).

We note that while the normality has been used freely, the cancellation property has not yet appeared in the proofs.

(4.3) If \( F(p) = p \) and if a convergent sequence \( F^{n_i} \) with the nonconstant limit \( F^* \) exists, then \( F \) has an inverse in \( N \).

The proof is somewhat similar to the preceding one, but \( F' \) is defined slightly differently and the cancellation property III is essential.

We select again a subsequence \( n'_i \) such that
(a) \( d_i = n'_{i+1} - n'_i - 1 \) is strictly increasing, and
(b) \( F^{d_i} \) is convergent with \( \lim F^{d_i} = F' \).

From these assumptions we derive the equality \( F^* = F^* F' F \), for
\[
\lim F^{n_i} = \lim F^{n'_{i+1}} = \lim F^{n'_i} F^{d_i} F = (\lim F^{n'_{i+1}})(\lim F^{d_i}) F.
\]

Writing this in the form \( F^* I = F^* (F' F) \) and using the fact that \( F^* \) is not a constant, we obtain, by virtue of the cancellation law, \( F' F = I \).

In \( F' \) we have, therefore, the inverse of \( F \), the existence of which was asserted in our theorem.

The principal theorem follows now as indicated before. If \( F(p) = p \), then \( F^n(p) = p \) shows that a convergent sequence \( F^{n_i} \) exists. If the limit is constant, then (4.2) implies that \( F \) is nilpotent; if it is not, then (4.3) shows that \( F \) has an inverse and consequently is a rotation (about \( p \)).

As an interesting corollary we obtain the following:

(4.4) A transformation with two different fixed points \( p \) and \( q \) is a rotation.

In the classical case one knows more: the rotation \( F \) is the identity. This generalization suggests itself as an additional axiom, which (see the end of the paper) permits a more precise description of \( S \) and \( N \). In view of the theory of Kerékjártó we call attention to the fact that instead of \( N \) we could have studied the subsystem formed by powers of \( F \) and their limits.

**Part II**

5. New restrictions on \( S \) and \( N \). From now on we shall use more freely the topological terminology, indicated by the words open, neighborhood, closed; closure \( \overline{A} \), boundary \( \text{Bd}(A) \) of a set \( A \); connected, component, locally con-
connected; separate, cut point; semicompact, locally compact, (perfectly) separable, and metrizable.

The set $AB$ is the common part and $A + B$ the union of the two sets, and $S - A$ is the complement of $A$ in $S$.

[5.1] $S$ is now a metrizable space with the following additional properties:

I. $S$ is connected and contains more than one point.
II. $S$ is locally connected.
III. $S$ has no cut points; that is, for all points $x$ the set $S - \{x\}$ is connected.

In Part II we shall use III generally for $x = p$, where $p$ is arbitrary but fixed.

Before we set down the restrictions on $N$ we define the geometrical concepts "circle" and "closed circle."

[5.2] The component of $S - L_q$ which contains $p$ is called the circle with center $p$ determined by $q$ and is denoted by $C_q$.

In other words, the circle is the largest connected subset of the complement of the circumference $L_q$ which contains $p$. If (and only if) $q$ is identical with $p$, then $C_q$ is empty.

We note that this describes the interior of the circular area determined by a circular curve in euclidean geometry, which has $p$ as center and $q$ on the curve.

[5.2.1] The closure $\overline{C_q}$ of $C_q$, comprising $C_q$ and all its limit points, is called a closed circle.

The restrictions on $N$ are now phrased as properties of circles and circumferences.

[5.3] $N$ is from now on a family of transformations which has not only the properties I–IV of [2.1] but the following:

V. If $q \neq p$, then $L_q$ separates $S$; that is, $S - L_q$ is not connected.
VI. The space $S$ is not representable as a finite sum of circles (with possibly different centers).†

These two axioms constitute very heavy restrictions on $N$, but in exchange we obtain a quite rich geometry (topologically speaking) for $S$.

6. Circles and circumferences. We can make the following assertion:

(6.1) The circles $C_q$ are open and connected.

† This property was not contained in the before mentioned note; my proof for the central theorem, loc. cit. (II, 5), contained a mistake which was pointed out to me by Mr. Hall and which I was not able to correct without a new assumption. The particular form of VI has been chosen since it is also useful for the justification theorem (§8).
If \( q = p \), then \( L_q = \{ p \} \), and \( S - L_q \) does not contain \( p \). In this case we have to interpret \( C_q \) as the empty set, which may be considered open and connected.

If \( q \neq p \), then \( p \) is in \( S - L_q \), since \( R(q) = p \) implies \( q = R^{-1}(p) = p \).

\( L_q \) is closed (even compact); its complement \( S - L_q \) is consequently open. Now \( S \) is locally connected; that is, in every neighborhood \( U_x \) (open set containing \( x \)) there exists a neighborhood \( V_x \) which is connected.

It follows that a component (largest connected subset) of any open set in a locally connected space is open; \( C_q \) is such a component, hence it is open; it is connected by definition. If it is not empty, it contains \( p \).

(6.2) The circles \( C_q \) and their boundaries \( Bd(C_q) \) are invariant.

