RESTRICTED LIE ALGEBRAS OF CHARACTERISTIC \( p \)

BY

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In an earlier paper\(^{(1)}\) [3] we noted certain identities which connect addition, scalar multiplication, commutation \( ([ab] = ab - ba) \), and \( p \)-th powers in an arbitrary associative algebra of characteristic \( p \) \((\neq 0)\). These lead naturally to the definition of a class of abstract algebras called restricted Lie algebras which in many respects bear a closer relation to Lie algebras of characteristic 0 than ordinary Lie algebras of characteristic \( p \).

As is shown in the present paper any restricted Lie algebra \( \mathfrak{g} \) may be obtained from an associative algebra by using the operations mentioned above. In fact \( \mathfrak{g} \) determines a certain associative algebra \( \mathfrak{U} \), called its \( u \)-algebra, such that \( \mathfrak{g} \) is isomorphic to a subalgebra of \( \mathfrak{U} \), the restricted Lie algebra defined by \( \mathfrak{g} \); and if \( \mathfrak{B} \) is any associative algebra such that \( \mathfrak{B} \) contains a subalgebra homomorphic to \( \mathfrak{g} \) and \( \mathfrak{B} \) is the enveloping algebra of this subset then \( \mathfrak{U} \) is homomorphic to \( \mathfrak{B} \). The algebra \( \mathfrak{U} \) has an anti-automorphism relative to which the elements corresponding to those in \( \mathfrak{g} \) are skew. For ordinary Lie algebras an algebra having these properties has been defined by G. Birkhoff [2] and by Witt [5]. In their case however, the associative algebra has an infinite basis even when the Lie algebra has a finite basis whereas here \( \mathfrak{U} \) has a finite basis if and only if \( \mathfrak{g} \) has. Consequently every restricted Lie algebra \( \mathfrak{g} \) with a finite basis has a (1-1) representation by finite matrices. The theory of representations of \( \mathfrak{g} \) can be reduced to that of the associative algebra \( \mathfrak{U} \). Thus, for example, there are only a finite number of inequivalent irreducible representations.

The most natural way to obtain a restricted Lie algebra is as a derivation algebra of an arbitrary algebra \( \mathfrak{A} \), i.e., as the set of transformations \( D : a \rightarrow aD \) in \( \mathfrak{A} \) such that

\[
(a + b)D = aD + bD, \quad (a\alpha)D = (aD)\alpha, \quad (ab)D = (aD)b + a(bD).
\]

If \( \mathfrak{A} = \mathfrak{g} \) is itself restricted \((ab = [ab])\) the derivations which satisfy

\[
a^pD = [\underbrace{[aD, a] a \cdots a}] \quad (p - 1)
\]

are called restricted. They are precisely the derivations of \( \mathfrak{g} \) which can be extended to derivations of the \( u \)-algebra \( \mathfrak{U} \). Hence their totality is a restricted Lie algebra \( \mathfrak{D}_0 \). Using \( \mathfrak{D}_0 \) and \( \mathfrak{g} \) we may define a restricted holomorph \( \mathfrak{S}_0 \) of \( \mathfrak{g} \). \( \mathfrak{D}_0 \) is a restricted Lie algebra.

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\(^{(1)}\) Numbers in brackets refer to the bibliography at the end of the paper.

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The considerations in the present paper apply for the most part to restricted Lie algebras with an infinite basis as well as to those with a finite basis. A special result, however, for Lie algebras with a finite basis is that the nilpotency of \( \mathfrak{g} \) implies that of the \( u \)-algebra (§6). In a later paper we hope to discuss certain classes of simple restricted Lie algebras with a finite basis.

1. Restricted Lie algebras. Definitions. If \( \mathfrak{A} \) is an associative algebra and the commutator \( [ab] = ab - ba \), it is well known that

\[
\begin{align*}
[ab] &= -[ba], \\
[a[bc]] + [b[ca]] + [c[ab]] &= 0.
\end{align*}
\]

If, in addition \( \mathfrak{A} \) has characteristic \( p \neq 0 \) then the following identities hold:

\[
(a + b)^p = a^p + b^p + s(a, b),
\]

where \( s(a, b) = s_1(a, b) + s_2(a, b) + \cdots + s_{p-1}(a, b) \) and the \( (p-i)s_i(a, b) \) is the coefficient of \( x^{i-1} \) in

\[
[\cdots [a, \lambda a + b], \lambda a + b], \cdots, \lambda a + b]
\]

([3], and [6]). We are thus led to define a restricted Lie algebra \( \mathfrak{g} \) over a field \( \Phi \) of characteristic \( p \) as a vector space over \( \Phi \) in which operations \( [ab] \) and \( a^p \) are defined such that

1. \( [ab] = -[ba], \quad [a[bc]] + [b[ca]] + [c[ab]] = 0, \)
2. \( [a, b_1 + b_2] = [ab_1] + [ab_2], \)
3. \( [ab] = [a, \lambda b] = [a\alpha, b], \quad a \in \mathfrak{A}, \)
4. \( (a + b)^p = a^{[p]} + b^{[p]} + s(a, b), \)
5. \( (a\alpha)^{[p]} = (a^{[p]})\alpha^{[p]}, \)
6. \( [\cdots [ab]b] \cdots b] = [ab^{[p]}]. \)

A subspace \( \mathfrak{B} \) of \( \mathfrak{g} \) is a subalgebra if \( \mathfrak{B} \supseteq a^{[p]} \) and \( [b]^{[p]} \) for all \( a, b \) in \( \mathfrak{B} \). \( \mathfrak{B} \) is an ideal if it contains also \( [ab] \) for all \( b \) in \( \mathfrak{B} \), \( a \in \mathfrak{g} \). A correspondence \( a \rightarrow a^S \) between two restricted Lie algebras is a homomorphism if

\[
(a + b)^S = a^S + b^S, \quad (a\alpha)^S = a^S\alpha, \quad [ab]^S = [a^Sb^S],
\]

If \( S \) is (1-1) it is an isomorphism. If besides \( S \) is a correspondence within \( \mathfrak{g} \) it is an automorphism.

