EXPONENT TRAJECTORIES IN SYMBOLIC DYNAMICS*

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1. Introduction. Morse, Hedlund,† and others have developed the theory of dynamics from the symbolic point of view. This theory is concerned in the main with the periodicity, recurrence, and transitivity properties of symbolic trajectories and rays. Morse has made use of exponents on symbols. Unless a trajectory \( T \) is of a very special type, it can be shown that the exponents on the symbols in a symbolic trajectory \( T \) form a symbolic trajectory \( T_e \) termed the "exponent trajectory" of \( T \). The trajectory \( T_e \) is uniquely determined by \( T \). Similar considerations hold for rays. In the present paper we are concerned with relations between a trajectory or ray and the associated exponent trajectory or ray. In particular we prove that a periodic or recurrent trajectory \( T \) has a periodic or recurrent exponent trajectory \( T_e \) respectively, while a transitive ray \( R \) has an exponent ray \( R_e \) which is in a sense also transitive. Further, if a trajectory \( T \) is periodic, \( T \) is distinct from its exponent trajectory. There exist, however, trajectories identical with their exponent trajectories, and in the case of trajectories generated by the symbols 1, 2 only, there is one and only one such trajectory. The term "identical" is used here in the usual sense, and will be defined explicitly in the next section. In the paper referred to above, Morse and Hedlund have given some methods of constructing recurrent trajectories from a given recurrent trajectory. The introduction of exponent trajectories yields another method of constructing such trajectories. Whether or not there exist recurrent trajectories identical with their exponent trajectories is still an open question.

2. Definitions and conventions. We shall use the term "symbolic trajectory" in a slightly more general sense than that employed by Morse and Hedlund in that we shall allow an infinite set of generating symbols. Let \( S_1 \) denote a sequence \( abc \cdots \) of symbols \( a, b, c, \cdots \) which may or may not be taken from a finite set of distinct symbols, and let \( S_2 \) denote a second such sequence \( \alpha\beta\gamma \cdots \). Let \( S_2^{-1} \) denote the sequence \( \cdots \gamma\beta\alpha \) of symbols obtained from \( S_2 \) by reversing the order of the symbols in \( S_2 \). The sequence \( S_2^{-1}S_1 \), given by \( \cdots \gamma\beta\alpha abc \cdots \), is termed a symbolic trajectory, or simply a trajectory. The sequence \( S_1 \) (also \( S_2^{-1} \)) is termed a ray. The symbol \( a \) in \( S_1 \) is termed the initial symbol of the ray \( S_1 \). We shall have occasion to use the notation

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$S_1 = abc \cdots$ meaning that $S_1$ is the sequence $abc \cdots$. A finite sequence $ab \cdots k$ of symbols is termed a block. If there are $n$ symbols in the set $a, b, \cdots, k$, the block $ab \cdots k$ is said to be of length $n$, and will be called an $n$-block. If $B$ is a block, the length of $B$ will be denoted by $l(B)$. We shall write $B = ab \cdots k$ to indicate that $B$ is the block $ab \cdots k$. If $B_1 = a_1 \cdots a_m$, $B_2 = b_1 \cdots b_n$, then $B_1B_2$ is the block $a_1 \cdots a_mb_1 \cdots b_n$. The blocks $B_1$ and $B_2$ are the same if $m = n$ and the symbol $a_i$ is identical with the symbol $b_i$ for each $i$ in the range $1, 2, \cdots, n$. In a block $C = a_{-n} \cdots a_0a_1 \cdots a_n$ of odd length, we term $a_0$ the central symbol of $C$. A trajectory $T$ can be written as

$$\cdots a_{-2}a_{-1}a_0a_1a_2 \cdots .$$

The symbols $a_i$ and $a_j$ are said to be in different positions in $T$ if $i \neq j$. If $i = j$ these elements are in the same position in $T$. Let $a_0$ denote a symbol in a fixed position in a trajectory $T_1$. The trajectory $T_1$ is said to be identical with a trajectory $T_2$ if $T_2$ contains the symbol $a_0$ in a fixed position so that for each $n$ the block $A_n$ in $T_1$ of length $2n+1$ containing $a_0$ as central symbol is identical with the $(2n+1)$-block $B_n$ of $T_2$ containing $a_0$ as central symbol.

Sequences of consecutive symbols of a trajectory $T$ (or ray $R$ or block $B$) which form a block or ray we term a subblock or subray of $T$ (or $R$ or $B$), and they are said to be contained in $T$ (or $R$ or $B$). As remarked above the symbols in a trajectory $T$ (or ray $R$ or block $B$) are taken from a finite or infinite set $S$ of distinct symbols, which will be termed the generating symbols of $T$ (or $R$ or $B$). A block $a \cdots a$ formed by repeating the symbol $a$ $n$ times is written as $a^n$. The symbol $n$ in $a^n$ is termed the exponent of $a$ in $a^n$, and $a$ is termed the base in $a^n$. We term $a^n$ a power. We write a block $B$ as a sequence of powers such that the bases in consecutive powers are distinct. The exponents then form the exponent block $B_e$ of $B$. Unless a trajectory $T$ contains a subray formed by only one generating symbol, $T$ can be written as a sequence

$$(1) \quad \cdots a^pb^qc^r \cdots ,$$

where no two consecutive bases are identical. The exponents in (1) form a trajectory $\cdots pqr \cdots$, which we term the exponent trajectory $T_e$ of $T$. Similarly, if a ray $R$ does not contain a subray formed by one generating symbol, the ray $R$ can be written as $a^pb^qc^r \cdots$, where consecutive bases are distinct. The exponents then form the exponent ray $R_e$ of $R$. A trajectory $T$ (or ray $R$) will be termed admissible if it has an exponent trajectory (or ray); that is, $T$ (or $R$) does not contain a subray of the form $aaa \cdots$ or $\cdots aaa$.

