ON THE GROWTH PROPERTIES OF A FUNCTION OF
TWO COMPLEX VARIABLES GIVEN BY ITS
POWER SERIES EXPANSION

by

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1. Introduction. One of the most fundamental formulas in the theory of
functions of one complex variable is the Cauchy integral formula. It is of par-
ticular value in the Weierstrass-Hadamard approach, i.e., in obtaining prop-
erties of a function from the coefficients of its power series expansion. A
similar formula cannot be obtained for functions of two complex variables
for an arbitrary four-dimensional domain, as is obtained, for instance, for the
bicylinder, where the integration is taken over a two-dimensional surface on
the boundary. Bergman(1) has shown, however, that for certain domains far
more general than those previously considered, i.e., domains bounded by a
finite number of analytic hypersurfaces, an analogous formula does exist, the
double integral being taken essentially over the two-dimensional surface com-
mon to two or more of the analytic bounding hypersurfaces(2).

In this paper we shall obtain growth properties in terms of the coefficients
of the power series expansion of a function \( f(z_1, z_2) \) of two complex variables
analytic in special domains of the type mentioned above; first, with the aid
of Bergman's integral formula, along the two-dimensional surfaces common to
the bounding hypersurfaces, and then, along a class of two-dimensional sur-
faces lying in only one of the bounding hypersurfaces and having a line of
contact with another bounding hypersurface. We also obtain a mapping theo-
rem which determines from the coefficients a convex region in the \( \mathbb{F}^2 \)-plane,
\( f(z_1, z_2) = f_1 + i f_2 \), which must be contained in the smallest convex region of the
mapping on the \( \mathbb{F}^2 \)-plane of the surfaces considered.

2. Properties of \( f \) associated with \( G^2(r) \). Let us consider a finite four-di-
mensional domain \( \mathbb{M}^4 \) which is bounded by the hypersurfaces

\[
\begin{align*}
S_1(r) & = E[z_2 = re^{\lambda_1}, \ 0 \leq \lambda_1 \leq 2\pi], \\
S_2(r) & = E[z_1 = re^{\lambda_2} + p(\lambda_2)z_2 = h(\lambda_2, z_2), \ 0 \leq \lambda_2 \leq 2\pi],
\end{align*}
\]

and which depends on a positive parameter \( r \); \( p(\lambda_2) \) is assumed merely to have

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(1) Bergman [2, 3]. See the bibliography at the end of this paper.

(2) Bergman calls such surfaces “distinguished boundary surfaces.”
a first derivative. Let \( G^2(r) \) be the two-dimensional surface on the boundary of \( W^4 \) which is the common part of the bounding hypersurfaces, i.e.,

\[
G^2(r) = s_1 \cdot s_2.
\]

**Theorem I.** Given a function \( f(z_1, z_2) = \sum_{m,n=0}^{\infty} a_{mn} z_1^m z_2^n \) regular in the domain \( \Omega^4(r) \); if \( M(r) \) is the maximum-modulus of \( f(z_1, z_2) \) on \( G^2(r) \), then

\[
M(r) \geq \max_{m,n} \frac{r^{m+n}}{G(m,n;p) B(p)},
\]

where \( m \) and \( n \) range over all non-negative integral values, \( B(p) \) is a constant depending upon \( p \), and \( G(m, n; p) \) is a function of \( m, n, \) and \( p \), given by

\[
1 + \int_0^{n+1} \left(1 + x \frac{1 + \log m}{m} \right) \max |\lambda n | \frac{m}{\log p} - \frac{m}{1 + \log m},
\]

when \( \max |\lambda n | < 1, m \geq 1 \),

\[
1 + \int_0^{n+1} \left(1 + x \frac{1 + \log m}{m} \right) \max |\lambda n | \frac{m}{\log p} - \frac{m}{1 + \log m},
\]

when \( \max |\lambda n | \geq 1, m \geq 1 \),

\[
1 - \max |\lambda n | ^{n+1} \frac{1}{1 - \max |\lambda n |},
\]

when \( m = 0 \) for all \( p \).