If \( x_1, x_2, \cdots \) (possibly infinite) is a basis for \( \mathfrak{g} \), then \( [x_i, x_j] = \sum x_i \gamma_{ij} x_j \), \( x_i^{[p]} = \sum x_i \mu_{i[p]} \) (finite sums) where

\[
\begin{align*}
\gamma_{ij} &= -\gamma_{ji}, \quad \sum \gamma_{rqk}\gamma_{qij} + \sum \gamma_{rqi}\gamma_{qji} + \sum \gamma_{rqi}\gamma_{oji} = 0, \\
\sum \gamma_{qij}\mu_{r[j]} &= \sum \gamma_{qij}\mu_{r[p-1]}\gamma_{q_{p-1}q_{p-2}} \cdots \gamma_{q_{i+1}}.
\end{align*}
\]
These equations are equivalent to

$$[x,x_i] = - [x, x_i],$$

$$[[x,x_i],x_i] + [[[x,x_i],x_i] + [[[x,x_i],x_i] = 0, \quad [x, x_i^{[p]}] = [\cdots [x, x_i] \cdots x_i],$$

respectively. The $\gamma$'s and $\mu$'s are the constants of multiplication of $G$. If $a \to a^G$ is an isomorphism between $G$ and $G^G, x_1^G, x_2^G, \cdots$ form a basis for $G^G$ and the $x_i^G$ have the same constants of multiplication as the $x_i$'s. On the other hand if $\mathfrak{M}$ is any restricted Lie algebra with basis $y_1, y_2, \cdots$ in (1-1) correspondence with the $x_i$ such that $[y_i, y_j] = \sum y_{i+j} y_{i-j}, y_i^{[p]} = \sum y_{i+p} y_i$ then it is readily seen that the correspondence $\sum x_i\alpha_i \to \sum y_i\alpha_i$ is an isomorphism.

We have noted above that any associative algebra $\mathfrak{A}$ of characteristic $p$ becomes a restricted Lie algebra $\mathfrak{A}_i$ when $[ab] = ab - ba$ and $a^{[p]} = a^p$. A homomorphism between $G$ and a subalgebra of $\Phi_n, \Phi_n$ the algebra of $n \times n$ matrices, is called a representation. Irreducibility, decomposability, equivalence, etc., of representations are defined as usual. As is well known these depend on the irreducibility, etc., of the enveloping algebra of the representing matrices.

2. The $u$-algebra of a restricted Lie algebra. Let $x_1, x_2, \cdots$ (possibly infinite) be a basis over $\Phi$ of a vector space $G$. We set $[x,x_i] = \sum x_{i+j} x_{i-j}$ and $x_i^{[p]} = \sum x_{i+p} x_i$ where the $\gamma$'s are $\mu$'s satisfy (8) and (9). Then for $a = \sum x_i\alpha_i$, $b = \sum x_i\beta_i$ (finite sums) we set $[ab] = \sum x_i\gamma_{i+j} \alpha_i \beta_j$. Evidently (8) implies that $G$ is a Lie algebra relative to $[ab]$. Equation (9) is equivalent to

$$[x_i x_j^{[p]}] = [\cdots [x_i x_j] \cdots x_j].$$

We shall show that $G$ is a restricted Lie algebra relative to a suitable definition of $a^{[p]}$.

Let $\mathfrak{A}$ be the vector space with the basis $x_1^\alpha x_2^\beta \cdots x_n^\nu, \kappa_i \geq 0$ integers and at least one $\kappa_i > 0, n = 1, 2, \cdots$. If only a finite number of monomials are being considered we may write them in terms of the same $x$'s. A product

$$(x_1 x_2 \cdots x_n)(x_1 x_2 \cdots x_n)$$

is defined by repeated "straightenings," i.e., substitutions for $x_i x_j$ when $i > j$ of the expression $x_i x_j + \sum x_i x_{i+j}$. It has been shown by G. Birkhoff [2] and by Witt [5] that this product is uniquely defined in $\mathfrak{A}$ and is associative.

Let $\mathfrak{S}$ be the ideal in $\mathfrak{A}$ having the basis $y_i = x_i^p - x_i^{[p]}$. Since

$$[b x_i^p] = [\cdots [b x_i] x_i] \cdots x_i] = [b x_i^{[p]}],$$

$y_i$ commutes with every linear $b$ and hence with every element of $\mathfrak{S}$.

