ON THE NUMBER OF PARTITIONS OF A NUMBER INTO UNEQUAL PARTS(1)

BY

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1. Introduction. Let \( q(n) \) be the number of partitions of an integer \( n \) into unequal parts, or into odd parts(2). Then

\[
q(n) = \frac{1}{(1 - x)(1 - x^4)(1 - x^9) \cdots}
\]

Hardy and Ramanujan(3) indicated that by their fundamental analytic method one can obtain the following result:

\[
q(n) = \frac{1}{2^{1/2}} \frac{d}{dn} J_1[i\pi \left\{ \frac{1}{3}(n + \frac{1}{2}) \right\}^{1/2}]
\]

\[+ 2^{1/2} \cos \left( \frac{2}{3} \pi n - \frac{1}{3} \pi \right) \frac{d}{dn} J_2 \left[ i\pi \left\{ \frac{1}{3}(n + \frac{1}{2}) \right\}^{1/2} \right] + \cdots
\]

\[+ \text{to } \lfloor an^{1/2} \rfloor \text{ terms } + O(1)
\]

where \( \alpha \) is an arbitrary constant. This result is less satisfactory than that concerning the number \( p(n) \) of partitions (unrestricted) of \( n \), since in the latter case the error term approaches zero with increasing \( n \). Recently Rademacher(4) obtained an equality for \( p(n) \). The object of the present paper is to find an equality for \( q(n) \). The work of this paper is a straightforward application of Hardy-Ramanujan's method with two modifications. These modifications are Kloosterman's sum and Rademacher's "Farey dissection of infinite order."

The present method may also be applied to find the explicit formula for

\[
\sum_{x=1}^{\lfloor n^{1/2} \rfloor} p(n-x^2)
\]

where \( p(n) \) is the number of unrestricted partitions of \( n \).

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(1) This paper was accepted by Acta Arithmetica before the war.


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2. Statement of the result. Let

\[
\epsilon_{h,k} = \begin{cases} 
\exp\left(-\pi i \left(\frac{h'^2 - 1}{8} \left(\frac{1 - hh'}{k} - 1\right) + \frac{h'(1 - hh')}{8k}\right)\right), & \text{for } 2 \mid k, \\
n\exp\left(\frac{\pi i}{24} \left(k + \frac{1 - hh'}{k}\right)(h + h' - h^2 h')\right), & \text{for } 2 \nmid k, 2 \nmid h, \\
\exp\left(-\frac{\pi i}{8} \left(k^2 - 1 - hh k + \frac{1}{3} (h + h') (hh'k - \frac{hh' - 1}{k})\right)\right), & \text{for } 2 \nmid k, 2 \mid h,
\end{cases}
\]

and

\[
\omega_{h,k} = \begin{cases} 
\epsilon_{h,k} \exp\left(-\frac{\pi i}{12k} (h + h')\right), & \text{for } 2 \mid k, \\
\epsilon_{h,k} \exp\left(-\frac{\pi i}{24k} (2h - h')\right), & \text{for } 2 \nmid k,
\end{cases}
\]

where \(hh' \equiv 1 \pmod{k}\), \(h \equiv h' \pmod{2}\).

**Theorem.** The number of partitions of an integer \(n\) into unequal parts is given by

\[
q(n) = \frac{1}{2^{1/2}} \sum_{k=1, k \text{ odd}} ^\infty \sum_{(h,k)=1, 0 \leq h \leq k} \omega_{h,k} e^{-2\pi i nk/k} \frac{d}{dn} J_0 \left(\frac{\pi}{k} \left\{\frac{3}{2} (n + \frac{1}{k})\right\}^{1/2}\right),
\]

where \(J_0(x)\) is the Bessel function of the 0th order.

3. Farey dissection. By means of Cauchy's integral formula we obtain for (1.1)

\[
q(n) = \frac{1}{2\pi i} \int_c \frac{f(x)}{x^{n+1}} dx.
\]

The path of integration may be the circle defined as \(|x| = e^{-2\pi N^{-2}}\) where \(N\) is a certain positive integer at our disposal. In the usual way we divide the circle into Farey arcs \(\xi_{h,k}\) of order \(N\). The Farey arc \(\xi_{h,k}\) is defined by

\[
x = \exp \left(2\pi i h/k - 2\pi N^{-2} + 2\pi i\theta\right), \quad (h, k) = 1,
\]

and

\[
-\theta_1(h, k) = \frac{h + h_1}{k + k_1} \leq \theta \leq \frac{h + h_2}{k + k_2} - \frac{h}{k} = \theta_2(h, k)
\]
where \( h_1/k_1, h/k, h_2/k_2 \) are three consecutive fractions in the Farey sequence of order \( N \). It is well known that

\[
\left( \frac{1}{k(N + k)} \right) \leq \vartheta_1(h, k) < \frac{1}{k(N + 1)}, \quad \frac{1}{k(N + k)} \leq \vartheta_2(h, k) < \frac{1}{k(N + 1)}.
\]