A trajectory $T$ is periodic if it can be written as a sequence

$$(2) \quad \cdots BBB \cdots$$
of blocks identical with a block \( B \). If \( B \) is a block of shortest length such that \( T \) can be written as (2), the block \( B \) is said to be a period block of \( T \), and its length is termed the period of \( T \). A trajectory \( T \) is termed recurrent if for each \( n \) there exists an \( m \) such that each block of length \( n \) in \( T \) is contained in each \( m \)-block of \( T \). If \( T \) is recurrent, for each \( n \) there exists a least \( m \) such that each \( m \)-block of \( T \) contains each \( n \)-block of \( T \). We write \( R(n) = m \), and term \( R(n) \) the recurrency function of \( T \). A ray \( R \) is said to be transitive if every possible block that can be formed from the generating symbols of \( R \) is a sub-block of \( R \).

3. Periodicity, recurrence, and transitivity of exponent trajectories. We shall now prove the following theorem.

**Theorem 1.** If a trajectory \( T \) in two or more generating symbols is periodic, \( T \) is admissible and the exponent trajectory \( T_e \) is periodic.

Let \( B \) represent a period block of \( T \) so that \( T \) is given by (2). Suppose that \( B \) begins with the symbol \( a \) and is preceded by \( a \) in \( T \). Then \( B \) is of the form \( a^b c^d \cdots c^a u \), where no two consecutive symbols in the set \( a, b, \cdots, c, a \) are identical. The block \( C = a^w b^c \cdots c^t \), where \( w = u + r \), is then also a period block of \( T \). The block \( C_e = w s \cdots t \) thus occurs in \( T_e \), and \( T_e \) is of the form \( \cdots C_e C_e C_e \cdots \), whence the theorem is proved.

**Theorem 2.** The exponent trajectory \( T_e \) of an admissible periodic trajectory \( T \) is distinct from \( T \).

As noted above \( T \) contains a period block \( C = a^w b^c \cdots c^t \), where \( a \neq c \), and the exponent block \( C_e = w s \cdots t \) of \( C \) is a subblock of \( T_e \). Evidently \( C_e \) or a subblock of \( C_e \) is a period block of \( T_e \). The period of \( T \) is \( \omega = w + s \cdots + t \). The period of \( T_e \) is no greater than the length \( L \) of \( C_e \). If at least one of the symbols in \( C_e \) is greater than 1, we have \( \omega > L \). If all of the symbols in \( C_e \) equal 1, the period of \( T_e \) is 1 and certainly less than \( \omega \).

Morse and Hedlund* have exhibited a nonperiodic recurrent trajectory \( T \) in four symbols with the property that consecutive symbols in \( T \) are distinct. If follows that in this case \( T_e \) is of the form

\[
\cdots 111 \cdots
\]

Since (3) is periodic, there exist nonperiodic trajectories whose exponent trajectories are periodic. That this is not true of trajectories with two generating symbols is stated in the theorem which follows.

**Theorem 3.** An admissible trajectory \( T \) with two generating symbols is periodic if and only if its exponent trajectory \( T_e \) is periodic.

* See the reference to Morse and Hedlund above, p. 844.
Let the generating symbols be denoted by 1, 2. Let the period of \( T_e \) be denoted by \( \xi \), and a period block of \( T_e \) by \( B_e = a_1 \cdots a_\xi \). Let \( B \) be a block of \( T \) with exponent block \( B_e \). If \( \xi \) is even, the first and last symbols of \( B \) are distinct, for \( B = 1a_12a_1a_3 \cdots 2a_\xi \) or \( 2a_1a_1a_3 \cdots 1a_\xi \). Hence \( T \) is given by (2), and \( T \) is periodic. If \( \xi \) is odd, the first and last symbols of \( B \) are identical. It follows that \( T \) is given by

\[
\cdots B_1B_2B_3B_4B_5\cdots,
\]

where \( B_1 = 1a_12a_1a_3 \cdots 1a_\xi \), \( B_2 = 2a_1a_1a_3 \cdots 2a_\xi \). Hence \( T \) is periodic.

**Theorem 4.** If the exponent trajectory \( T_e \) of a periodic trajectory \( T \) in two generating symbols has the period block \( a_1a_2 \cdots a_\xi \), the trajectory \( T \) has the period \( \omega \), where

\[
\omega = \sum_{j=1}^{\xi} a_j,
\]

\[
\omega = 2 \left( \sum_{j=1}^{\xi} a_j \right),
\]

according as \( \xi \) is even or odd.

From the proof of Theorem 3 it follows in the case where \( \xi \) is even that the trajectory \( T \) is given by (2), where \( B = 1a_12a_1a_3 \cdots 2a_\xi \) or \( 2a_1a_1a_3 \cdots 1a_\xi \). Hence \( \omega \leq (a_1+ \cdots + a_\xi) \). It is no restriction to suppose that \( B = 1a_12a_1a_3 \cdots 2a_\xi \). If \( B \) is not a period block, a subblock \( 1a_12a_1a_3 \cdots 2a_j \), \( j < \xi \), of \( B \) is a period block of \( T \). Then \( T_e \) is given by \( \cdots B'_eB'_eB'_e\cdots \), where \( B'_e = a_1a_2 \cdots a_j \). The trajectory \( T_e \) thus has a period less than \( \xi \), which is impossible. It follows that (5) is valid.