If the term $x_1^\alpha \cdots x_n^\nu$ has degree $\geq p$ in $x$, we may replace $x_i^p$ by $y_i + x_i^{[p]}$. After a finite number of such substitutions we may write any $a = \sum x_i^\alpha \cdots x_n^\nu$.
in the form \(\sum x_1^{\lambda_1} \cdots x_n^{\lambda_n} u_{\lambda_1} \cdots \lambda_n\) where \(\lambda_i < p\) and the \(u's\) are polynomials in \(y_1, y_2, \cdots, y_n\). Thus any \(b = \sum \alpha_i x_i + \sum \gamma_i \alpha_i\) in \(\mathfrak{g}\) has the form 
\[
\sum x_1^{\lambda_1} \cdots x_n^{\lambda_n} v_{\lambda_1} \cdots \lambda_n, \quad v \text{ a polynomial in } y_1, y_2, \cdots, y_n \text{ with no constant term.}
\]
Now \(x_1 x_2 \cdots x_n = x_1 (x_1 - x_1) \cdots x_n (x_n - x_n)\)
\[
= x_1 + m_1 \cdots x_n + \cdots
\]
where the terms not indicated have degree \(\leq \sum \lambda_i + p \sum m_i\). Consider the terms of maximum degree \(N = \sum \lambda_i + p \sum m_i\) in \(b\) where we suppose \(b \neq 0\) and hence one of the \(v's\) is \(\neq 0\). Since at least one \(m > 0\), \(N > p\). Two terms of maximum degree are different if \((\lambda_1, \cdots, \lambda_n; m_1, \cdots, m_n) \neq (\lambda'_1, \cdots, \lambda'_n; m'_1, \cdots, m'_n)\).

Hence these terms occur only once and can not cancel off. It follows that when \(b\) is written in its normal form \(x_1 \cdots x_n x_1^{\lambda_1} \cdots x_n^{\lambda_n}\) at least one of the \(x's\) has degree \(> p\). Thus the classes \(\{x_1^{\lambda_1} \cdots x_n^{\lambda_n}\}, \lambda_i < p\), omitting \(\{x_1^p \cdots x_n^p\}\), determined by the elements \(x_1^{\lambda_1} \cdots x_n^{\lambda_n}\) modulo \(\mathfrak{g}\) form a basis for the difference algebra \(U = \mathfrak{g}/\mathfrak{g}\).

Since the classes \(\{x_1\}, \{x_2\}, \cdots\) are linearly independent and \([\{x_i\} \{x_j\}] = \sum \{x_i\} y_{i \otimes j}\) the correspondence \(\sum x_1^{\lambda_1} \cdots x_n^{\lambda_n} \alpha_i \cdots \alpha_n\) is an isomorphism between \(\mathfrak{g}\) and an ordinary Lie subalgebra \(\{\mathfrak{g}\}\) of \(U\). Since \(\{x_i\} = \{x_i^p\}\) is a (restricted) subalgebra of \(U\).

Hence if we define \((\sum x_1^{\lambda_1})^{[p]}\) as the element corresponding to \(\sum x_1^{\lambda_1}\) \(p\), \(\mathfrak{g}\) becomes a restricted Lie algebra in which \(x_1^{[p]}\) is as originally given.

If \(\mathfrak{g}\) is a restricted Lie algebra to begin with, then the correspondence \(\sum x_1^{\lambda_1} \cdots x_n^{\lambda_n} \alpha_i \cdots \alpha_n\) is an isomorphism. Now suppose \(\sum x_1^{\lambda_1} \cdots x_n^{\lambda_n}\) is a homomorphism between \(\mathfrak{g}\) and \(\mathfrak{g}\), a subalgebra of \(\mathfrak{g}\), where we suppose that any element of \(\mathfrak{g}\) is a polynomial in the elements of \(\mathfrak{g}\). We have

\[
\tilde{x}_i \tilde{x}_j = \tilde{x}_j \tilde{x}_i + \sum \tilde{x}_q y_{q \otimes i j}, \quad \tilde{x}_i^p = \sum \tilde{x}_r y_{r \otimes i}
\]

The first set of equations implies that \(\sum \tilde{x}_1^{\alpha_1} \cdots \tilde{x}_n^{\alpha_n} \rho_{a_1} \cdots \rho_{a_n} \rightarrow \sum \tilde{x}_1^{\alpha_1} \cdots \tilde{x}_n^{\alpha_n} \rho_{a_1} \cdots \rho_{a_n}\) is a homomorphism between \(\mathfrak{g}\) and \(\mathfrak{g}\) \((\dagger)\). Because of the second set of equations \(x_i^p - \tilde{x}_i^p\) are mapped into 0 and so our correspondence induces a homomorphism between \(U = \mathfrak{g}/\mathfrak{g}\) and \(\mathfrak{g}\). This mapping is an extension of the homomorphism between \(\mathfrak{g}\) and \(\mathfrak{g}\). Thus we have proved the following theorem.

**Theorem 1.** If \(\mathfrak{g}\) is a restricted Lie algebra there exists an associative algebra \(U\) having the following properties: 1. \(\mathfrak{g}\) is isomorphic to a subalgebra \(\{\mathfrak{g}\}\) of \(U\). 2. \(U\) is the enveloping algebra of \(\{\mathfrak{g}\}\). 3. If \(\mathfrak{g}\) is homomorphic to a subalgebra \(\mathfrak{g}\) of any \(\mathfrak{g}\) where \(\mathfrak{g}\) is the enveloping algebra of \(\mathfrak{g}\), then \(U\) is homomorphic to \(\mathfrak{g}\).

This theorem shows that conditions (1) to (6) are characteristic of the functions \(a + b, a a, [a b]\) and \(a^p\) in an associative algebra of characteristic \(p\).

The above considerations show also that equations (8) and (9) on the con-

\((\dagger)\) See [2] or [5].
stays $\gamma$ and $\mu$ insure that the vector space $\mathfrak{g}$ be a restricted Lie algebra. If $P$ is an extension of the field $\Phi$ the extended vector space of elements of the form $\sum x_i \xi_i$, $\xi_i$ in $P$, is a Lie algebra over $P$. We denote this extended algebra as $\mathfrak{g}P$ since, as is easily shown, it does not depend on the particular choice of basis. In the remainder of the paper we shall denote $a^{[p]}$ by $a^P$ when there is no risk of confusion and shall call $\mathfrak{u}$ the $u$-algebra of $\mathfrak{g}$.