This is a consequence of the following group of statements:

(6.2.1) \( R(A + B) = R(A) + R(B) \); \( R(AB) = R(A)R(B) \); \( R(S - A) = S - R(A) \).

This holds for subsets \( A, B \) of \( S \) and for any (1-1) mapping of \( S \) on itself, in particular, for a rotation.

(6.2.2) \( R(A) = \overline{R(A)} \).

This holds at least for topological mappings (where \( R \) and \( R^{-1} \) are continuous).

(6.2.3) \( R(Bd(A)) = Bd(R(A)) \).

The boundary, as the set of all points which are limit points of sequences from \( A \) but not in \( A \), can be written as

\[ Bd(A) = \overline{A} - A. \]

(6.2.3) follows algebraically from this definition and the preceding identities.

(6.2.4) Any function of invariant sets \( A, B, C \) which is composed from sums, products, complements, and closures is invariant.

For example the "frontier \( Fr(A) \) of \( A \)" is equal to \( R(Fr(A)) \) because by definition

\[ Fr(A) = \overline{A(S - A)}, \quad R(\overline{A} \cdot S - A) = R(\overline{A})R(S - A), \]
\[ R(\overline{A}) = R(A) = \overline{A}, \quad R(S - A) = R(S - A) = R(S) - R(A) = S - A. \]

Also

\[ R(Bd(A)) = R(\overline{A} - A) = R(\overline{A}) - R(A) = \overline{R(A)} - R(A) = Bd(R(A)). \]

In order to derive (6.2) we have only to go back to the definition of \( C_q \). The set \( L_q \) is invariant; hence \( S - L_q \) is invariant; a rotation \( R \) maps \( S - L_q \)
topologically on itself, a connected subset on a connected subset, a largest connected subset on a (possibly different) subset of the same character, and, since \( R(p) = p \), the component \( C_q \) of \( p \) on itself.

The boundary \( \text{Bd}(C_q) \) is invariant as a function of an invariant set; this invariance we use now for the determination of \( \text{Bd}(C_q) \).

(6.3) The boundary \( \text{Bd}(C_q) \) is exactly \( L_q \), if \( q \neq p \); if \( q = p \) it is, of course, empty.

If \( q = p \), then \( C_q = 0 \); hence \( \text{Fr}(C_q) = \text{Fr}(0) = 0 \). Hence we assume \( C_q \neq 0 \). Since \( q \) is in \( L_q \), \( q \) is not in \( C_q \) and \( C_q \) is not equal to \( S \).

The set \( C_q \) could not be closed, for an open and closed set in a connected space \( S \) is either \( 0 \) or \( S \). Hence there is a point which is limit point for \( C_q \) but not in \( C_q \); let \( r \) be such a boundary point. The point \( r \) cannot be in \( S - L_q \), for \( C_q \) is a component of \( S - L_q \); hence it contains all its limit points in \( S - L_q \), and it is "relatively closed" with respect to \( S - L_q \). The point \( r \), that is, any boundary point of \( C_q \), is therefore in \( L_q \).

The boundary is not only a non-empty subset of \( L_q \), it is also invariant. Since \( 0 \) and \( L_q \) are the only invariant subsets of \( L_q \), the boundary \( C_q \) is exactly \( L_q \).

The connectedness of \( S \) and \( C \) will be used so often in the proofs to come that we deem it advisable to insert the following theorem:

(6.3.1) The connectedness of a space is equivalent to the following implications:
(a) If a set \( A \) is open and closed, it is either \( 0 \) or the whole space.
(b) If an open set \( A \) has no boundary, then it is \( 0 \) or the whole space.
(c) If one knows, for an open set \( A \), that \( \text{Bd}(A) \subset A \), then \( A \) is \( 0 \) or the whole space.
(d) The space is not the sum of two disjoint open proper subsets.

These statements are trivial consequences of the following definition:

[6.3.2] A space \( S \) is connected if \( A + B = S \) and \( \overline{A} \cap \overline{B} = 0 \) imply that either \( \overline{A} \) or \( \overline{B} \) is empty.

Since \( \text{Bd}(A) = \overline{A} - A \), we get from (6.3) the corollary:

(6.3.3) \( \overline{C_q} = C_q + L_q \), if \( p \neq q \).

For \( p = q \) this is not true since \( \overline{C_q} = 0 \); but \( \overline{C_q} \subset C_q + L_q \) is always true.

(6.3.4) \( L_q \) is also, for \( q \neq p \), equal to the frontier \( \text{Fr}(C_q) \).

We show that every point of \( L_q \) is a limit point of \( S - \overline{C_q} \). Since \( S - \overline{C_q} \) is invariant, we need this for one single point \( r \) of \( L_q \).
We know that $C_q$ is an open and closed set with respect to $S - L_q$; its complement in $S - L_q$ is exactly $(S - L_q) - C_q = S - (L_q + C_q) = S - C_q$; such a complement is also open and closed in $S - L_q$. Therefore $S - C_q$ has no limit points in $C_q$. It is not empty since, because of property V, $S - L_q$ is not connected whereas $C_q$ is connected.