It follows from the proof of Theorem 3 that if \( \xi \) is odd the period of \( T \) is not greater than \( 2(a_1+a_2+ \cdots + a_\xi) \), and that \( T \) is given by (4). By the argument of the preceding paragraph the period of \( T \) cannot be less than \( (a_1+a_2+ \cdots + a_\xi) \). Hence \( T \) has the period block \( 1a_12a_1a_3 \cdots 1a_\xi 2a_1a_3 \cdots 2a_j \) or the equivalent block with the symbols 1 and 2 interchanged. Thus \( T_e \) is given by \( \cdots B''_eB''_eB''_e\cdots \), where \( B''_e = a_1a_2 \cdots a_4a_1a_2 \cdots a_j \). Since \( \xi \) divides the length of \( B_e \), we have \( j = \xi \), whence (6) is valid.

From Theorem 4 it is evident that the number of periodic trajectories of period \( \omega \) with two generating symbols is the number of solutions of

\[
\sum_{j=1}^{2n} a_j = \omega, \quad 2 \left( \sum_{j=1}^{2n+1} a_j \right) = \omega,
\]

where the \( a \)'s and \( n \) are integers, and the blocks \( a_1a_2 \cdots a_\xi \) (\( \xi = 2n, 2n+1 \)) are not of the form \( DD \cdots D \), that is, formed by the repetition of a block.
Lemma 1. A recurrent trajectory $T$ with two or more generating symbols is admissible.

Since $T$ contains a block $ab$, where $a$ and $b$ are distinct, and this block cannot be contained in a subray with one generating symbol, it follows that each exponent is finite, and the exponent trajectory $T_e$ exists.

Lemma 2. If an admissible trajectory $T$ is recurrent, its exponent trajectory contains a finite number of generating symbols.

Consider again a subblock $ab$ of $T$ where $a \neq b$. If there exists in $T$ a sequence of blocks $a_1^{n_1}, a_2^{n_2}, \ldots$, where the sequence $n_1, n_2, \ldots$ is unbounded, then there exists an arbitrarily long block which does not contain $ab$. Hence $T$ is not recurrent. Thus Lemma 2 is proved.

Theorem 5. If an admissible trajectory $T$ is recurrent, the exponent trajectory $T_e$ of $T$ is recurrent.

Consider a block $B_e = ps \cdots q$ of $T_e$. There is a corresponding block $B = a^b c^d$ of $T$ bordered on the left and right by symbols $g$ and $h$ respectively, where $g \neq a$ and $h \neq c$. Since $T$ is recurrent, the block $gBh$ occurs in each block of $T$ of length $R(n)$, where $n$ is the length of $gBh$, and $R(n)$ is the recurrency function of $T$. Thus in each subblock $B'$ of $T$ of length $R(n)$ there occurs a block $g^aB^\beta h$, where $\alpha \geq 1$, and $\beta \geq 1$. Each block $B'$ is contained in a block $B''$, where $B''$ is preceded in $T$ by a symbol distinct from the first symbol of $B''$, and followed by a symbol distinct from the last symbol of $B''$, and the exponent block of $B''$ has the same length as the exponent block of $B'$. Evidently, $B_e$ is contained in the block of exponents of each block $B''$. Let $t$ be the maximum length of the exponent blocks of the blocks of type $B''$. We denote the exponent block of a block $B''$ by $B''_{e}$. Each exponent block $C_e$ in $T_e$ of length $t$ corresponds to a block $C$ of $T$ which contains a block $B''$ as subblock. It follows that each block of $T_e$ of length $t$ contains $B_e$. Let $r$ denote the length of $B_e$. There are a finite number of blocks $B_{e1}, B_{e2}, \ldots, B_{ep}$ in $T_e$ of length $r$. There exist numbers $t_1, t_2, \ldots, t_p$ such that for each $i$ ($i = 1, 2, \ldots, p$) $B_{ei}$ is contained in each $t_i$-block of $T_e$. Let $R_e(r)$ denote the maximum of the numbers $t_1, t_2, \ldots, t_p$. Then each $r$-block of $T_e$ is contained in each $R_e(r)$-block. Thus $T_e$ is recurrent.

Corollary 1. If $T$ is a recurrent nonperiodic trajectory in two generating symbols, the exponent trajectory $T_e$ of $T$ is a recurrent nonperiodic trajectory.

It is obvious that a non-recurrent trajectory $T$ may have a recurrent exponent trajectory $T_e$. It is necessary even in the case of two generating symbols to impose an additional restriction on $T_e$ to insure the recurrence of $T$. We shall give the additional restriction for the case of two generating symbols.
We say that a trajectory $T$ in two generating symbols is strongly recurrent if for each $n$ and $n$-block $B$ in $T$ there exists an integer $R(n)$ such that if $B_1, B_2$ are any nonoverlapping blocks of length $R(n)$, the block $B_1$ contains a block $B$ whose first symbol is separated from the first symbol of a block $B$ in $B_2$ by an odd number of symbols. An immediate result is the following theorem.

**Theorem 6.** An admissible trajectory $T$ in two generating symbols is recurrent if and only if its exponent trajectory is strongly recurrent.

Certain inequality relations exist between the recurrency function of a recurrent trajectory $T$ and the recurrency function of the exponent trajectory $T_*$ of $T$. For the sake of brevity these relations will be omitted.

That the following theorem is true appears from the definition of transitivity.