**Proof of Theorem I.** Keeping \( z_2 \) constant, say equal to \( t_2 \), we obtain for a particular value of \( z_1 \), say \( t_1 \),

\[
f(t_1, t_2) = \frac{1}{2\pi i} \int_0^{2\pi} \frac{f[h(t_1, t_2), t_2][\lambda t_1 + p'(\lambda t_2)t_2]}{[(e^{i\lambda t_1} + p(\lambda t_2)) - t_1][e^{i\lambda t_1} - t_1]} d\lambda_1.
\]

Since the numerator of the integrand is an analytic function of \( t_2 \), we again apply the Cauchy integral formula and obtain

\[
f(t_1, t_2) = \frac{1}{(2\pi i)^2} \int_0^{2\pi} \int_0^{2\pi} \frac{f[h(t_1, re^{i\lambda}), re^{i\lambda}]}{[(e^{i\lambda (t_1 + p(\lambda t_2))} - t_1)[e^{i\lambda (t_1 - t_2)}]}
\]

\[
\cdot [e^{i\lambda t_1} + p'(\lambda t_2)re^{i\lambda}] e^{i\lambda t_1} d\lambda_1 d\lambda_2.
\]

For the \( m \)th derivative of \( f(t_1, t_2) \) with respect to \( t_1 \), we obtain

\[
\frac{\partial^m f(t_1, t_2)}{\partial t_1^m} = \frac{m!}{(2\pi i)^2} \int_0^{2\pi} \int_0^{2\pi} \frac{f[h(t_1, re^{i\lambda}), re^{i\lambda}]}{[(e^{i\lambda (t_1 + p(\lambda t_2))} - t_1)[e^{i\lambda (t_1 - t_2)}]}
\]

\[
\cdot [e^{i\lambda t_1} + p'(\lambda t_2)re^{i\lambda}] e^{i\lambda t_1} d\lambda_1 d\lambda_2.
\]

Let

\[
H_1 = (re^{i\lambda t_1} + p(\lambda t_2) - t_1), \quad H_2 = re^{i\lambda t_1} - t_2.
\]
For the $n$th derivative of $1/H_1^{m+1}H_2$ with respect to $t_2$, we obtain by Leibniz' rule

$$\frac{m!n!}{(2\pi i)^2} \int_0^{2\pi} \int_0^{2\pi} \left[ \frac{(m + \nu)!}{m!\nu!} \left( \frac{H_2}{H_1} \right)^n \right] d\lambda_1 d\lambda_2. \tag{2.8}$$

Hence we obtain for $\frac{\partial^{m+n} f(t_1, t_2)}{\partial t^m \partial t^n}$ the expression

$$\frac{m!n!}{(2\pi i)^2} \int_0^{2\pi} \int_0^{2\pi} \frac{f[h, \lambda_1][ire^{\lambda_1} + \rho'(\lambda_2)re^{\lambda_1}][ire^{\lambda_1}]}{[(re^{\lambda_1} + \rho(\lambda_2)t_2 - t_1)^{m+1}[re^{\lambda_1} - t_2]^{n+1}}} \cdot \left[ 1 + \sum_{r=1}^{n} \frac{(m + \nu)!}{m!\nu!} \left( \frac{H_2}{H_1} \right)^n \right] d\lambda_1 d\lambda_2. \tag{2.9}$$

Now

$$a_{mn} = \frac{\partial^{m+n} f(0, 0)}{m!n! \partial t^m \partial t^n}. \tag{2.10}$$

Hence

$$a_{mn} = \frac{1}{(2\pi i)^2} \int_0^{2\pi} \int_0^{2\pi} \frac{f[h, \lambda_1][ire^{\lambda_1} + \rho'(\lambda_2)re^{\lambda_1}][ire^{\lambda_1}]}{r^{m+n+2} \exp \left\{ i(m\lambda_2 + n\lambda_1 + \lambda_2 + \lambda_1) \right\}} \cdot \left[ 1 + \sum_{r=1}^{n} \frac{(m + \nu)!}{m!\nu!} (\rho'(\lambda_1 - \lambda_2))^{r} \right] d\lambda_1 d\lambda_2. \tag{2.11}$$

Taking the absolute value of $a_{mn}$ we get

$$|a_{mn}| \leq \frac{1}{4\pi^2} \frac{M(r) \max_{0 \leq \lambda_1 \leq 2\pi} \left[ 1 + |\rho'(\lambda_2)| \right]}{r^{m+n}} \cdot \left[ 1 + \sum_{r=1}^{n} \frac{(m + \nu)!}{m!\nu!} (\max \rho) \right] 4\pi^2. \tag{2.12}$$

Now for $m \geq 1$, it can be shown that

$$1 + \sum_{r=1}^{n} \frac{(m + \nu)!}{m!\nu!} \max |\rho(\lambda_2)|^r \leq 1 + \frac{1 + \log m}{m} \cdot \rho^r. \tag{2.13}$$