In $S$ itself $S - C_q$ is open; it could not be closed because it is neither empty nor equal to $S$. There must be a boundary point $r$, and this point is necessarily on $L_q$.

We shall now have to derive a series of relations between different circles and circumferences; it will be convenient to write $L_1, L_2, L_i, C_1, C_2, C_i$ instead of $L_q, C_q$, and so on; it is always understood that $C_i$ is the circle determined by $L_i$.

(6.4) The product $L_1C_2$ is either empty or $L_1$.

For $L_1C_2$, as a product of invariant sets, is invariant; 0 and $L_1$ are the only invariant subsets of $L_1$.

(6.4.1) $L_2 \subset C_v$ and $x \in C_v$ are equivalent.

A non-trivial statement is the following:

(6.5) $L_1C_2 = 0$ implies $C_2 \subset C_1$.

We shall derive this by showing that the product $C_1C_2$ is equal to $C_2$. If $C_2$ is empty, then $C_2 \subset C_1$ is trivially true. If not, we shall see that $C_1C_2$ is a non-vanishing open and relatively closed subset of $C_2$; $C_1C_2 = C_2$ follows because $C_2$, as a circle, is connected.

To this purpose we determine the relative boundary of $C_1C_2$ in $C_2$, that is, the set of all limit points of $C_1C_2$ which are in $C_2$ but not in $C_1C_2$; in other terms, the product $C_2\text{Bd}(C_1C_2)$. Here and later we shall often use the following formulas:

(6.5.1) $\text{Bd}(A + B) \subset \text{Bd}(A) + \text{Bd}(B); \text{Bd}(AB) \subset \text{Bd}(A) + \text{Bd}(B)$.

Now we have $\text{Bd}(C_1C_2) \subset \text{Bd}(C_1) + \text{Bd}(C_2) \subset L_1 + L_2$; consequently,

$$C_2\text{Bd}(C_1C_2) \subset C_2L_1 + C_2L_2.$$ 

The set $C_2L_2$ is always empty; $C_2 \subset S - L_2; C_2L_1$ is empty by assumption. Hence $C_2\text{Bd}(C_1C_2) = 0$. Since $C_1C_2$, absolutely open, as a product of open sets in $S$, is a fortiori relatively open in $C_2$, it is either empty or equal to $C_2$. How could $C_1C_2$ be empty? Only if one of the factors is empty, for otherwise both will contain the point $p$. The case that $C_2$ is empty has been disposed of; if $C_1$ were empty, $L_1 = \{p\}$ would imply $L_1C_2 = L_1 \neq 0$, contrary to our assumption.

Property VI has not been used yet.
(6.6) \( L_1 C_2 = L_1 \) implies \( C_1 \subset C_2 \).

Considering (6.5) we see that it suffices to prove \( L_2 C_1 = 0 \). The proof is indirect and based on property VI.

Suppose that \( L_2 C_1 \neq 0 \); then it is equal to \( L_2 \) and \( L_2 \subset C_1 \).

Now consider the (open) set \( C_1 + C_2 \) and in particular, its boundary \( \text{Bd}(C_1 + C_2) \). The relation

\[
\text{Bd}(C_1 + C_2) \subset \text{Bd}(C_1) + \text{Bd}(C_2) \subset L_1 + L_2
\]

implies together with

\[
L_1 \subset C_2, \quad L_2 \subset C_1, \quad L_1 + L_2 \subset C_1 + C_2
\]

the fact that the open set \( C_1 + C_2 \) contains its boundary. Hence it is equal to \( S \) or to 0. Since \( L_1 \) is in \( C_2, C_2 \), and a fortiori \( C_1 + C_2 \), are not empty, and in this way we have derived from the assumption \( L_2 C_1 \neq 0 \) that the space \( S \) is a sum of a finite number of circles \( S = C_1 + C_2 \). That is excluded by property VI; hence \( L_2 C_1 \neq 0 \) is wrong, \( L_2 C_1 = 0 \) is true, and that implies \( C_1 \subset C_2 \), as we know from the preceding theorem.

As an immediate formal consequence of (6.4), (6.5), and (6.6) we obtain the next theorem:

(6.7) If \( C_1 \) and \( C_2 \) are two circles (as always with center \( p \)), then at least one of the inclusions \( C_1 \subset C_2, C_2 \subset C_1 \) is true.

The next theorem states the equivalence of several other inclusion relations, which we have to use later on:

(6.8) The following properties are equivalent:

(a) \( L_x \subset C_y \) (we know that this is equivalent to \( x \in C_y \)).

(b) \( \overline{C}_x \subset C_y \), and \( C_y \) is not empty.

(c) \( L_y \subset S - \overline{C}_x \), and if \( x = p \) then \( y \neq x \).

(d) \( C_y \subset C_x \).

We show that every one of these relations implies the succeeding one and that the last implies the first.

(a) implies (b). \( L_x \subset C_y \) shows that \( C_y \) is not empty. From (6.6) we get \( C_x \subset C_y \); consequently, \( \overline{C}_x = C_x + \text{Bd}(C_x) \subset C_x + L_x \subset C_y + C_y = C_y \).