**Theorem 7.** A transitive ray in two or more generating symbols is admissible.

**Theorem 8.** The exponent ray $R_*$ of a transitive ray $R$ in two or more generating symbols is transitive.

It is evident that $R_*$ has the infinite set $1, 2, 3, \cdots$ of generating symbols. We denote this set by $S$. Let $l, m, n, \cdots, p$ be an arbitrary subset of $S$ containing $\mu$ symbols not necessarily distinct, and let $q, r, s, \cdots, t$ be a second subset of $\mu$ symbols in $S$ not necessarily distinct. By assumption $R$ contains at least two distinct generating symbols $a, b$. Since $R$ is transitive, $R$ contains the block $\alpha B \gamma$, where

$$B = \alpha a_1 a_2 \cdots a_q a_{q+1} \cdots a_{q+r} a_{q+r+1} \cdots a_{q+r+s}$$

$$\cdots \cdots a_{q+r+s+t+u+1} \cdots a_{q+r+s+t+u+1},$$

$\alpha \neq a_1, \gamma \neq a_{q+r+s+t+u+1}$, the exponent block of $B$ is $B_* = \ell m n^* \cdots p^*$, and the $a$'s are alternately equal to $a$ and $b$ so that $a_1 = a$, $a_2 = b$, $a_3 = a$, $\cdots$. Thus $R_*$ contains each block $B_*$ that can be formed from the symbols in $S$, whence $R_*$ is transitive.

Theorem 8 can be extended to “transitive trajectories” with no subray generated by one symbol only.

4. **A trajectory identical with its exponent trajectory.** In Theorem 2 we noted that a periodic trajectory is distinct from its exponent trajectory. That this is not true for trajectories in general is a consequence of the theorem which follows.

**Theorem 9.** There exists a trajectory identical with its exponent trajectory.

We let $B_0$ denote the block 212, and let $B_1 = 2$. We form the trajectory
where $B_i$ is the exponent block of $B_{i+1}$ for each $i > 0$, the last symbol of $B_i$ is distinct from the first symbol of $B_{i+1}$ for each $i > 0$, and $B_i^{-1}$ denotes the block obtained from $B_i$ by reversing the symbols in $B_i$. We illustrate by giving some of the blocks $B_i$ explicitly:

$$B_2 = 11, B_3 = 21, B_4 = 221, B_5 = 22112, B_6 = 1122121.$$

We note that for $i > 0$ the block $B_i^{-1}$ is the exponent block of $B_i^{-1}$. Thus (7) is the sequence

$$
\cdots, 21122, 122, 12, 11, 2, 212, 2, 11, 21, 221, 22112, \cdots,
$$

where we have separated the blocks $B_i$ and $B_i^{-1}$ by commas. The exponent block of $B_i^{-1}B_0B_1$ is $B_6$. From this statement and the definition of (7), it appears that the exponent block of $B_1^{-1} \cdots B_0^{-1}B_1B_2 \cdots B$, is the block $B_1^{-1} \cdots B_0^{-1}B_1B_2 \cdots B_{r-1}$. Thus (7) has an exponent trajectory and is identical with it.

Employing the same technique as that used in constructing (7) and using more than two symbols, one can construct an unlimited number of trajectories identical with their exponent trajectories. We shall prove later the uniqueness of (7) for the class of trajectories in two generating symbols 1, 2.

5. Proper exponent blocks and join-blocks in trajectories with generating symbols 1, 2. Consider an arbitrary subblock $B$ of a trajectory $T$ in generating symbols 1, 2 where the exponent trajectory $T_e$ of $T$ contains the same generating symbols. The block $B$ has an exponent block $B_e$ which does not necessarily occur as a subblock of the exponent trajectory $T_e$ of $T$ since $B$ may be preceded by or followed by a symbol identical with the first or last symbol of $B$ respectively. For this reason we associate with $B$ a new type of exponent block. Consider the block $B_e$ of exponents of $B$ which occur in $T_e$ and can be determined without reference to $T$ from $B$ alone and the fact that the exponents equal 1 or 2. We term $B_e$ the proper exponent block of $B$. We similarly speak of a proper exponent ray. We let $C_1, C_2$ be consecutive subblocks of the trajectory $T$ so that $C_1C_2$ is a subblock of $T$. We denote the proper exponent blocks of $C_1$ and $C_2$ by $D_1$ and $D_2$ respectively. The proper exponent block of $C_1C_2$ is a block $D_2JD_2$. We shall say that $J$ is the exponent block due to the join of $C_1$ and $C_2$. Obviously, $J$ is either vacuous, or is one of the blocks 1, 2, or 11.

**Theorem 10.** Let $T_e$ be the exponent trajectory of a trajectory $T$, and suppose that $T$ and $T_e$ have the same generating symbols 1, 2. The length of the proper exponent block $B_e$ of a block $B$ in $T$ satisfies the formula
If $B \neq \alpha$, $\alpha^2$ ($\alpha = 1, 2$). If $B$ has an intermediate block $1^2$ or $2^2$, then

(10) \[ L(B_e) \leq L(B) - 3. \]

In any case $L(B_e) \leq L(B) - 1$. We write $B = a_1 a_2 a_3 \cdots a_n$, $n \geq 2$, where the $a_i$'s are distinct and alternate between 1 and 2. Obviously, $L(B_e) = n - 2$, and $L(B) \geq n$. If $B = a_1^2 a_2^1 a_3^1 \cdots a_n^1$, then $L(B_e) = n - 1$, $L(B) = n + 2$. Thus (9) is valid. The validity of (10) is obvious.

Theorems 11–13 to follow will be needed in a later section.