When $|\rho| < 1$ we have

$$1 + \sum_{r=1}^{n} \frac{(m + \nu)!}{m!\nu!} \max |\rho(\lambda_2)|^r \leq 1 + \int_1^{m} \left[ 1 + x \frac{1 + \log m}{m} \right] \max |\rho|^r dx - \frac{m}{\log \rho} - \frac{m}{1 + \log m},$$

and when $|\rho| \geq 1$.\pagebreak
(2.14) \[ 1 + \sum_{r=1}^{n} \frac{(m + \nu)!}{m!\nu!} \max |p|^r \leq \int_{0}^{\max|p|^{n+1}} \left[ 1 + x - \frac{1 + \log m}{m} \right] \max |p|^x dx. \]

When \( m = 0 \), \((1 - \max|p|^n)/(1 - \max|p|)\) is the exact value of the left-hand side of (2.14) for all \( p \).

Therefore for all differentiable \( p(\lambda_2) \) and non-negative integral values of \( m \) and \( n \) we have

\[
(2.15) \quad |a_{mn}| \leq \frac{M(r)B(p)}{m+n} G(m, n; p),
\]

where \( B(p) = \max_{0 \leq \lambda_2 \leq 2\pi} (1 + |p'|) \), or

\[
(2.16) \quad M(r) \geq \frac{m+n}{B(p)G(m, n; p)} |a_{mn}|.
\]

To find those values of \( m \) and \( n \), say \( \mu(r) \) and \( \nu(r) \), for which the right-hand expression in (2.16) is maximum for a given \( r \), we take the logarithm of the expression, letting \(-\log|a_{mn}| = g_{mn}\) and employ a generalized Newton polygon method. Then

\[
(2.17) \quad g_{mn} - (m + n) \log r + \log B + \log G(m, n) \geq g_{\mu\nu} - (\mu + \nu) \log r + \log B + \log G(\mu, \nu) = C.
\]

We choose \( m, n, \) and \( g_{mn} \) as the \( x-, y-, \) and \( z-\)axes, respectively, and plot the points \((m, n, g_{mn})\). Then the \( m \) and \( n \) of the first point which lies in the surface \( z = x \log r + y \log r - \log G(x, y) - \log B + k \) as this surface is translated along the \( z \)-axis from \(-\infty\) by varying \( k \), i.e., until \( k = C \), are the \( \mu \) and \( \nu \) which give the right-hand side of (2.16) a maximum. If there is more than one point lying on the surface, the one with the smaller \( m \) is chosen; if the \( m \)'s are the same, the one with the smaller \( n \) is chosen. \( \mu \) and \( \nu \) are obviously functions of \( r \).

We then have

\[
(2.18) \quad M(r) \geq \frac{r^{n+s}|a_{\mu\nu}|}{BG(\mu, \nu)}.
\]

This gives a lower bound for the growth of \( f(z_1, z_2) \) along the hypersurface \( g^2 = S_{r=r_0} G^2(r) \), where \( r \) varies continuously.

3. The mapping of the surface \( G^2(r) \). Let us introduce the function

\[
(3.1) \quad F(f, \alpha) = e^{-i\alpha f(z_1, z_2)} = \sum_{r, s=0}^{\infty} A_{rs} z_1^r z_2^s,
\]

where \( 0 \leq \alpha \leq 2\pi \) and \( f \) is defined as in the previous sections. The coefficients \( \{A_{rs}\} \) are functions of \( \alpha \) and a combination of the \( a_{mn} \)'s such that \( m \leq r \) and \( n \leq s \). We define the region \( R^2(r) \) as the product of the half-planes.
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(3.2) \[ f_1 \cos \alpha + f_2 \sin \alpha \leq Q(\alpha, r), \quad 0 \leq \alpha < 2\pi, \]

where \( f_1 \) and \( f_2 \) are cartesian coordinates in the \( f_1f_2 \)-plane, and

(3.3) \[ Q(\alpha, r) = \log | A_{\mu}(\alpha, \alpha) | + (\mu + \nu) \log r - \log G(\mu, \nu) - \log B. \]

**Theorem II.** Let \( f(z_1, z_2) = f_1 + i f_2 \). Then the smallest convex domain enclosing the mapping of \( G^2(r) \) on the \( f_1f_2 \)-plane contains the closed convex region \( R^2(r) \) which depends only on the coefficients of the expansion of \( f(z_1, z_2) \) and the surface \( G^2(r) \).

This gives a lower bound, so to speak, of the mapping of \( G^2(r) \) on the \( f_1f_2 \)-plane.