(b) implies (c). \( C_y \) is not empty; hence if \( x = p \), \( y \) is not equal to \( x \), for \( C_y \) is empty. In both cases the set \( L_y \overline{C}_x \) is invariant, and hence either 0 or \( L_y \). If it is \( L_y \), then \( L_y \subset \overline{C}_x, \overline{C}_x \subset C_y \) would yield the contradiction \( L_y \subset C_y \). Hence \( L_y \overline{C}_x = 0 \) or \( L_y \subset S - \overline{C}_x \).

(c) implies (d). In view of (6.7) let us show that \( C_y \subset C_x \) is impossible. Indeed, if \( x = p \), \( C_x \) is empty and \( C_y \subset C_x \) would make \( C_y \) empty, whereas \( y \) is...
not $x$. If $x \neq p$ and $y \neq p$, then $L_y \subset S - \overline{C}_z$, $L_y \subset \overline{C}_y \subset \overline{C}_z$ constitutes a contradiction. If $x \neq p$ and if $y = p$, $L_y \subset C_z$ would contradict the assumption $L_y \subset S - \overline{C}_z$.

(d) implies (a). From $C_y \neq C_z$ we infer that $C_z \subset C_y$, but not $C_z = C_y$, also that $C_y$ is not empty, $\overline{C}_y \supset L_y$. Consequently,

$$\overline{C}_z \subset \overline{C}_y, \quad L_z \subset \overline{C}_y = C_y + L_y.$$ 

Hence we get for $L_z$

$$L_z = L_z C_y + L_z L_y.$$ 

The set $L_z L_y$ must be empty; for in the opposite case $L_z = L_y, C_z = C_y, C_y \subset C_z$ would ensue. It follows that $L_z = L_z C_y$, which is (a).

Abstract absolute values, symbols of the form $|x|$, where $x$ is a point in $S$, and the number 0 are now introduced by the following definition:

$$|x| < |y| \text{ or } |y| > |x| \text{ shall mean } L_x \subset C_y; \quad |x| = |y| \text{ shall mean } L_x = L_y; \quad |x| = 0 \text{ shall mean } x = p; \quad |x| > 0 \text{ shall mean } x \neq p; \quad |x| \geq |y| \text{ shall mean } |x| > |y| \text{ or } |x| = |y|.$$ 

(6.10.1) For any two points $x, y$ exactly one of the relations $|x| < |y|$, $|x| = |y|$, $|x| > |y|$ is true.

Suppose that neither $|x| < |y|$ nor $|x| > |y|$ is true; in other terms, neither $L_x \subset C_y$ nor $L_y \subset C_z$ is true. On account of (6.4) we have then $L_x L_y = L_y C_z = 0$; from (6.5) we conclude $C_x \subset C_y$ and $C_y \subset C_z$; hence $C_x = C_y$, $L_z = L_y$, or $|x| = |y|$, which was to be shown.

(6.10.2) $|x| < |y|, |y| < |z|$ imply $|x| < |z|.$

We know $L_x \subset C_y$ and (cf. (6.8)) $\overline{C}_y \subset C_z$; we have a fortiori $C_y \subset C_z$; hence $L_z \subset C_z$ or $|x| < |z|$ by definition.

(6.11) $\lim_{i \to \infty} x_i = p$ is true if and only if for every $|e| > 0$ an index $i^*$ can be found such that for $i > i^*$, $|x_i| < |e|$.

For the set of all points $x$ with $|x| < |e|$ is the circle $C_z$, which is, because of the relation $|e| > 0$, a neighborhood of $p$, and must contain almost all points of any sequence which converges towards $p$.

7. Geometrical version of the lemma. The following statement is of use:

(7.1) If $|x| < |y|$, then a $z$ exists which satisfies $|x| < |z| < |y|.$

The relation $|x| < |y|$ implies, as we know, $\overline{C}_z \subset C_y$, and $C_y$ is not empty. We maintain that $C_y - \overline{C}_z$ is not empty; for otherwise the open set $C_y$, neither empty nor $S$, would be equal to the closed set $\overline{C}_z$, which is impossible.
It is also impossible that $C_y - C_x$ is equal to the one-point set $\{p\}$, for $\{p\}$ is closed and a difference "open minus closed" is open. Since $x \neq y$, the set $\{p\}$ is not equal to $S$. Hence we see that $C_y - C_x$ is not only not empty but contains a point $z$ which is not $p$. Any such $z$ will do in (7.1) because $z \in C_y$ gives $L_z \subset C_y$ and $|z| < |y|$. On the other hand, $z$ is in $S - C_z$, hence $L_z \subset S - C_z$; and if $x = p$, then $z \neq x$, for we took $z \neq p$; (6.8c) reveals this as an equivalent of $|x| < |z|$.  

(7.1.1) For every $x$ there exists a $y$ with $|y| > |x|$, if $S$ has more than one point.

If $|x| = 0$, take $y \neq p$; if $|x| > |p|$, take any point from $S - C_z$, which is not empty since $S - L_x$ is not connected.

(7.2) If $\lim x_i = x$, $\lim y_i = y$, $|x| < |y|$, then there exists an index $i^*$ such that for $i > i^*$, $|x_i| < |y_i|$.