We obtain then

\[
q(n) = \frac{1}{2\pi i} \sum_{(h,k)=1, 0 < h \leq k \leq N} \int_{\xi_{h,k}} f(x) x^{n+1} dx.
\]

Let \( I_1 \) and \( I_2 \) denote the sums of those terms satisfying \( 2 \mid k \), and \( 2 \nmid k \), respectively. Then, by (3.4), we have

\[
q(n) = I_1 + I_2.
\]

4. Lemmas on Kloosterman's sums.

**Lemma 4.1**. Let

\[
g(N, \vartheta, h, k) = \begin{cases} 
1 & \text{for} \quad -\vartheta_1(h, k) \leq \vartheta \leq \vartheta_2(h, k), \\
0 & \text{otherwise}.
\end{cases}
\]

Then

\[
g = \sum_{r=1}^{k} b_r e^{2\pi i h'/k}
\]

where \( h' \) is an integer satisfying

\[
hh' \equiv 1 \pmod{k},
\]

and \( b_r \) is independent of \( h \) and

\[
\sum_{r=1}^{k} |b_r| < \log 4k.
\]

**Lemma 4.2.** Let \( a \) be an absolute constant. Then

\[
\sum_{0 < h \leq ak, (h, ak) = 1} \exp \left( \frac{2\pi i}{ak} (mk + mh') \right) = O(k^{2/3 + \varepsilon}(n, k)^{1/3}).
\]

**Lemma 4.3.** If \( k \) is even and \( \omega_{h', k} \) as defined in \( \S 2 \), then

\[
S_k = \sum_{1 \leq h \leq k, (h, k) = 1, hh' = 1} \omega_{h', k} e^{2\pi i (nh + mh')/k} = O(k^{2/3 + \varepsilon}(n, k)^{1/3}).
\]

**Proof.** For the sake of simplicity I give here only the proof of the case \( 24 k \).

\(^{(c)}\) T. Estermann, Abhandlungen aus dem Mathematischen Seminar der Hamburgischen Universität, vol. 7 (1929), pp. 93, 94.
Then

\[ S_k = \sum_{1 \leq i \leq 24, (i, 24) = 1} \sum_{1 \leq h \leq k, (h, k) = 1, hh' = 1, h = i (24)} \omega_{h, k} e^{2\pi i (nh + mh')/k}. \]

The inner sum becomes a Kloosterman's sum as in Lemma 4.2. Therefore we have

\[ S_k = O(k^{2/3 + \varepsilon} (n, k)^{1/3}). \]

As to the proof of the other cases, nothing is difficult but a little complicated, and the following fact is used: let

\[ F(h, k) = \omega_{h, k} e^{2\pi i (nh + mh')/k}, \]

then \( F(h + k, k) = F(h, k) \).

**Lemma 4.4.** Let \( 2 \mid k \) and \( \omega_{h, k} \) be as defined in §2, then

\[ S = \sum_{1 \leq h \leq k, (h, k) = 1, hh' = 1 (k), h' \text{ odd}} \omega_{h, k} e^{2\pi i (2nh + mh')/k} = O(k^{2/3 + \varepsilon} (h, k)^{1/3}). \]

The proof is similar to that of Lemma 4.3, only notice that

\[ \omega_{h, k} e^{2\pi i (nh + mh')/k} = 0 \text{ if } h' \text{ odd}. \]

5. Lemmas from the theory of the linear transformation of the elliptic modular functions.

**Lemma 5.1.** Suppose that \( 2 \mid h, 2 \mid k \); that \( h' \) is a positive integer satisfying \( hh' = 1 \pmod{k} \); that \( \omega_{h, k} \) is defined in §2; and that

\[ x = \exp \left( - \frac{2\pi z}{k} + \frac{2\pi i k}{k} \right), \quad x' = \exp \left( - \frac{2\pi z}{k} - \frac{2\pi i k}{k} \right), \]

where the real part of \( z \) is positive. Then

\[ f(x) = \omega_{h, k} \exp \left( - \frac{\pi}{12k} + \frac{\pi z}{12k} \right) f(x'). \]