Suppose $x_{i_1} \cdots x_{i_m}$ is a monomial in $\mathfrak{g}$ and the first $x_1$ in this product occurs in the $r$th place. Then we may straighten this term by interchanging $x_1$ successively with the $r_1 - 1$ terms in front of it and obtain a polynomial in the $x$'s having one term $x_1 x_{i_2} \cdots x_{i_m}$ of degree $m$. We define the rank of $x_{i_1} \cdots x_{i_m}$ inductively as $(r_1 - 1)$ plus rank of $x_{i_2} \cdots x_{i_m}$. A monomial of rank 0 is said to be in canonical form. Then $i_1 \geq i_2 \geq \cdots \geq i_m$. Consider the correspondence $a = \sum x_i^{p_1} \cdots x_i^{p_n} \xi_1 \cdots \xi_m \rightarrow a^J = \sum (-1)^{s_1 + \cdots + s_n} x_i^{p_1} \cdots x_i^{p_n}$. Evidently $J$ is linear. We wish to show that it is an anti-automorphism. For this purpose it suffices to prove that

$$(x_{i_1} \cdots x_{i_m})^J = (-1)^{m} x_{i_m} \cdots x_{i_1}.$$ 

Suppose this holds for all products of $(m-1)$ or less $x$'s and also for products of $m$ $x$'s whose ranks are less than those of the given monomial. Then

$$(x_{i_1} \cdots x_{i_m}) = (x_{i_1} \cdots x_{i_{j+1}} x_{i_{j+1}} \cdots x_{i_m}) + (x_{i_1} \cdots [x_{i_{j+1}} x_{i_{j+1}}] \cdots x_{i_m}),$$

where we may suppose that if the rank $r$ of the original term is $>0$ that of $x_{i_1} \cdots x_{i_{j+1}} x_{i_{j+1}} \cdots x_{i_m}$ is $r-1$. We have

$$(x_{i_1} \cdots x_{i_m})^J = (x_{i_1} \cdots x_{i_{j+1}} x_{i_{j+1}} \cdots x_{i_m})^J + (x_{i_1} \cdots [x_{i_{j+1}} x_{i_{j+1}}] \cdots x_{i_m})^J$$

$$= (-1)^m (x_{i_m} \cdots x_{i_{j+1}} x_{i_{j+1}} \cdots x_{i_1})$$
$$+ (-1)^{m-1} (x_{i_m} \cdots [x_{i_{j+1}} x_{i_{j+1}}] \cdots x_{i_1})$$
$$= (-1)^m (x_{i_1} \cdots x_{i_m}).$$

Since $(x_i^{p_i} - x_i^p)^J = - (x_i^{p_i} - x_i^p)$, $J$ sends the ideal $\mathfrak{g}$ into itself and therefore induces an anti-automorphism in $\mathfrak{u} = \mathfrak{g}/\mathfrak{g}$. The elements $\sum x_i \alpha_i$ of $\{\mathfrak{g}\}$ are skew relative to the anti-automorphism.

By property (3) of $\mathfrak{u}$ any representation of $\mathfrak{g}$ determines a representation of $\mathfrak{u}$ and conversely. Questions of irreducibility, equivalence, etc., for $\mathfrak{g}$ are reducible to the corresponding questions for $\mathfrak{u}$. If $\mathfrak{g}$ has a finite basis $x_1, x_2, \cdots, x_n$, $\mathfrak{u}$ has the basis $\{x_1^{p_1} \cdots x_n^{p_n} \mid \lambda_i < p\}$ of $p^n - 1$ elements. Since $\mathfrak{u}$ has a (1-1) representation in some $\Phi_m, m \leq p^n$, the same is true for $\mathfrak{g}$.

**Theorem 2.** Every restricted Lie algebra with a finite basis has a (1-1) representation.

The number of inequivalent irreducible representations ($\neq 0$) of $\mathfrak{u}$ is equal to the number of simple components of $\mathfrak{u}/\mathfrak{R}$, $\mathfrak{R}$ the radical of $\mathfrak{u}$. This implies
Theorem 3. There are only a finite number of inequivalent irreducible representations of a restricted Lie algebra with a finite basis.

Example. \( \mathfrak{g} \), the restricted Lie algebra with the basis \( x, y, z \) such that

\[
[xy] = z, \quad [xz] = [yz] = 0, \quad x^p = y^p = z^p = z.
\]

\( \mathfrak{u} \) has the basis \( x'y'^i, i, j, k < p \) such that the above relations hold, it being understood that \([xy] = xy - yx\), etc. It is readily proved that \( \mathfrak{u} \) is a direct sum of \( p \) algebras with bases \( x'y' \) such that

\[
x'y' = y'x', \quad x'^p = y'^p = z,
\]

where \( z = 0, 1, \ldots, p - 1 \) in turn. If \( z = 0 \) this algebra is nilpotent. Otherwise it is isomorphic to \( \Phi_p \). Hence there are \((p - 1)\) inequivalent irreducible representations of \( \mathfrak{g} \).

Theorem 2 is valid for Lie algebras of characteristic 0 though its proof given by Ado [1] is considerably more complicated than the present one. Theorem 3 is not true for algebras of characteristic 0. It is not known whether either of these results holds for ordinary Lie algebras of characteristic \( p \).