Theorem 11. Let $T_{ee}$ and $T_e$ be the exponent trajectories of trajectories $T$, and $T$ respectively, and suppose that $T$, $T_{ee}$, and $T_e$ have the generating symbols 1, 2. Let $JED$ be a subblock of $T$, and suppose that the blocks $J$, $E$, and $D$ are so related that $E$ is the proper exponent block of $D$, while $J$ is the exponent block due to the join of $E$ and $D$. If $L(D) \geq 4$, then

(11) \[ L(JE) < L(D). \]

We write $D = GH$, where $G$ is a block of length 4. We let $J_e$ denote the exponent block due to the join of $G$ and $H$, and let $G_e$, $H_e$ denote the proper exponent blocks of $G$ and $H$ respectively. We have the following relations:

(12) \[ L(JE) = L(J) + L(E), \]

(13) \[ L(E) = L(G_e) + L(J_e) + L(H_e). \]

We consider first the case where $G$ begins with the block $\alpha^2$. By the assumption $L(G) = 4$ we have $G \neq \alpha$, $\alpha^2$, whence by Theorem 10 the relation $L(E) \leq L(D) - 2$ follows. Since $D$ begins with $\alpha^2$, the block $J$ contains no exponent arising from $D$. Hence $J$ is vacuous or 1, whence $L(J) \leq 1$. It follows by (12) that (11) is valid.

Next, we suppose that $G$ begins with $\alpha \beta$ ($\alpha \neq \beta$). If $G = \alpha \beta \alpha \alpha$, then $G_e = 12$. If $H$ is vacuous, $E = 12$ and $ED = 12 \alpha \beta \alpha \alpha$. If $\alpha = 1$, the proper exponent block of $12 \alpha \beta \alpha \alpha$ contains a subblock $1^3$, which is impossible in view of the fact that $T_{ee}$ contains only the symbols 1, 2. Hence $\alpha = 2$, and $J = 2$. Thus $L(JE) = 3$, and (11) holds. If $H$ is not vacuous, the block $GH$ begins with $\alpha \beta \alpha \alpha \beta$ since $T_e$ contains only the generating symbols 1, 2. By Theorem 10, $L(E) \leq L(D) - 3$. Since $L(J) \leq 2$, formula (11) is valid. We now let $G = \alpha \beta \beta \alpha$. Since we have an intermediate block $\beta^2$, by Theorem 10 we have $L(E) \leq L(D) - 3$, whence (11) holds. If finally $G = \alpha \beta \beta \beta$, $G$ is preceded in $T$ by $\alpha$ since we cannot have a block $1^3$ in $T_e$. Then $J = 2$, and $L(J) = 1$. By Theorem 10 we have $L(E) \leq L(D) - 2$, whence (11) holds. Thus in any case (11) is valid.
Theorem 12. Let $T_e$ be the exponent trajectory of a trajectory $T$, and let $T$ and $T_e$ be trajectories in the generating symbols 1, 2. Let $JED$ be a subblock of $T$, where $J$, $E$, and $D$ are related as in Theorem 11. If $L(D) \geq 4$, then $L(JE) \geq 2$.

If the leading 4-block of $D$ is of the form $\alpha\alpha\beta\beta$, $\alpha\beta\alpha\alpha$, or $\alpha\beta\beta\alpha$ ($\alpha \neq \beta$), the proper exponent block of this block is of length 2, whence $L(JE) \geq 2$. If the leading 4-block of $D$ is of the form $\alpha\beta\beta\alpha$, this block has the proper exponent block 2, whence $L(E) \geq 1$. The leading symbol $\alpha$ of $D$ will yield an exponent in $J$. Thus in any case $L(JE) \geq 2$.

Theorem 13. Let $T$, $T_e$, $J$, $E$, and $D$ be defined as in Theorem 12. If $J$ is non-vacuous, then $E$ is non-vacuous.

6. Subrays of a trajectory identical with its exponent trajectory. The theorem which follows is valid for trajectories based on an arbitrarily given set of generating symbols, and is not restricted to the 1, 2 case.

Theorem 14. If a trajectory $T$ is identical with its exponent trajectory $T_e$, the trajectory $T$ does not contain two identical subrays $R_1$, $R_2$ with initial elements in different positions in $T$.

Suppose that the rays $R_1$, $R_2$ are directed to the right in the sense that $R_1 = R_2 = abc \ldots$. The rays $R_1$ and $R_2$ overlap, whence it is no restriction to suppose that $R_1$ overlaps $R_2$. Let the subblock of $R_1$ which precedes $R_2$ in $R_1$ be denoted by $B$. Since $R_1 = R_2$, the ray $R_2$ contains a subray $R_3$ identical with $R_2$ and preceded in $R_2$ by the block $B$. Thus $T$ contains the subray $N = BBB \ldots$. Since $T = T_e$, the trajectory $T_e$ contains a subray $N_e$ identical with the ray $N$. Let $N_e$ denote the proper exponent ray of $N$. The rays $N_1$ and $N_e$ overlap in $T_e$. Therefore the ray $N_e$ contains a subray $N_2$ identical with $N$. Clearly, $N_2$ is the exponent ray of a subray $N_3 = B_1B_1B_1 \ldots$ of $N$ where $B$ is the exponent block of $B_1$. Since the ray $N_3$ is a subray of the ray $N$, and $l(B_1) \geq l(B)$, we can write $B_1 = B_{11}B'B_{12}$, where $B_{11}B_{11} = B$, and $r \geq 0$. If $r = 0$ it is understood that the block $B'$ is vacuous. Thus the trajectory $T_1 = \ldots B_1B_1B_1 \ldots$ obtained by continuing $N_3$ to the left is identical with the trajectory $T_2 = \ldots BBB \ldots$. But $T_2$ is the exponent trajectory of $T_1$, whence by Theorem 2 we have arrived at a contradiction.

7. The uniqueness of a trajectory identical with its exponent trajectory in the case of generating symbols 1, 2. We shall prove in this section that the trajectory (7) is the only one of its kind for trajectories in generating symbols 1, 2. We let $T^{-1}$ denote the trajectory obtained from a trajectory $T$ by reversing the order of the symbols in $T$.