**Proof of Theorem II.** Let

(3.4) \[ P(r) = \max |e^{-iaf(z_1,z_2)}| \]

on the surface \( G^2(r) \); then from (3.4) and (2.18)

\[
\log P(r) = \log | \exp \{ e^{-iaf^*(z_1,z_2)} \} | \\
= \log | \exp \{ (f_1^* \cos \alpha + f_2^* \sin \alpha) \} | \\
\cdot | \exp \{ -i(f_1^* \sin \alpha - f_2^* \cos \alpha) \} | \\
= f_1^* \cos \alpha + f_2^* \sin \alpha \\
\geq \log | A_{\mu}(\alpha, \alpha) | + (\mu + \nu) \log r - \log G(\mu, \nu) - \log B = Q(\alpha, r),
\]

where the * indicates that value of \( f \) which gives \( |P| \) its maximum, for a given \( \alpha \). Now, for each \( \alpha \), \( Q(\alpha, r) \) has a fixed value (depending on \( r \)). It is clear from (3.6) that at least one point of the mapping, namely, \( (f_1^*, f_2^*) \), will lie in the half-plane

(3.7) \[ f_1 \cos \alpha + f_2 \sin \alpha \geq Q(\alpha, r). \]

The region \( R^2(r) \) will therefore be contained in the smallest convex domain containing the mapping of \( G^2(r) \) on the \( f_1f_2 \)-plane. Theorem II is then proved.

It is clear that a similar theorem will hold for any surface for which we have a lower bound for the maximum of the function \( f(z_1, z_2) \) on the surface. For example, we can state similar theorems for the surfaces considered in §§4 and 5.

4. **Further properties of the function on other surfaces of the type \( G^2(r) \).**

Let us consider the finite four-dimensional region \( \mathbb{R}^4(r) \) bounded by the three infinite hypersurfaces:

\[
\begin{align*}
3_s(r) & = E[z_2 = re^{\alpha_1}, \ 0 \leq \lambda_1 \leq 2\pi], \\
3_s^2(r) & = E[z_1 = re^{\alpha_2} + C_2z_2, \ 0 \leq \lambda_2 \leq 2\pi], \\
3_s^3(r) & = E[z_1 = re^{\alpha_3} - C_3z_2, \ 0 \leq \lambda_3 \leq 2\pi],
\end{align*}
\]
where, as above, \( r \) is a parameter and \( C_2 \) and \( C_3 \) are positive constants less than unity. This restriction on \( C_2 \) and \( C_3 \) is necessary in order that the hypersurfaces of (3.1) form the boundary of a finite closed domain. Let \( G^2_{22}(r) \) be that part of \( S^2_{22}(r) \cdot S^2_{2} \( which belongs to the boundary of \( \mathcal{M}^4 \). Now let

\[
G^2_{22}(r) = G^2_{12}(r) + G^2_{13}(r) + G^2_{23}(r).
\]

Let also \( g^2_{22} = S^2_{22} G^2_{22}(r) \), and \( g^2_{23} = S^2_{22} G^2_{23}(r) \), where \( r \) varies continuously and \( r_1 < \infty \).

Let \( f(z_1, z_2) \), as before, be an analytic function regular in \( \mathcal{M}^4 \). We now apply Bergman’s integral formula\(^*\) for functions of two complex variables which states that at a point \((t_1, t_2)\) in \( \mathcal{M}^4 \),

\[
f(t_1, t_2) = \frac{1}{2} \sum_{k,s} M_{ks}(t_1, t_2)
\]

\[
= \frac{1}{2(2\pi)^2} \sum_{k,s} \iint_{B^2_{ks}} f(\phi_{ks}^{(1)}, \phi_{ks}^{(2)}) B_{ks}(t_1, t_2, \lambda_k, \lambda_s) d\lambda_k d\lambda_s,
\]

\[
B_{ks}(t_1, t_2, \lambda_k, \lambda_s) = \frac{Z_{ks}(t_1, t_2, \lambda_k, \lambda_s)}{(\phi_{ks}^{(1)} - t_1)(\phi_{ks}^{(2)} - t_2)}, \quad k \neq s,
\]

\[
Z_{ks}(t_1, t_2, \lambda_k, \lambda_s) = \frac{D(\phi_{ks}^{(1)}, \phi_{ks}^{(2)})}{D(\lambda_k, \lambda_s)} \left[ \Phi_k(t_1, t_2, \lambda_k) \Phi_s(t_1, t_2, \lambda_s) \right.\]