Choose a $z$ exactly as before; then $|x| < |z|$ yields $x \in C_z$, and $|z| < |y|$ implies $y \in S - C_z$. The sets $C_z$ and $S - C_z$ are open; consequently, there exists an index $i^*$ such that for $i > i^*$

$$x_i \in C_z, \quad y_i \in S - C_z.$$  

The first formula is equivalent to $|x_i| < |z|$, the second to $|z| < |y|$ since $z \neq p$. The transitive law (6.10.2) furnishes $|x_i| < |y_i|$, which was to be proved.

We may state the following corollary:

(7.2.1) If the sequences $x_i, y_i$ are convergent, $|x_i| = |y_i|$ for all $i$ implies $\lim x_i = \lim y_i$.

[7.3] If $F$ is a (single-valued) mapping of $S$ in itself, then $S = S_1 + S_2 + S_3$, where $S_1$, $S_2$, $S_3$ in this order are defined by the relations $|F(x)| < |x|$, $|F(x)| = |x|$, $|F(x)| > |x|$.

The geometric version of Schwarz' lemma is a statement about the $S_i$ of a transformation $F$ in $N$ with $F(p) = p$. We derive first, with the aid of (7.2), a simple statement for continuous transformations.

(7.4) If $F$ in [7.3] is continuous, then $S_1$ and $S_3$ are open sets.

We prove that $S_1$ is open; the proof for $S_3$ is virtually the same.

For a point $x$ in $S_1$ we have, by definition, $|F(x)| < |x|$. Let $\lim x_i = x$; then we have to show that for almost all indices $i$, $|F(x_i)| < |x_i|$. This follows from (7.2) if we define $y = F(x)$, $y_i = F(x_i)$, and use the relation (continuity) $\lim F(x_i) = F(x)$.

Again we note without proof that $S_2$ is closed.
(7.5) If \( F \) is a continuous mapping of \( S \) in itself and if neither \( S_1 \) nor \( S_3 \) is empty, then there exist at least two points \( p, q \) with \( p \neq q \) in \( S_2 \).

This is a well known theorem about continuous functions coupled with the fact that \( S \) has no cut points. If \( S_2 \) were empty, \( S = S_1 + S_3 \) would be a non-trivial decomposition of \( S \) into two disjoint open sets, which does not exist in a connected space. If \( S_2 = \{ p \} \), then \( S_1 + S_3 \) would be a non-trivial decomposition of \( S - \{ p \} \) into open sets, and \( p \) would be a cut point of \( S \).

(7.6) Let \( F \) be in \( N \), \( F(p) = p \), such that \( S_2 \) contains \( p \). If now \( S_2 \) contains another point \( q \), \( q \neq p \), \( |F(q)| = |q| \), then \( F \) is a rotation.

For a rotation \( |F(x)| = |x| \) is identically true and \( S_1 \) and \( S_3 \) are both empty.

**Proof.** Since \( F(q) \) and \( q \) are in the same circumference, there exists a rotation \( R \) such that \( R(F(q)) = q \). What do we know about the transformation \( RF \)? The relations \( RF(p) = R(p) = p \), \( RF(q) = q \) show that \( RF \), which is in \( N \), has two different fixed points. The corollary (4.4) tells us that \( RF \) is a rotation \( R_1 \). From \( RF = R_1 \) we get \( F = R_1^{-1}R_1 \), which is a rotation since it is the product of two rotations.

(7.7) If \( F \) is in \( N \) and \( F(p) = p \), then one of the sets \( S_1 \) and \( S_3 \) is empty.

Indeed, if none were empty there would exist two different points in \( S_2 \) (cf. statement (7.5)), and \( F \) would have to be a rotation; \( S_1 \) and \( S_3 \) would be empty.

(7.8) If \( F \) is in \( N \), \( F(p) = p \), then \( S_3 \) is empty. In other words, \( |F(x)| \leq |x| \) for all \( x \).

This is the geometrical version of the Schwarz lemma.

**Proof.** If \( S_3 \) is not empty, then \( S_1 \) is empty on account of (7.7). The set \( S_2 \) is not empty, for it contains the point \( p \). It does not contain any others, for in that case \( F \) would be a rotation and \( S_3 \) would be empty (as well as \( S_1 \)). Therefore the inequality \( |F(x)| > |x| \) would hold whenever \( |x| \neq 0 \). But this contradicts the topological alternative (cf. (4.1)) that \( F \) is either a rotation or nilpotent. Indeed, \( F \) is not a rotation, and \( |F(x)| > |x| \) for all \( |x| > 0 \) is incompatible with nilpotency. For \( |x| > 0 \) implies (by mathematical induction) \( |F^n(x)| \neq 0 \) and \( |F^n(x)| \geq |x| \); and this would show that the Cauchy condition (6.11) for \( \lim F^n(x) = p \) cannot be fulfilled with \( |e| = |x| \). If our theorem were wrong, we should have a contradiction; therefore, assertion (7.8) is true.