Lemma 3. If a trajectory $T$ is identical with its exponent trajectory $T_e$, and $T$
contains the generating symbols 1, 2 only, the trajectory \( T \) or \( T^{-1} \) contains a subray

\[
R = B'_1 B'_2 B'_3 \ldots ,
\]

where \( B'_i \) is the exponent block of \( B'_{i+1} \), and the last symbol of \( B'_i \) is different from the first symbol of \( B'_{i+1} \) for each \( i \).

Let \( a \) denote a symbol of \( T \) in a fixed position in \( T \). The corresponding symbol \( a \) of the exponent trajectory \( T_e \) is the exponent of a symbol \( b \) in \( T \) so that the block \( b^a \) occurs in \( T \). It is no restriction to assume that the block \( b^a \) is not to the left of the symbol \( a \) in \( T \). We suppose first that the block \( b^a \) of \( T \) does not contain the symbol \( a \), so that \( b^a \) is to the right of \( a \) in \( T \).

We let \( B'_i \) denote the block of symbols in \( T \) starting with \( a \) and ending with the symbol preceding the block \( b^a \) in \( T \). Since \( T = T_e \), the symbol \( a \) in \( T_e \) is the initial symbol of a block \( B'_i \) in \( T_e \). The block of \( T \) starting with \( b^a \) and having \( B'_i \) as exponent block is unique since consecutive exponents in \( T \) are exponents on distinct bases alternating between the symbols 1, 2. We emphasize that \( T \) is of the form

\[
\ldots a^{-2}a^{-1}a^2a^{-1}a^{-2} \ldots .
\]

We denote the block of \( T \) starting with \( b^a \) and having exponent block \( B'_i \) by \( B'_i \). Thus \( T \) contains the block \( B'_i B'_i \). We assume now that \( T \) contains the block \( B'_i B'_i \cdots B'_i \) where \( B'_i B'_i \cdots B'_{i-1} \) is the exponent block of \( B'_i B'_i \cdots B'_i \). Since \( T = T_e \), the block \( B'_i B'_i \cdots B'_{i-1} \) in \( T_e \) is followed by \( B'_i \), whence \( B'_i B'_i \cdots B'_i \) in \( T \) is followed by a block \( B'_{i+1} \) whose exponent block is \( B'_i \), and the first symbol of \( B'_{i+1} \) is distinct from the last symbol in \( B'_i \). Thus \( T \) contains the subray \( R \).

Finally, we suppose that the block \( b^a \) of \( T \) contains the symbol \( a \) of \( T \). If \( a = 1 \), then \( b^a \) is the block \( 1^1 \). Since the bases alternate between 1 and 2, the block \( b^a \) is preceded and followed in \( T \) by the base 2. Thus \( T \) contains the block \( B_0 = 2a2 = 212 \), where \( B_0 \) is the block \( B_0 \) occurring in (7). Since \( T = T_e \), the symbol \( a \) in \( T_e \) is preceded and followed by 2 in \( T_e \), whence \( a \) is the central symbol in a block \( B_0 \) of \( T_e \). It follows that \( B_0 \) is the exponent block of a block \( B_1^{-1}B_0B_1 \) in \( T \) with central symbol \( a \) and \( B_1 = 2 \). Making use of the equality \( T = T_e \) and developing \( T \) to the right and left of \( B_1^{-1}B_0B_1 \) as in §4 we obtain (7). The subray

\[
B_1B_2B_3 \cdots
\]

of (7) is clearly a subray of the type (14). If now \( a = 2 \), the symbol \( a \) in \( T \) is either the leading or final symbol in the block \( b^a \), so that \( b^a \) is either \( a2 \) or \( 2a \). If \( b^a = a2 \), then since \( a = 2 \), the block \( b^a \) is preceded in \( T \) by the symbol 1,
and thus this block is preceded in $T_*$ by the symbol 1. Thus the block $1^1$ precedes $b^a$ in $T$, and since both base and exponent in $1^1$ are followed by $a$ in $T$ and $T_*$ respectively, the base and exponent in the power $1^1$ are corresponding symbols. The argument thus reduces to the preceding case where $a=1$. If now $a=2$, while $b^a=2a$, the block $b^a$ is followed in $T_*$ by the symbol 1, so that the block $b^a$ in $T$ is followed by the power $1^1$. Clearly the base and exponent in this power are corresponding symbols, whence we have again reduced the argument to the case where $a=1$. Thus $T$ contains the subray $14$, and Lemma 3 is proved, for if $b^a$ is to the left of $a$ in $T$, then $b^a$ is to the right of $a$ in $T^{-1}$.

We remark that the exponent ray $R_*$ of $B_1' B_2' B_3' \cdots$ in (14) is the ray $R$. We consider now a trajectory $T$ with $T=T_*$, whence by the lemma just proved $T$ contains the subray $R$ of (14). We let $E_0$ be a block such that the ray $E_0 B_1' B_2' \cdots$ is the proper exponent ray of $R$ in (14). The block $E_0$ may be vacuous. Since $T=T_*$, the trajectory $T$ contains (16) as a subray. We let $J_0$ denote the exponent block due to the join of $E_0$ and $B_1'$ in $E_0 B_1'$. It is clear that $T$ contains the subray $J_0 E_0 B_1' B_2' \cdots$. For $i>0$, we let $E_i$ denote the proper exponent block of a block $J_{i-1} E_{i-1}$, and $J_i$ the exponent block due to the join of $E_i$ and $J_{i-1} E_{i-1}$. In the following lemma we use $G_i$ to denote the block $J_i E_i \cdots J_{i-1} E_{i-1} E_0$, and $R$ as in (14). Here $G_{i+1}=G$, if $J_{i+1} E_{i+1}$ is vacuous.