3. Ideals. Nilpotency. If \( \mathfrak{g} \) is an ideal in \( \mathfrak{u} \) we define the sum, scalar product and commutator of the classes \( \bar{a} \) modulo \( \mathfrak{u} \) as usual by

\[
\bar{a}_1 + \bar{a}_2 = \bar{a}_1 + \bar{a}_2, \quad \bar{a}\alpha = a\alpha, \quad [\bar{a}_1\bar{a}_2] = [\bar{a}_1\bar{a}_2].
\]

If \( a_1 - a_2 = b \in \mathfrak{u} \) then \( a_1^p = a_2^p + b^p + s(a_2, b) \). Since \( b^p \) and \( s(a_2, b) \in \mathfrak{g} \) we see that \( a_1 = a_2(\mathfrak{g}) \) implies \( a_1^p = a_2^p(\mathfrak{g}) \). Hence the definition \( a^p = \bar{a}^p \) is unambiguous and together with the above operations it defines \( \mathfrak{g}/\mathfrak{u} \), the difference algebra of \( \mathfrak{g} \) relative to \( \mathfrak{u} \), as a restricted Lie algebra. The correspondence \( a \rightarrow \bar{a} \) is a homomorphism between \( \mathfrak{g} \) and \( \mathfrak{g}/\mathfrak{u} \). Conversely we may show in the usual manner that if \( \mathfrak{g} \) is homomorphic to the restricted Lie algebra \( \mathfrak{g}/\mathfrak{u} \), \( \mathfrak{g}/\mathfrak{u} = \mathfrak{g}/\mathfrak{u} \) where \( \mathfrak{u} \) is the set of elements mapped into 0 by the homomorphism.

If \( \mathfrak{u}_1 \) and \( \mathfrak{u}_2 \) are subspaces of \( \mathfrak{g} \) we denote their sum as \( \mathfrak{u}_1 + \mathfrak{u}_2 \) and their commutator, i.e., the smallest space containing all \([\mathfrak{u}_1, \mathfrak{u}_2]\), \( \mathfrak{u}_1 \in \mathfrak{u}_1, \mathfrak{u}_2 \in \mathfrak{u}_2 \) by \([\mathfrak{u}_1, \mathfrak{u}_2]\). Conditions (1), (2) and (3) imply

\[
[\mathfrak{u}_1, \mathfrak{u}_2] = [\mathfrak{u}_1, \mathfrak{u}_2], \quad [\mathfrak{u}_1, [\mathfrak{u}_2, \mathfrak{u}_3]] = [\mathfrak{u}_2, [\mathfrak{u}_1, \mathfrak{u}_3]] + [\mathfrak{u}_3, [\mathfrak{u}_1, \mathfrak{u}_2]].
\]

Set \( \mathfrak{g}^{[1]} = [\mathfrak{g}], \ldots, \mathfrak{g}^{[i]} = [\mathfrak{g}^{[i-1]}] \) and define \( \mathfrak{g}^i \) to be the smallest subspace containing all \( a^{[i]} \) where \( a^p = (a^{p^{i-1}})^p \). Thus \( \mathfrak{g} \geq \mathfrak{g}^{[1]} \geq \mathfrak{g}^{[2]} \geq \cdots \) and \( \mathfrak{g} \geq \mathfrak{g}^p \geq \mathfrak{g}^p \geq \cdots \). Hence if we define

\[
\mathfrak{g}_i = \mathfrak{g}^{[i]} + (\mathfrak{g}^{[i-1]})^p + (\mathfrak{g}^{[i-2]})^{p^2} + \cdots + \mathfrak{g}^{[i-1]}
\]

we have \( \mathfrak{g} = \mathfrak{g}_1 \geq \mathfrak{g}_2 \geq \mathfrak{g}_3 \geq \cdots \). By induction on \( j \) one readily establishes

\[
[\mathfrak{g}^{[i]}\mathfrak{g}^{[j]}] \leq \mathfrak{g}^{[i+j]}.
\]

Since
\[ [b_1^{p^k}, b_2^{p^l}] = [\cdots [b_1^{p^k}b_2^{p^l}] \cdots b_2] \]

we have

\[ [\mathfrak{g}(i-k)^{p^k}\mathfrak{g}(j-l)^{p^l}] \leq \mathfrak{g}(i-k+j-l)^{p^l} \leq \mathfrak{g}(i+j)^{p^l}. \]

This leads readily to

\[ [\mathfrak{g}, \mathfrak{g}_i] \leq \mathfrak{g}_{i+1}, \quad \mathfrak{g}_i^p \leq \mathfrak{g}_{i+1}. \]

In particular, \( \mathfrak{g}_i \) is an ideal in \( \mathfrak{g} \) and \( \mathfrak{g}_i \geq [\mathfrak{g}_{i-1}\mathfrak{g}_i] + \mathfrak{g}_{i-1}^p \). On the other hand, from the definition of \( \mathfrak{g}_i \), we have \( \mathfrak{g}_i \leq [\mathfrak{g}_{i-1}\mathfrak{g}_i] + \mathfrak{g}_{i-1}^p \). Hence \( \mathfrak{g}_i = [\mathfrak{g}_{i-1}\mathfrak{g}_i] + \mathfrak{g}_{i-1}^p \). It follows that if \( \mathfrak{g}_{i-1} = \mathfrak{g}_i \), \( \mathfrak{g}_{i-1} = \mathfrak{g}_i = \mathfrak{g}_{i+1} = \cdots \). If \( \mathfrak{g}_N = 0 \) for \( N \) sufficiently large, \( \mathfrak{g} \) is nilpotent. The smallest \( N \) for which this holds is called the index of nilpotency. If \( \mathfrak{g}_N = 0 \), \( \mathfrak{g}^{\mathfrak{g}_N} = 0 \) and \( \mathfrak{g}^{(N)} = 0 \). Conversely if \( \mathfrak{g}^{(r)} = 0 \) and \( \mathfrak{g}^p = 0 \) it is readily seen that \( \mathfrak{g}_t = 0 \) for \( t = r + s - 1 \).