Combining (7.6) and (7.8), we formulate the final geometrical theorem, (7.9), which corresponds to the classical Schwarz lemma together with its standard corollary.
(7.9) If $F$ is in $N$, $F(p) = p$, then for all $x$ in $S$ we have $|F(x)| \leq |x|$. If equality holds for one point distinct from $p$, then it holds throughout. In the latter case $F$ has an inverse which is an element of $N$.

The classical lemma would be a consequence provided we know that the abstract relation $|x| < |y|$ is equivalent to the analytically defined inequality $|x| < |y|$. In §8 we shall prove a theorem to the effect that the analyticity of linear homogeneous functions together with simple topological properties of the euclidean circles make the abstract and analytical order relations equivalent.

**Part III**

8. **Characterization of circumferences.** Our definitions of absolute value relations are such that if $S$ is the unit circle in the plane of the complex numbers, $p$ the origin, $L_q$ the circular curve through $q$ with center $p$, and $C_q$ the interior of the corresponding circular area, then $|x| < |y|$ is equivalent to saying that the classical absolute value of $x$ is less than the classical absolute value of $y$.

But we wish to know if the euclidean circumferences are circumferences in the sense of our definition. Of course it is well known that in the classical case an abstract rotation is an ordinary rotation; but this is usually shown as an application of the Schwarz lemma, or at least derived in an analogous fashion.

Let us therefore denote a euclidean circumference with $K$ and the corresponding circle with $E$; and let us discuss the case where $K$ contains a point $z$ but not the point $p$.

If we use the analyticity of linear homogeneous transformations, we see immediately that, $N$ being the set of all analytical mappings of $S$ in itself, $K$ is rotatory; that is, that there exists a topological mapping in $N$ which carries $p$ into itself and a preassigned $x$ on $K$ into an arbitrary $y$ on $K$. Applying some elementary topology of the euclidean plane, we can make the following assertion:

(8.1) (a) $K \subset S$ is not empty; it contains a point $z$ but not the point $p$.
(b) $S - K$ is not connected; the component of $S - K$ which contains $p$ is $E$.
(c) $K$ is the boundary $Bd(E)$ of $E$.
(d) $K$ is rotatory.
(e) $E = E + K$ is compact.

We maintain that from these statements and the properties I–VI of $N$ and I–III of $S$ it follows that $K$ is a circumference. (The case $K = \{p\}$ is trivial, since $L_p = p$.)
Let us forget the euclidean origin of (8.1) and make the following definition:

\[ 8.2 \] "\((K, E, z)\) is circular" shall mean that the sets \( K \subset S, \ E \subset S, \) and the point \( z \) satisfy the relations (8.1).

The "justification theorem" in question is now simply the following:

\[ 8.3 \] Let \( N \) and \( S \) be as in Part II. If \( (K, E, z) \) is circular, then \( K \) is a circumference and \( E \) a circle; in short \( K = L_x, \ E = C_x \).

Due to the definition (in (8.1a)) of \( E \) and [5.2] of \( C_x \) it is sufficient to show \( K = L_x \).

The proof is arranged backwards:

\[ 8.4 \] If \( (K, E, z) \) is circular and if no point of the circumference \( L_x \) is in \( E \), then \( K = L_x \).

Consider the set \( C_xE \); this set, the product of two open sets, is open. (The set \( E \) is open since \( K \), being a boundary, is closed.) The set \( C_xE \) is not empty because \( \rho \) is in \( C_x \) and in \( E \).

We study, as we always did in questions of this type, the relative boundary of \( C_xE \), this time with respect to both \( C_x \) and \( E \).

Note that \( K \) is a subset of \( L_x \), for it contains \( z \) and is rotatory. We get \( \text{Bd}(C_xE) \subset \text{Bd}(C_x) + \text{Bd}(E) \subset L_x + K \subset L_x \). Of course we cannot conclude directly that equality holds, for we do not know yet that \( K \) is invariant. But at least we can say that the relative boundaries \( C_x\text{Bd}(C_xE) \) and \( E\text{Bd}(C_xE) \) are empty. That \( C_xL_x = 0 \) follows from the definition of \( C_x \); whereas \( L_xE = 0 \) is an assumption of our theorem. Hence the relative boundaries of \( C_xE \) with respect to the (connected) sets \( C_x \) and \( E \) are empty as subsets of \( C_xL_x \) and \( EL_x \), respectively. Since \( C_xE \) is not empty, we obtain \( C_xE = C_x, \ C_xE = E; \) hence \( C_x = E \). Taking boundaries on both sides, we have \( L_x = K \), which was to be proved.

\[ 8.5 \] If \( (K, E, z) \) is circular and if \( S \) is not compact, then \( L_x \) has no point in common with \( E \).

The proof is indirect: If \( q \) is in \( L_xE \), then \( L_q = L_x \), and to every point \( x \) in \( L_q = L_x \) there will exist a rotation \( R^{(x)} \) such that \( R^{(x)}(q) = x \). The open set \( E \) is transformed into open sets \( R^{(x)}(E) \), and \( q \in E \) implies \( R^{(x)}(q) = x \in R^{(x)}(E) \).