**Lemma 4.** If a trajectory $T$ is equal to its exponent trajectory $T_*$, and $T$ contains the subray

\[(17) \quad G_i R,\]

the trajectory $T$ contains the subray

\[(18) \quad G_{i+1} R,\]

where $J_{i+1} E_{i+1}$ in $G_{i+1}$ may be vacuous.

We assume that $T$ contains the subray (17). The proper exponent ray of (17) is the ray

\[(19) \quad E_{i+1} G_i R.\]

Since $T=T_*$, the trajectory $T$ contains the subray (19). Evidently the proper exponent ray of (19) contains the subray (18).

**Theorems 11 and 12 yield at once the following lemma.**

**Lemma 5.** If the subray (17) in a trajectory $T$ with $T=T_*$ is continued to the left, one arrives at a block $J_i E_i$ of length 2 or 3, provided the trajectory $T$ contains a subblock $J_i E_i$ of length at least 2.
Lemma 6. If a trajectory $T$ in generating symbols 1, 2 is identical with its exponent trajectory $T_e$, the trajectory $T$ or $T^{-1}$ contains a subray identical with the subray (15) of (7).

By Lemma 3, the trajectory $T$ or $T^{-1}$ contains a subray (14). Suppose that $T$ contains (14). We continue the subray (14) of $T$ to the left to obtain a subray (17) of $T$ where $E_{i+1}$ is vacuous. We suppose first that the subray (17), which is explicitly the ray

$$J_iE_i \cdots J_1E_1J_0E_0B_1' B_2' B_3' \cdots$$

contains a block $J_iE_i$ of length at least 2, whence by Lemma 5 the ray (17) contains a subblock $J_iE_i$ of length 2 or 3.

We assume that $L(J_iE_i) = 2$. If $J_iE_i = 22$, then $E_{i+1} = 2$, and $E_{i+1}J_iE_i$ contains the subblock $2^3$. Hence $J_iE_i \neq 22$. Suppose that $J_iE_i = 12$. We cannot have $J_i = 12$, since by Theorem 13 the block $E_i$ is then not vacuous. Also, we cannot have $E_i = 12$ since the symbol 2 in $E_i$ yields an exponent in $J_i$ due to the join of $E_i$ with the block following $E_i$ in $T$, whence $J_i$ is not vacuous. Thus $J_i = 1$, and $E_i = 2$. Then $J_iE_i$ is followed by a block 122 in $T$. Now $J_iE_i122$ has the proper exponent block 112 = $J_iE_i$. Since $T = T_e$, the trajectory $T$ contains the subray $R_a = 112G_{a-1}R_a$, where if $a = 0$, we understand that $G_{a-1} = G^{-1}$ is vacuous. The leading block 11 in $R_a$ has exponent 2, whence $T$ contains the subray $R_a = KGGR_a$, where $K = 21$. The proper exponent ray of $R_b$ is $R_b$ itself. We write $B_1$ for the symbol 2 in $K$, and $B_2$ for the block 11 in $KJ_i$. The exponent of $B_2$ in $R_b$ is the initial symbol of the proper exponent ray of $R_b$. If we define $B_3$, $B_4$, $\cdots$ as in §4, it is clear that $R_b$ is identical with the ray (15). If $J_iE_i = 11$, then $E_{i+1} = 2$. Writing $B_1 = B_{i+1}, B_2 = J_iE_i$, defining $B_i$ ($i > 2$) as in §4, and using the fact that (19) is the proper exponent ray of (17), we find that in this case (17) with $i = \sigma + 1$ is identical with the subray (15) of (7). Finally, we write $J_iE_i = 21$. If $E_i = 21$, the symbol 1 in $E_i$ yields an exponent so that $J_i$ is not vacuous. Hence $J_i = 2, E_i = 1$. Now $J_iE_i$ is followed in $T$ by the block 121. Writing $B_1$ for $J_i$, and $B_2$ for the block 11 which follows $J_i$ in $J_iE_i121$, and defining $B_i$ ($i > 2$) as in §4, we find that in this case (17) with $i = \sigma$ is identical with (15).

If $J_iE_i = 221$, then $E_{i+1} = 2$ and $E_{i+1}J_iE_i = 2^5$ which is impossible. If $J_iE_i = 211$, then $E_{i+1} = 2, J_{i+1} = 2$, which in the paragraph above was proved impossible. If $J_iE_i = 212$, then $E_{i+1} = J_{i+1} = 1$, which case was treated above. If $J_iE_i = 121$, then $E_{i+1} = 1, J_{i+1} = 2$, which was also treated above. If $J_iE_i = 112$, then $E_{i+1} = 2$, and $J_{i+1}$ is vacuous. Writing $B_1 = E_{i+1}$, and $B_2$ for the leading block 11 of $J_iE_i$, and defining $B_i$ ($i > 2$) as in §4, it is clear that (17) with $i = \sigma + 1$ is in this case identical with (15). If $J_iE_i = 122$, then

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\[ J_{\sigma+1} E_{\sigma+1} = 12, \] which case was treated above. This completes the cases where \( L(J_\sigma E_\sigma) = 3 \), since the blocks 111 and 222 cannot occur in \( T \).

Suppose now that \( T \) contains no block \( J_\sigma E_\sigma \) with \( L(J_\sigma E_\sigma) \geq 2 \). Assume that \( T \) contains a subblock \( J_\sigma E_\sigma \) with \( L(J_\sigma E_\sigma) = 1 \). We cannot have \( J_\sigma E_\sigma = 1 \), whence \( E_\sigma = 1 \), since \( E_\sigma \) is the proper exponent block of 121 or 212, and the block \( E_\sigma 121 \) or \( E_\sigma 212 \) yields a non-vacuous block \( J_\sigma \) due to the join of \( E_\sigma \) with 121 or 212. If \( J_\sigma E_\sigma = 2 \), then \( E_\sigma = 2 \), and \( E_\sigma \) is followed by 11 in \( T \). Writing \( B_1 = J_\sigma E_\sigma, B_2 = 11 \), and defining \( B_i \) \((i > 2)\) as in \$4\), we find that \( T \) contains the subray (15).