4. Restricted derivations. The most natural instances of restricted Lie algebras are the derivation algebras (3). We recall that if \( \mathfrak{A} \) is an arbitrary algebra (not necessarily associative) then a derivation \( D \) is defined to be a transformation \( a \to aD \) in \( \mathfrak{A} \) such that

\[ (a + b)D = aD + bD, \quad (aa)D = (aD)a, \quad (ab)D = (aD)b + a(bD). \]

If \( \mathfrak{A} \) has characteristic \( p \) the set \( \mathfrak{D} \) of these transformations is closed under addition, scalar multiplication, commutation and \( p \)th powers. Thus \( \mathfrak{D} \) is a restricted Lie algebra. If \( \mathfrak{A} \) is associative

\[ a^pD = (aD)a^{p-1} + a(aD)a^{p-2} + \cdots + a^{p-1}(aD) = [\cdots [aD, a]a] \cdots a. \]

It is therefore natural to confine our attention in the case that \( \mathfrak{A} = \mathfrak{g} \) is a restricted Lie algebra to the derivations called restricted such that

\[ a^pD = [\cdots [aD, a]a] \cdots a. \]

Suppose \( D \) is a linear transformation in \( \mathfrak{g} \) such that \( [x_i x_j]D = [x_i, x_j]D, \quad x_1, x_2, \cdots \) a basis for \( \mathfrak{g} \). Then \( D \) is a derivation. If \( x_1^{k_1} \cdots x_n^{k_n}, \quad k_1 = 1, 2, \cdots, \) is a basis for the Birkhoff-Witt algebra \( \mathfrak{A} \) we define the linear transformation \( D \) in \( \mathfrak{A} \) by setting

\[ b^{p-1} + aba^{p-2} + \cdots + a^{p-1}b = [\cdots [b, a]a] \cdots a. \]

See [3, p. 209].

(?) Cf. Jacobson [3].

(?) In general

\[ ba^{p-1} + ab^{p-2} + \cdots + a^{p-1}b = [\cdots [ba, a]a] \cdots a. \]
\[(x_1^i \cdot \cdots x_n^i)D = (x_iD)x_1^{i-1} \cdots x_n^i + \cdots + x_1^{i-1}(x_iD)\cdots x_n^i + \cdots + x_1^i \cdots x_n^{i-1}(x_nD).\]

By an induction similar to that of §2 we can show that
\[(x_1 \cdots x_n)D = (x_iD)x_1 \cdots x_{i-1} + \cdots + x_1 \cdots (x_{i-1}D),\]
holds. Hence \(D\) is a derivation.

Now suppose
\[x_i^{[p]}D = [\cdots [x_iD, x_i] \cdots x_1].\]
Then \((x_i^{[p]} - x_i^p)D = 0\) and \(D\) maps the ideal \(\mathcal{I}\) whose basis is \(x_i^{[p]} - x_i^p\) into itself. It follows that \(D\) induces a derivation in \(\mathcal{U} = \mathcal{A}/\mathcal{I}\) and hence is a restricted derivation in \(\mathcal{I}\). We have shown also that any restricted derivation is determined by a derivation of the \(u\)-algebra. The converse of this is clear. Hence we have proved

**Theorem 4.** A linear transformation \(D\) in \(\mathcal{I}\) is a restricted derivation if and only if \([x_iD, x_j] = [x_jD, x_i] = 0\) and \(x_i^{[p]}D = [\cdots [x_iD, x_i] \cdots x_1]\) for any basis \(x_1, x_2, \ldots\). Every restricted derivation is induced by a derivation of the associative \(u\)-algebra \(\mathcal{U}\) of \(\mathcal{I}\). The set of restricted derivations forms a restricted Lie algebra.

The last statement follows immediately from the second. The set of restricted derivations will be denoted by \(D_0\). A consequence of the first part of Theorem 4 is that the restricted derivation algebra of \(\mathcal{I}\) is \(D_0\).

For any three elements \(a, b, l\) in \(\mathcal{I}\) we have
\[\left[a + b, l\right] = [a, l] + [b, l], \quad [a, [a, l]] = [a, [a, l]] + [a, [b, l]],\]
\[a[[a, l]] = [\cdots [a] \cdots a].\]
Thus the transformations \(L: a \rightarrow [a, l]\) are restricted derivations which we call inner. The other parts of (1) to (6) show that the derivations corresponding to \(l_1 + l_2, l_\alpha, [l_\alpha l_\beta], l^p\) are respectively \(L_1 + L_2, L_\alpha, [L_\alpha L_\beta], L^p\). Hence the inner derivations form a subalgebra \(\mathfrak{g}\) of \(D_0\) and \(\mathcal{I}\) is isomorphic to \(\mathfrak{g}\) under the correspondence \(l \rightarrow L\). The elements \(c\) mapped into 0 are those which satisfy \([ac] = 0\) for all \(a\) and form the center \(\mathfrak{C}\) of \(\mathcal{I}\). Hence \(\mathfrak{g} \cong \mathcal{I}/\mathfrak{C}\). Since
\[\left[a[lD, l] - [aD, l]\right] = [a, lD],\]
\[[lD] \in \mathfrak{g}\] for every \(L\) in \(\mathfrak{g}\) and \(D\) in \(D_0\), i.e., \(\mathfrak{g}\) is an ideal.