\[
- \Phi_k(t_1, t_2, \lambda_k) \Phi_s(t_1, t_2, \lambda_s) \bigg],
\]

where \( B^2_{ks} \) is the surface range of integration. We have in our case

\[
\Phi_1 = z_2 - re^{\alpha_1},
\]

\[
\Phi_2 = z_1 - re^{\alpha_2} - C_2 z_2,
\]

\[
\Phi_3 = z_1 - re^{\alpha_3} + C_3 z_2;
\]

\[
\begin{cases}
\phi_{12}^{(1)} = re^{\alpha_2} + C_2 re^{\alpha_1}, \\
\phi_{12}^{(2)} = re^{\alpha_3}, \\
\phi_{13}^{(1)} = re^{\alpha_2} - C_2 re^{\alpha_1}, \\
\phi_{13}^{(2)} = re^{\alpha_3}.
\end{cases}
\]

\[
\begin{cases}
\phi_{22}^{(1)} = \frac{1}{C_2 + C_3} [C_2 re^{\alpha_1} + C_3 re^{\alpha_1}], \\
\phi_{22}^{(2)} = \frac{1}{C_2 + C_3} [re^{\alpha_2} - re^{\alpha_1}];
\end{cases}
\]

\(^*\) Bergman [2, p. 97] and [3, p. 861].
and consequently,

\[(4.5) \quad f(t_1, t_2) = M_{12}(t_1, t_2) + M_{13}(t_1, t_2) + M_{23}(t_1, t_2)\]

\[= \frac{1}{(2\pi i)^2} \int \int_{B_{12}^2} f(\phi_{12}, \phi_{13})(r e^{i(\lambda_1 + \lambda_2)}) d\lambda_1 d\lambda_2\]

\[+ \frac{1}{(2\pi i)^2} \int \int_{B_{12}^2} f(\phi_{13}, \phi_{13})(r e^{i(\lambda_1 + \lambda_2)}) d\lambda_1 d\lambda_3\]

\[+ \frac{1}{(2\pi i)^2} \int \int_{B_{23}^2} f(\phi_{23}, \phi_{23})(r e^{i(\lambda_1 + \lambda_2)}) d\lambda_2 d\lambda_3.\]

As in §2 we have that

\[(4.7) \quad a_{mn} = \frac{\partial^{m+n} f(0, 0)}{m! n! \partial t_1^m \partial t_2^n} = \frac{\partial^{m+n} [M_{12}(t_1, t_2) + M_{13}(t_1, t_2) + M_{23}(t_1, t_2)]}{m! n! \partial t_1^m \partial t_2^n} \bigg|_{t_1, t_2=0},\]

\[(4.8) \quad \frac{\partial^{m+n} M_{12}(t_1, t_2)}{\partial t_1^m \partial t_2^n} = \frac{m!}{(2\pi i)^2} \int \int_{B_{12}^2} \frac{\partial^n}{\partial t_2^n} \left[ f(\phi_{12}, \phi_{12})(r e^{i(\lambda_1 + \lambda_2)}) \right] d\lambda_1 d\lambda_2\]

so that

\[(4.9) \quad \frac{\partial^{m+n} M_{12}(0, 0)}{\partial t_1^m \partial t_2^n} = \frac{m! n!}{(2\pi i)^2} \int \int_{B_{12}^2} f(\phi_{12}, \phi_{12})(r e^{i(\lambda_1 + \lambda_2)}) d\lambda_1 d\lambda_2\]

This yields, by a process analogous to that used in §2,

\[(4.10) \quad \frac{1}{m! n!} \left| \frac{\partial^{m+n} M_{12}(0, 0)}{\partial t_1^m \partial t_2^n} \right| \leq \frac{B_{12}(G_{12}(m, n))}{M^{m+n}},\]

where $M(r)$ is the maximum-modulus of $f$ on $G^2(r)$, $B(G_{12})$ is a constant depending on the hypersurface $G_{12} \subset \Sigma_{12}$, $G_{12}(m, n)$ is a function of $m$ and $n$, also depending on $G_{12}$ and is defined in a way similar to $G(m, n)$ of §2.