We suppose, finally, that \( T \) contains no subray (17) with a block \( J_\sigma E_\sigma \) for which \( L(J_\sigma E_\sigma) \geq 1 \). Thus the subray (14) of \( T \) cannot be continued to the left. The block \( B'_i \) cannot be of length greater than or equal to 3 since then \( B'_i \) would yield a non-vacuous block \( E_0 \). In the same way \( B'_i \neq 11, 22 \). If \( B'_i = 12 \), then \( B'_i = 122 \), and \( E_0 = 1 \), whereas if \( B'_i = 21 \), then \( B'_i = 221 \), and \( E_0 = 1 \). Thus \( L(B'_i) = 1 \). If \( B'_i = 1 \), then \( B'_i = 2, B'd = 11 \). In this case dropping the first block in (14), we obtain the subray (15) of (7) as a subray by writing \( B_i = B'_i + 1, i \geq 1 \). If \( B'_i = 2 \), writing \( B_i = B'_i \) we obtain (15) from (14).

Thus in any case the trajectory \( T \) or \( T^{-1} \) contains the subray (15) of (7).

**THEOREM 15.** There is one and only one trajectory \( T \) in generating symbols 1, 2 identical with its exponent trajectory.

By Theorem 14 and Lemma 6 the trajectory \( T \) or \( T^{-1} \) contains the subray (15) of the trajectory (7) exactly once. Suppose that (15) is a subray of \( T \). If (15) is preceded by the symbol 2 in \( T \), the subray \( \rho_1 = 2B_1 B_2 \cdots \) of \( T \) must be preceded by the symbol 1, since no block \( 2^k \) can occur in \( T \). Thus \( T \) contains the subray \( \rho_2 = 12 B_1 B_2 \cdots \). The proper exponent ray of \( \rho_1 \) is \( \rho_1 \) itself. In particular since the proper exponent ray \( \rho_1 \) of \( \rho_2 \) is preceded by the symbol 1, the ray \( \rho_2 \) is preceded in \( T \) by the symbol 2, so that \( 212 B_1 B_2 \cdots \) occurs in \( T \). We write \( B_0 = 212 \), whence the ray \( \rho_3 = B_0 B_1 B_2 \cdots \) occurs in \( T \). Since \( \rho_3 \) occurs in \( T_\varepsilon \), the ray \( \rho_4 = 2212 B_1 B_2 \cdots \) occurs in \( T \), whence \( B^{-1}_1 B_0 B_1 B_2 \cdots \) occurs in \( T \). By induction, since

\[
B_r^{-1} \cdots B_1^{-1} B_0 B_1 B_2 \cdots
\]

occurs in \( T \), the same ray occurs in \( T_\varepsilon \), and is the exponent ray of the ray

\[
B_r^{-1} \cdots B_1^{-1} B_0 B_1 B_2 \cdots .
\]

Thus \( T \) is identical with (7).

If, on the other hand, the subray (15) is preceded by 1 in \( T \), the trajectory \( T \) contains either the subray \( R_1 = 21 B_1 B_2 \cdots \) or the subray \( R_2 = 11 B_1 B_2 \cdots \). The proper exponent rays of \( R_1 \) and \( R_2 \) are respectively \( R_2 \) and \( R_1 \). Since
T = T_*$, the trajectory $T$ then contains both $R_1$ and $R_2$. Since $R_1 \neq R_2$, the sub-ray (15) in $R_1$ and $R_2$ occurs twice in $T$, contradicting Theorem 14.

If $T$ is the trajectory (7), then $T = T^{-1}$. Thus Theorem 15 is proved.

**Theorem 16.** The trajectory (7) is the exponent trajectory of two distinct trajectories in generating symbols 1, 2.

Theorem 16 states that (7) is not symmetric in the symbols 1, 2. Suppose, on the contrary, that (7) is unchanged when we interchange the symbols 1 and 2. Let $C_i$ be the block obtained from $B_i$ by interchanging 1 and 2 in $B_i$. Then we have the trajectory

$$\cdots C_2^{-1}C_1^{-1}C_0C_1C_2 \cdots .$$

We remark that the block $C_i$, $i \geq 2$, has the exponent block $B_{i-1}$, whence $C_i^{-1}$ has the exponent block $B_{i-1}^{-1}$. The block $C_1^{-1}C_0C_1$ has the exponent block $B_0$. We note that the symbol 2 in $C_1^{-1}C_0C_1 = 11211$ yields the exponent 1 in $B_0$. Now the ray $C_1^{-1}C_0C_1C_2 \cdots$ has the proper exponent ray $\sigma = B_0B_1B_2 \cdots$. If the trajectory (20) is identical with the trajectory (7), the trajectory (20) contains a subray $\sigma' = B_1^{-1}\sigma$ with proper exponent ray $\sigma$, where the symbol 1 in the subblock $B_0$ of the exponent ray $\sigma$ is the exponent of the symbol 1 in the subblock $B_0$ of the ray $\sigma'$. Thus the exponent trajectory (7) of (20) contains the subray $\sigma$ twice with initial symbols of each $\sigma$ in different positions in (7). By Theorem 14 we have arrived at a contradiction.

Although (7) is the exponent trajectory of two distinct trajectories $T_1$ and $T_2$ in generating symbols 1, 2, the trajectories $T_1$ and $T_2$ are equivalent in the sense that these trajectories differ only in the notation used for the generating symbols.

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