Let \(\mathcal{S}_0\) be the vector space which is a direct sum of \(\mathfrak{g}\) and \(D_0\). The elements \(U\) of \(\mathcal{S}_0\) are uniquely representable in the form \(a + D, a\) in \(\mathfrak{g}\), \(D\) in \(D_0\). Hence if \(x_1, x_2, \cdots\) is a basis for \(\mathfrak{g}\) and \(D_1, D_2, \cdots\) one for \(D_0, x_1, x_2, \cdots; D_1, D_2, \cdots\) is a basis for \(\mathcal{S}_0\).
We define commutation in $\mathfrak{g}_0$ by

\begin{equation}
[a + D, b + E] = [ab] + aE - bD + [DE].
\end{equation}

It is readily seen that this satisfies conditions (1), (2), (3)(6). We also have

\begin{align*}
\prod [a x_i] \cdots x_i &\equiv [a x_i^p], \\
\prod [D x_i] \cdots x_i &\equiv [D x_i^p], \\
\prod [a D_i] \cdots D_i &\equiv [a D_i^p], \\
\prod [D D_i] \cdots D_i &\equiv [D D_i^p],
\end{align*}

or

\begin{align*}
\prod [U x_i] \cdots x_i &\equiv [U x_i^p], \\
\prod [U D_i] \cdots D_i &\equiv [U D_i^p],
\end{align*}

for all $U$ in $\mathfrak{g}_0$. Hence, by §2, the definition

\begin{equation}
(a + b)^{\mathfrak{g}_0} = a^{\mathfrak{g}_0} + b^{\mathfrak{g}_0}, \quad (a b)^{\mathfrak{g}_0} = a^{\mathfrak{g}_0} b^{\mathfrak{g}_0}, \quad [a b]^{\mathfrak{g}_0} = [a^{\mathfrak{g}_0} b^{\mathfrak{g}_0}].
\end{equation}

Then

\begin{align*}
\prod [x a] \cdots a^{\mathfrak{g}_0} &\equiv \prod [x^s a^s] \cdots a^s = [x^s (a^s)^p], \\
[x, a^p]^{\mathfrak{g}_0} &\equiv [x^s (a^p)^s].
\end{align*}

Hence $(a^p)^{\mathfrak{g}_0} - (a)^{\mathfrak{g}_0} \in \mathfrak{g}_2$ the center of $\mathfrak{g}_2$. If $\mathfrak{g}_2 = 0$, $S$ is a homomorphism. Next we suppose $D$ is a derivation in $\mathfrak{g} = \mathfrak{g}_0$. This implies that

\begin{align*}
\prod [x a] \cdots a D &\equiv \prod [xD a] \cdots a + \prod [[x, a D] a] \cdots a \\
&\quad + \cdots + \prod [[x a] a] \cdots a D
\end{align*}

\begin{align*}
&= [xD, a^p] + [x, [\cdots [a D, a] \cdots a]],
\end{align*}

since if $A$ is the transformation $x \rightarrow [x a]$ and $B$ is the transformation $x \rightarrow [x, a D]$,

\begin{equation}
BA^{p-1} + ABA^{p-2} + \cdots + A^{p-1} B = \cdots [[BA] A] \cdots A
\end{equation}

On the other hand $[x a^p] D = [xD, a^p] + [x, a^p D]$. Hence $a^p D - [\cdots [a D, a] \cdots a] = 0$ if $\mathfrak{g}_a = 0$ every derivation is restricted.

If $\mathfrak{B}$ is a subset of $\mathfrak{g}$ closed with respect to addition, scalar multiplication and commutation, then $\mathfrak{B}^* = \mathfrak{B} + \mathfrak{B} D + \mathfrak{B}^2 + \cdots$ is the smallest subalgebra of $\mathfrak{g}$ containing $\mathfrak{B}$. If $[\mathfrak{B} \mathfrak{g}] \leq \mathfrak{B}$, $\mathfrak{B}^*$ is an ideal.

(\textsuperscript{6}) Cf. Zassenhaus [6, p. 57].

(\textsuperscript{8}) See Footnote 4.
An ordinary Lie algebra $\mathfrak{g}$ whose center is 0 may always be imbedded in a restricted Lie algebra. For let $\mathfrak{d}$ be the derivation algebra of $\mathfrak{g}$. Since the center of $\mathfrak{g}$ is 0 the set $\mathfrak{g}$ of inner derivations forms an ideal of $\mathfrak{d}$ (regarded as an ordinary Lie algebra) isomorphic to $\mathfrak{g}$. If $\mathfrak{g}^* = \mathfrak{g} + \mathfrak{g} + \cdots$, $\mathfrak{g}^*$ is a restricted Lie algebra containing $\mathfrak{g}$. Since $[u^p v^p] = [ \cdots [u \cdots [uv] \cdots ]$, this is true for any $u, v$ in $\mathfrak{g}$, the ordinary Lie algebra $\mathfrak{g}^*/\mathfrak{g}$ is commutative.

Suppose $\mathfrak{g}$ is a restricted Lie algebra and $\alpha \to A$ is an absolutely irreducible representation of the ordinary Lie algebra determined by $\mathfrak{g}$, i.e., $a \alpha \to A \alpha$, $a + b \to A + B$, $[ab] \to [AB]$. We assume also that the representing matrices all have trace 0 and $p \nmid m$ where $m \times m$ are the dimensions of the matrices. We assert that our representation is one of the restricted Lie algebra. For if $a^p \to B$ we have $[XB] = [ \cdots [XA] \cdots A] = [X A^p]$. Hence $A^p - B = 1p$ and since $tr A^p = tr B = 0$, $A^p = B$.