In the same way we obtain

\[(4.11) \quad \frac{1}{m! n!} \left| \frac{\partial^{m+n} M_{13}(0, 0)}{\partial t_1^m \partial t_2^n} \right| \leq \frac{B_{13}(G_{13})}{M^{m+n}}.\]
From (4.3), we have

\[ \Phi_2(t_1, t_2) = (t_1 - re^{\alpha_1} - C_3 t_2), \quad \Phi_3(t_1, t_2) = (t_1 - re^{\alpha_2} + C_3 t_2). \]

Hence

\[ \frac{\partial^m}{\partial t_1^m} \left[ \frac{1}{\Phi_2 \Phi_3} \right] = (-1)^m m! \sum_{r=0}^{m} \frac{1}{\Phi_{2,2}^{r+1} \Phi_{3,3}^{m-r+1}}, \]  

and

\[ \frac{\partial^m}{\partial t_1^m} \left[ \frac{1}{\Phi_2 \Phi_3} \right] = (-1)^{m+n} m! \sum_{r=0}^{n} \sum_{\mu=0}^{n} \frac{(m+n-v-\mu)! C_2^{\mu} C_3^{n-\mu}}{(m-\mu)! \Phi_{2,2}^{r+1} \Phi_{3,3}^{m+n-v-\mu+1}} \]

\[ \cdot \sum_{r=0}^{m} \frac{(m+n-v-\mu)! (C_2 \Phi_3)^\mu}{(m-v)!} \]  

Then

\[ \frac{1}{m! n!} \left| \frac{\partial^{m+n} \frac{M_{23}(0, 0)}{C_3}}{\partial t_1^m \partial t_2^n} \right| \leq \frac{1}{4\pi^2} \int_{B_{23}} \int_{B_{23}} |f(\Phi_{23}^{(1)}, \Phi_{23}^{(2)})| \sum_{\mu=0}^{n} \frac{1}{(n-\mu)! (C_3)^\mu} \]

\[ \cdot \sum_{r=0}^{m} \frac{(m+n-v-\mu)! d\lambda_2 d\lambda_3}{(m-v)!} \]

\[ \leq \frac{B_{23}(\mu)}{r^{m+n}} \sum_{\mu=0}^{n} \frac{1}{(n-\mu)! (C_3)^\mu} \]

\[ \cdot \sum_{r=0}^{m} \frac{(m+n-v-\mu)!}{(m-v)!} \]  

The constant \( B_{23}(\mu) \) is given by \( (1/4\pi^2) \int_{B_{23}} d\lambda_2 d\lambda_3 \), where the precise limits of integration are obtained by a tedious process and can be omitted here since they are not necessary for our purpose; it may be noted, however, that \( 0 < B_{23} < 1 \). We shall denote by \( G_{23}(m, n) \) the expression

\[ \sum_{\mu=0}^{n} \frac{1}{(n-\mu)! (C_3)^\mu} \sum_{r=0}^{m} \frac{(m+n-v-\mu)!}{(m-v)!} \]  

This gives

\[ |a_{mn}| \leq \frac{1}{m! n!} \left( \left| \frac{\partial^{m+n} M_{13}(0, 0)}{\partial t_1^m \partial t_2^n} \right| + \left| \frac{\partial^{m+n} M_{13}(0, 0)}{\partial t_1^m \partial t_2^n} \right| + \left| \frac{\partial^{m+n} M_{23}(0, 0)}{\partial t_1^m \partial t_2^n} \right| \right) \]

\[ \leq \frac{M(r)}{r^{m+n}} \frac{1}{2} \sum_{k,s=1}^{3} B_{23}(g_{ks}) G_{23}(m, n). \]
Therefore
\[(4.19) \quad M(r) \geq \frac{r^{m+n} |a_{mn}|}{\frac{1}{2} \sum_{k=1}^{\infty} B(k)G_k(\mu, \nu)}.
\]

Those values of \(\mu\) and \(\nu\) which make the right-hand side of (4.19) a maximum can be obtained by a process similar to that employed in §2. Hence
\[(4.20) \quad M(r) \geq \frac{r^{\mu+\nu} |a_{\mu\nu}|}{\frac{1}{2} \sum_{k=1}^{\infty} B(k)G_k(\mu, \nu)}.
\]

We can then state

**Theorem III.** Given a function
\[f(z_1, z_2) = \sum_{m,n=0}^{\infty} a_{mn} z_1^m z_2^n\]
regular in the closed domain \(\overline{D}(r)\); then along
\[g = \sum_{r=r_0}^{r_1} G(r) = \sum_{r=r_0}^{r_1} (G_{12}(r) + G_{13}(r) + G_{23}(r)), \]
\[M(r) \geq \frac{r^{\mu+\nu} |a_{\mu\nu}|}{\frac{1}{2} \sum_{k=1}^{\infty} B(k)G_k(\mu, \nu)}.
\]