In other terms,

\[ L_x \subset \bigcup_{z \in L_x} R^{(z)}(E). \]

Now we have to use, for the first time, the metrizability of the space \( S \). Since
$L_z$ is a compact subset (cf. (3.5)) of a metrizable space, the Heine-Pincherle-Borel-Lebesgue theorem is valid, and already a finite number of sets $R^z(E)$, say $R_1(E), \ldots, R_n(E)$ covers $L_z$; that is,

$$L_z \subset \sum_1^n R_i(E).$$

We set $S' = \sum_1^n R_i(E)$ and propose to show that $S' = S$. This is again done with the standard device based on the connectedness of $S$.

The set $S'$ is open as a sum of open sets; it is not empty because it contains $L_z$. What is its boundary? We obtain

$$\text{Bd}(S') = \text{Bd}\left( \sum_1^n R_i(E) \right) = \sum_1^n \text{Bd}(R_i(E)) = \sum_1^n R_i(\text{Bd}(E)) = \sum_1^n R_i(L_z) \subset L_z.$$

(We have applied (6.6.1), (6.2.4), $K = \text{Bd}(E)$, $K \subset L_z$, and $R_i(L_z) \subset L_z$.) Isolating the first and the last terms, we have $\text{Bd}(S') \subset L_z$; and since $L_z \subset S'$ we see that the open, non-empty set $S'$ contains its boundary; $S$ is connected, hence (cf. (6.3.1)) $S' = S$.

From $S = \sum_1^n R_i(E)$ we obtain a fortiori $S = \sum_1^n R_i(E)$. Since $(K, E, z)$ is circular, $E$ and its topological images $R_i(E)$ are compact; the sum of a finite number of compact sets is compact; hence $S$ is compact, which contradicts the assumption of the theorem. Hence we have seen, indirectly, that if $S$ is compact, $L_z = 0$, which was to be shown.

Finally, we remove, in (8.6), the last condition.

(8.6) $S$ is not compact.

For if it were, it would have to be bicom pact, being metrizable. Consider the covering which is defined by assigning to $p$ the open set $C_p$ and to every other point $x$ the circle with center $x$ determined by $p$. If $S$ were bicom pact, a finite number of these circles would have the sum $S$, which is excluded by property VI.

With (8.6) the proof of the justification theorem (8.3) is completed.

9. **Separability and local compactness of $S$.** If we use the foregoing theory for variable centers $p$, we see that every point is contained in arbitrarily small neighborhoods with compact, metrizable and hence separable boundaries. From a theorem of F. B. Jones we could infer the next theorem:

---

† A space $S$ is called bicom pact or the Heine-Pincherle-Borel-Lebesgue theorem holds in $S$ if from every covering of $S$ by open sets a finite set of elements (open sets) can be extracted which has $S$ as its sum.

(9.1) \textit{The space \( S \) is (perfectly) separable.}

This result will also appear as a corollary of the theorem (9.8). Independently from (9.1) we are going to show that the closed circles \( \overline{C}_q \) are compact, and that the space \( S \) is representable as the sum of a countable number of circles.

(9.2) \textit{Let \( x_i \) be a sequence of points such that a point \( x \), a subsequence \( x'_i \) and a sequence of rotations \( R_i \) can be found with \( \lim R_i(x'_i) = x \). Then there exists also a limit point for the sequence \( x_i \).}

We select corresponding subsequences \( R'_i, x'_i \) such that \( \lim R'_i = R \) exists; we know then that \( \lim R'_i(x'_i) = x \), and from \( \lim R'_i(x'_i) = x \) it follows that \( x'_i = R'_i (R'_i^{-1}(x'_i)) \) is convergent (with the limit \( R^{-1}(x) \)).

(9.3) \textit{Suppose that the sequence \( x_i \) is such that sequences \( x'_i, R_i \), as described in (9.2), do not exist. Then for every point \( y \) in \( S \) there exists a neighborhood \( U_y \) and an index \( i^* \) such that for \( i > i^* \), \( U_y \) is completely in \( C_{x_i} \), or completely in the exterior of \( C_{x_i} \).}

In other terms, \( i > i^* \) implies that either \( U_y \subseteq C_{x_i} \) or \( U_y \subseteq S - C_{x_i} \).

As before, we shall write \( C_i \) for \( C_{x_{i^*}} \), \( L_i \) for \( L_{x_{i^*}} \).

We first choose a neighborhood \( V_y \) and an index \( i^* \) such that for all indices \( i > i^* \), \( V_i \cap L_i = \emptyset \), and in addition for \( i > i^* \), \( C_i \neq \emptyset \). Such a \( V_y \) exists; for \( L_i \) consists exactly of the points \( R(x_i) \), where \( R \) is arbitrary. With the first countability axiom of Hausdorff (a trivial consequence of the metrizability of \( S \)) a sequence \( R_i(x'_i) \) with limit \( y \) could be constructed. If \( C'_i = \emptyset \) for a subsequence \( C'_i \), then \( x'_i = p, R_i = I \) yields \( \lim R_i(x'_i) = p \).