6. Algebras with a finite basis. In this section we suppose $\mathfrak{g}$ has a finite basis. For any element $a$ there is a least integer $m$ such that $a, a^p, \cdots, a^{p-1}$ are linearly independent but $a^p$ depends on $a, \cdots, a^{p-1}$. Then $a^p + a^{p-1} \alpha_1 + \cdots + a \alpha_m = 0$ or $f(a) = 0$ where $f(\lambda) = \lambda^p + \lambda^{p-1} \alpha_1 + \cdots + \lambda \alpha_m$. It follows that $a^{p+1} = (a^p) \rho$, $a^{p+1} \cdots$ are linear combinations of $a$, $\cdots$, $a^{p-1}$ and hence these elements form a basis for the subalgebra generated by $a$.

A polynomial having the form of $f(\lambda)$ has been called a $p$-polynomial by Ore. The following facts were established by Ore: (1) A necessary and sufficient condition that $f(\lambda)$ be a $p$-polynomial is that its roots form a modulus (group under addition) with each root having multiplicity $p^k$, $k$ fixed. (2) Any polynomial $\phi(\lambda)$ is a factor of a $p$-polynomial $f(\lambda)$. The $f(\lambda)$ of least degree with leading coefficient 1 is unique and is a divisor of any other $p$ polynomial divisible by $\phi(\lambda)$. Thus suppose $a \to A$ is a (1-1) representation of $\mathfrak{g}$ and $\phi(\lambda)$ is the minimum polynomial of the linear transformation (or matrix) $A$. Then the $p$-polynomial $f(\lambda)$ associated with $a$ is the one of least degree divisible by $\phi(\lambda)$.

An element $a$ is nilpotent if $a^p = 0$ for some $p^n$. The least integer $p^n$ for which this holds is the index of $a$.

Theorem 5. If $\mathfrak{g}$ is a Lie algebra with a finite basis and contains only nilpotent elements, then $\mathfrak{g}$ is nilpotent.

Since $\mathfrak{g}$ has a finite basis the index of any $a$ is $\leq p^n$ where $n$ is the dimensionality of $\mathfrak{g}$. Hence $a^p = 0$ and if $A$ is the linear transformation $x \to [xa]$, $A^p = 0$. Thus $\mathfrak{g}^p = 0$ and as has been shown by Zorn [8], $\mathfrak{g}$ is nilpotent when regarded as an ordinary Lie algebra, i.e., $\mathfrak{g}^{[n]} = 0$. It follows as in §4 that $\mathfrak{g}$ is nilpotent.

The algebra $\mathfrak{g} \otimes \mathfrak{g} = \mathfrak{g}^{[1]} + \mathfrak{g}^p$. If $\mathfrak{m}$ is a subspace such that $\mathfrak{g} \supseteq \mathfrak{m} \supseteq \mathfrak{g}_2$, $\mathfrak{m}$ is a nilpotent ideal and $\mathfrak{g}/\mathfrak{m}$ is commutative and has all of its elements $\neq 0$ nil-

(7) Ore [4, p. 581].
potent of index \( p \). We choose \( \mathfrak{M} \) so that \( \dim \mathfrak{M} = n - 1 \) and let \( x_2, x_3, \ldots, x_n \) be a basis for \( \mathfrak{M} \) with \( d, x_2, \ldots, x_n \) a basis for \( \mathfrak{N} \).

**Theorem 6.** The \( u \)-algebra of a nilpotent Lie algebra with a finite basis is a nilpotent associative algebra.

The theorem is trivial if \( \mathfrak{N} \) has 1 dimension. Suppose it true for algebras of order \( n - 1 \). Choose \( \mathfrak{N} \) and \( d, x_2, \ldots, x_n \) as indicated. Then the \( u \)-algebra \( \mathfrak{U} \) is generated by \( d, x_2, \ldots, x_n \) and the \( u \)-algebra \( \mathfrak{B} \) of \( \mathfrak{M} \) is generated by \( x_2, \ldots, x_n \). \( \mathfrak{B} \) is nilpotent. The elements of \( \mathfrak{U} \) have the form

\[
u = \nu_0 + d\nu_1 + \cdots + d^{p-1}\nu_{p-1} + d\beta_1 + d^2\beta_2 + \cdots + d^{p-1}\beta_{p-1}
\]

where \( \nu_i \in \mathfrak{B} \) and \( \beta_i \in \mathfrak{N} \). The weight of \( d^{(\kappa_1)}\nu_2^{(\kappa_2)} \cdots \nu_s^{(\kappa_s)} \) where

\[
\nu^{(\kappa)} = \left[ \cdots \left[ d \cdots d \right] \cdots \right] \in \mathfrak{B}
\]

is defined to be \( \geq \kappa_1 + \cdots + \kappa_s \). Hence the weight of each term of \( \nu \) is \( \geq 1 \). Since

\[
v_1^{(\kappa_1)}v_2^{(\kappa_2)}\cdots\nu_s^{(\kappa_s)}d = d^{(\kappa_1)}v_1^{(\kappa_2)}v_2^{(\kappa_2)}\cdots\nu_s^{(\kappa_s)} + v_1^{(\kappa_1+1)}v_2^{(\kappa_2)}\cdots\nu_s^{(\kappa_s)}
\]

\[
\quad + \cdots + v_1^{(\kappa_1+1)}(\kappa_2)v_2^{(\kappa_2)}\cdots\nu_s^{(\kappa_s+1)}
\]

the weight of a product \( u_1u_2\cdots u_s \) is \( \geq k \). If the index of nilpotency of \( \mathfrak{B} \) is \( m \) and \( d^m = 0 \), then every term of weight \( \geq mp^t \) is 0. Thus \( \mathfrak{U} \) is nilpotent of index \( \leq mp^t \).

A consequence of this theorem is that the irreducible representations of a nilpotent restricted Lie algebra are all 0. Hence any representation has matrices in triangular form with diagonal elements 0 if the basis is properly chosen.

**Bibliography**

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