5. **Properties for the function on certain classes of surfaces lying in the boundary and different from \(G^2(r)\).** We next wish to consider the growth of the function \(f(z_1, z_2)\) over a special class of surfaces \(H^2(r)\) belonging to the boundary of \(\overline{D}(r)\). Let
\[(5.1) \quad H^2(r) = E[z_1 = \zeta(r, \lambda_1, \sigma), z_2 = re^{\lambda_1}, \lambda_1^{(1)} \leq \lambda_1 \leq \lambda_1^{(2)}, \sigma_1 \leq \sigma \leq \sigma_2] \]
where for all \(\sigma\) satisfying \(\sigma_1 \leq \sigma < \sigma_2\), and for any fixed \(\lambda_1\) in the range considered,
\[\zeta(r, \lambda_1, \sigma) \in \mathcal{S}_2(r, \lambda_1), \]
\[\mathcal{S}_2(r, \lambda_1) = A_2^2(r, \lambda_1) \cdot A_2^2(r, \lambda_1); \]
\[A_2^2(r, \lambda_1) = E[|z_1 - C_2z_2| \leq r, \ z_2 = re^{\lambda_1}], \]
\[A_2^2(r, \lambda_1) = E[|z_1 + C_2z_2| \leq r, \ z_2 = re^{\lambda_1}] ; \]
and for \(\sigma = \sigma_2\), with \(\lambda_1\) again fixed,
\[\zeta(r, \lambda_1, \sigma_2) \in \mathcal{S}_1(r, \lambda_1), \]
where \(\mathcal{S}_1(r, \lambda_1)\) is the boundary of \(\mathcal{S}_2(r, \lambda_1)\). It will be assumed that the set of all points of \(H^2(r)\) for which \(\lambda_1\) has an arbitrary fixed value in the range consid-
ered is a continuous curve $h^I(r)$ with an initial point $z_1 = \xi(r, \lambda_1, \tilde{\sigma}_1(\lambda_1))$, and a terminal point on $s^I_0(r, \lambda_1)$. The surface $H^I(r)$ lies completely in that part of $s^I_0(r, \lambda_1)$ which belongs to the boundary of the $\mathfrak{M}^I(r)$ of the previous section. A portion of the boundary of $H^I(r)$ lies on $G^I(r)$ of (4.2).

Let the maximum-modulus of $f(z_1, z_2)$ on $H^I(r)$ be $\gamma(r)$. We now map (using for simplicity the same notation for the mapped region) $\mathfrak{M}^I$ into the unit circle so that $z_1 = 0$ goes into itself and the direction of the real axis, $\mathfrak{R}(z_1) = 0$, at the point $z_1 = 0$, remains unchanged. The curve $h^I(r)$ maps into a segment of a continuous curve, its initial point determined by $\sigma = \tilde{\sigma}_1(\lambda_1)$ and its terminal point lies on the unit circle. Now let $\theta = |z_1| = |\xi(r, \lambda_1, \tilde{\sigma}_1(\lambda_1))|$ for $\lambda_1^{(1)} \leq \lambda_1 \leq \lambda_1^{(2)}$. The quantities $\theta$ and $\alpha = \lambda_1^{(2)} - \lambda_1^{(1)}$ were introduced by Bergman and are the characteristic numbers of the surface $^6$.

One form of the Milloux theorem is $^6$: Let $J$ be a continuous finite arc lying in the unit circle $|z| \leq 1$ joining a point $z_0$ within the circle to a point on the boundary. Let $W(z)$ be regular, single-valued, and $|W(z)| < 1$ inside the unit circle, and let $|W(z)| \leq \omega$ on $J$. Then

$$|W(0)| < \omega^{(2/\pi)} \sin^{-1} \left(1-\theta'/1+\theta'\right),$$

where $\theta' = |z_0|$.

Using this theorem for the mapped region $\mathfrak{M}^I$ with

$$W(z_1) = \frac{f(z_1, z_2^*)}{M(r)},$$

we have

$$|W(z_1)| = \frac{|f(z_1, z_2^*)|}{M(r)} \leq \frac{\gamma(r)}{M(r)} < 1,$$

and get, letting $\Theta = (2/\pi) \sin^{-1} (1-\Theta)/(1+\Theta)$,

$$|f(0, z_2)| < M^{1-\Theta} \gamma^\Theta,$$

where $M(r)$ is the maximum-modulus of $f(z_1, z_2)$ in $\mathfrak{M}^I$, $\lambda_1^*$ is an arbitrarily chosen value of $\lambda_1$ in the range considered, and $z_2 = z_2^* = r e^{i\lambda_1^*}$.