Since \( S \) is locally connected, \( V_y \) contains a connected neighborhood \( U_y \) of \( y \). Now consider the formula

\[
U_y = U_y C_i + U_y (S - C_i);
\]

if \( i > i^* \), then \( C_i \neq \emptyset \) and \( C_i = C_i + L_i \); hence

\[
U_y = U_y C_i + U_y L_i + U_y (S - \overline{C}_i).
\]

For \( i > i^* \), \( U_i L_i \) is \( \emptyset \); the resulting equation

\[
U_y = U_y C_i + U_y (S - \overline{C}_i)
\]

is a decomposition of \( U_y \) into two disjoint open sets. One of these must be empty, since \( U_y \) is connected and not empty; but that means that either \( U_y \subseteq C_i \) or \( U_y \subseteq S - \overline{C}_i \), which was to be shown.

(9.4) \textit{Let the sequence \( x_i \) be such that to every \( y \) in \( S \) there belongs a neighborhood \( U_y \) and an index \( i^* \) such that for \( i > i^* \) either \( U_y \subseteq C_i \) or \( U_y \subseteq S - \overline{C}_i \) is true. Then the set \( S' = \sum C_i \) (which is trivially open) is closed.}
In order to prove this we show that if \( y_i \) is a convergent sequence from \( S' \), its limit \( y \) is also an element of \( S' \).

Without loss of generality we may assume \( y_i \in C_i \); this corresponds to the deletion of some \( C \)'s and introduction of a new index, which does not affect the validity of our theorem.

Now let \( i^* \) be such that for \( i > i^* \) (a) \( y_i \in U_v \) and (b) either \( U_v \subset C_i \) or \( U_v \subset S - \overline{C}_i \).

(a) may be satisfied since \( \lim y_i = y \); (b) has been explicitly assumed. Since for \( i > i^* \), \( y_i \in U_v \), \( y_i \in C_i \), we see that \( y_i \in U_v C_i \); that decides the alternative (b) in favor of \( U_v \subset C_i \); but \( U_v \subset C_i \) (any special case such as \( i = i^* + 1 \) will do) implies \( y \in C_i \), and a fortiori \( y \in \sum C_i = S' \).

(9.5) If a sequence \( x_i \) has no limit point, then \( S = \sum C_{z_i} \) (\( = \sum C_i \)).

Since \( x_i = p \) can be true only a finite number of times, almost all \( C_i \) are not empty; a fortiori the open set \( S' = \sum C_i \) is not empty.

(9.2), (9.3), (9.4) together guarantee that \( S' \) is closed; the connectedness argument yields \( S' = S \).

(9.6) The circles \( C_q \) are limited, and their closures \( \overline{C}_q \) compact.

The proof is indirect. Let \( x_i \) be a sequence from \( C_q \). If it had no limit point, we would have \( S = \sum C_{z_i} \). On the other hand, \( x_i \in C_q \) implies \( C_{z_i} \subset C_{q} \), (cf. (6.6)) since \( C_{z_i} \neq 0 \). That would lead to the contradiction \( S \subset C_q \), since \( q \) is not in \( C_q \). Hence every sequence \( x_i \) from \( C_q \) must have a limit point, which was to be proved.

We could express and slightly generalize this in the following familiar form:

(9.6.1) If all points \( x \) of a set satisfy \( |x| \leq |q| \), then every infinite sequence \( x_i \) has a limit point.

As a consequence we have the statement:

(9.7) \( S \) is locally compact.

For if \( x \) is an arbitrary point, there exists a point \( y \) such that \( |x| < |y| \), \( x \in C_{y} \). Hence every point is contained in a limited open set.

(9.8) \( S \) is semicompact; that is, it is the sum of a sequence of compact sets \( \overline{C}_i \). (They will be closed circles.)

If \( S \) were compact (which is excluded by (8.6)), then it would be trivially semicompact. If it is not, then there exists a sequence \( x_i \) without limit points. In that case we have (cf. (9.5)) \( \sum C_{z_i} = S \) and a fortiori \( \sum \overline{C}_i = S \); and the \( \overline{C}_i \) are now known to be compact.
From the theorem (9.8) (all theorems in this section are proved without recourse to (9.1)) we get (9.1) as a trivial consequence, using metrizability.

**Conclusion.** It would be possible to obtain valuable new properties of \( S \) and \( N \) by adjunction of new postulates. We could demand that the circumferences be connected; this would permit us to conclude that for every pair \( x, y \) a center \( p \) and a rotation \( R \) exist such that \( R_p(x) = y \). If we postulate that a transformation with two fixed points is the identity, \( L_q \) would be homeomorphic to a connected compact continuous group. These groups are rather well known, and together with the fact that the abstract absolute values can be interpreted as real numbers, this additional axiom would heavily restrict the structure of the space \( S \). If, finally, \( S \) is supposed to be homeomorphic to the euclidean plane, the application of a theorem of Hilbert would show that the invertible transformations in \( N \) induce an absolute, that is, either euclidean or hyperbolic, geometry in \( S \). The decision as to whether these axioms together with a maximality axiom are categoric will largely depend, we believe, on the better understanding and proper generalization of the Schwarz lemma.

**University of California at Los Angeles, Los Angeles, Calif.**