Now

$$a_{0n} = \frac{1}{2\pi i} \left\{ \int_0^{\lambda_{1n}} \frac{f(0, re^{i\lambda_1})}{r^{n e^{i\lambda_1}}} \, d\lambda_1 + \int_{\lambda_{1n}}^{\lambda_{1n}^{(2)}} \frac{f(0, re^{i\lambda_1})}{r^{n e^{i\lambda_1}}} \, d\lambda_1 \right.$$

$$+ \int_{\lambda_{1n}^{(2)}}^{2\pi} \frac{f(0, re^{i\lambda_1})}{r^{n e^{i\lambda_1}}} \, d\lambda_1 \right\},$$

$^4$ The restriction that $h^I(r)$ be continuous is not essential since theorems of the Milloux type hold for more general one-dimensional sets.

$^6$ Bergman [1, pp. 347–348, Corollary]; and [4, pp. 200–201].

$^6$ R. Nevanlinna [5].
\[
|a_{0n}| \leq \frac{1}{2\pi} \left\{ \int_{\lambda_1^{(1)}} |f(0, re^{i\lambda})| \frac{1}{r^n} d\lambda_1 + \int_{\lambda_1^{(1)}} |f(0, re^{i\lambda})| \frac{1}{r^n} d\lambda_1 \right\},
\]
(5.8)

and

\[
|a_{0n}| \leq \frac{1}{2\pi r^n} \left( (2\pi - \alpha)M + \alpha M \left( \frac{\alpha}{M} \right)^{\theta} \right),
\]
(5.9)

where \(M = \max |f(0, z_2)| = \max |\sum_{n=0}^{\infty} a_{0n} z_2^n|\). Then

\[
\frac{2\pi}{\alpha} \left[ \frac{|a_{0n}| r^n}{M} - \frac{M}{M} \right] + \frac{M}{M} \leq \left( \frac{\gamma}{M} \right)^{\theta}.
\]
(5.10)

Let \(\Delta\) be defined by the equation

\[
\Delta = 1 - \frac{|a_{0n}| r^n}{M},
\]
(5.11)

\(\mu\) being that \(n\) which maximizes \(|a_{0n}| r^n\), and \(\mu\) depends on \(r\). Then \(\Delta\) is positive.

If \(\alpha > 2\pi \Delta\), we have that

\[
\gamma(r) \geq M(r) \left[ \frac{M}{M} \left( 1 - \frac{2\pi}{\alpha} \Delta \right) \right]^{\theta-1(\theta)},
\]
(5.12)

where the right-hand side is positive.

From these results we can state

**Theorem IV.** Given the function

\[
f(z_1, z_2) = \sum_{m,n=0}^{\infty} a_{mn} z_1^m z_2^n,
\]

regular in \(H^4(r)\). Let \(\max |f(z_1, z_2)| \leq \gamma(r)\) on the surface \(H^4(r)\) of (5.1) having the characteristic numbers \(\theta(r)\) and \(\alpha, \alpha = \lambda_1^{(0)} - \lambda_1^{(1)} > 2\pi \Delta\), where \(\Delta = 1 - |a_{0m}| r^n/M\); then

\[
\gamma(r) \geq M(r) \left[ \frac{M(r)}{M(r)} \left( 1 - \frac{2\pi \Delta}{\alpha} \right) \right]^{\theta-1(\theta)},
\]
(5.13)

where \(M = \max |f(z_1, z_2)|\) and \(M = \max |f(0, z_2)|\).

Since

\[
\frac{|a_{0n}| r^{n+1}}{\rho - r} > M(r), \quad \rho > r,
\]
(5.14)
a lower bound for $\gamma(r)$ can be obtained in terms of the coefficients of $f(z_1, z_2)$ by replacing in (5.12) $M(r)$ by the right-hand side of (4.20), $\overline{M}(r)$ by $|a_{0\nu}|r^\mu$, and the $M(r)$ in $\Delta$ by

$$\frac{|a_{0\nu}|^{\mu+1}}{\rho_1 - r}$$

where

$$|a_{0\nu}|^{\mu} = \max_{1+\epsilon \leq \mu \leq \epsilon} \left[ \max_n |a_{0n}| \right],$$

for an arbitrary positive $\epsilon$. $\rho_1$ is a function of $r$ and of the coefficients $\{a_{0n}\}$, and can be determined by a process similar to the Newton polygon method.

REFERENCES


5. R. Nevanlinna, Eindeutige analytische Funktionen, chap. 4, §5.

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