**Proof.** If we take \( a = h, b = -k, c = (1 - hh')/k, d = h' \), so that \( ad - bc = 1 \), and write

\[ x = q^2 = e^{2\pi i \tau}, \quad x' = Q^2 = e^{2\pi i T}, \]
\[ \tau = (h + iz)/k, \quad T = (-h' + i/z)/k, \]

then we can easily verify that

\[ T = \frac{c + d\tau}{a + b\tau}. \]
Also, in the notation of Tannery and Molk, we obtain
\[
f(x) = \frac{1}{2^{1/3}} g^{-1/12} \frac{\phi(t)}{\chi(t)}, \quad f(x') = \frac{1}{2^{1/3}} Q^{-1/12} \frac{\phi(T)}{\chi(T)}.
\]

Then
\[
f(x') = \frac{1}{2^{1/3}} Q^{-1/12} \frac{\phi(T)}{\chi(T)} = \exp \left( \pi i \left( \frac{1}{8} (d^2 - 1)(c - 1) + \frac{cd}{8} \right) - \frac{(b - c)(bcd - a)}{24} \right) \frac{1}{2^{1/3}} Q^{-1/12} \frac{\phi(t)}{\chi(t)}
\]
\[
= \exp \left( \pi i \left( \frac{1}{8} (d^2 - 1)(c - 1) + \frac{cd}{8} \right) - \frac{(b - c)(bcd - a)}{24} \right) \frac{1}{2^{1/3}} Q^{-1/12} f(x)
\]
\[
= \exp \left( \pi i \left( \frac{1}{8} (d^2 - 1)(c - 1) + \frac{cd}{8} \right) - \frac{(b - c)(bcd - a)}{24} \right) \cdot \exp \left( \frac{\pi i}{24k} (h + h') \right) f(x).
\]

**Lemma 5.2.** Suppose that \(2 \nmid h \) and \( hh' \equiv 1 \pmod{2k} \), that
\[
f_1(x) = \prod_{1}^{\infty} (1 + x^{n-1/2}) = 1 + \sum_{n=1}^{\infty} q_1(n)x^{n/2}.
\]

Then
\[
f(x) = \frac{\omega_{h,k}}{2^{1/2}} \exp \left( \frac{\pi i}{24k} \left( z + \frac{1}{2z} \right) \right) f_1(x').
\]

**Proof.** As in Lemma 5.1, we have
\[
f_1(x) = f_1(q^2) = \prod (1 + q^{2n-1}) = 2^{1/8} q^{1/24} \frac{1}{\chi(T)}
\]
\[
f_1(x') = 2^{1/8} Q^{1/24} \frac{1}{\chi(T)} = 2^{1/8} Q^{1/24} \exp \left( \frac{(b - c)(abc - d)}{24} \pi i \right) \frac{\phi(T)}{\chi(T)}
\]
\[
= 2^{1/8} Q^{1/24} \exp \left( \frac{(b - c)(abc - d)}{24} \pi i \right) 2^{1/3} Q^{1/12} f(x)
\]
\[
= \exp \left( - \frac{(b - c)(abc - d)}{24} \pi i \right) 2^{1/2}
\]
\[
\cdot \exp \left( \frac{\pi i}{24k} \left( - \frac{h'}{k} + \frac{i}{kz} + \frac{2h}{k} + \frac{2iz}{k} \right) \right) f(x).
\]
Lemma 5.3. Suppose that $2 \mid h, 2 \nmid k, hh' \equiv 1 \pmod{k}$, $2 \not\mid h'$ and suppose that

$$f_2(x) = \prod_{n=1}^{\infty} (1 - x^{n-1/2}) = 1 + \sum q_2(n) x^{n/2}.$$ 

Then

$$f(x) = \frac{\omega_{h,k}}{2^{1/2}} \exp \left( \frac{\pi}{12k} \left( z + \frac{1}{2z} \right) \right) f_2(x').$$

Proof. We take

$$a = -h, \quad b = k, \quad c = (hh' - 1)/k, \quad d = -h'.$$

Then

$$f_2(x') = f_2(Q^2) = 2^{1/8} Q^{1/24} \frac{\psi(T)}{\chi(T)} \frac{\phi(t)}{\chi(t)}$$

$$= 2^{1/8} Q^{1/24} \exp \left( \frac{\pi i}{2} \left( \frac{b^2 - 1}{4} + \frac{ab}{4} - \frac{(a + d)(abd - c)}{12} \right) \right) \frac{\phi(t)}{\chi(t)}$$

$$= 2^{1/8} \exp \left( \frac{\pi i}{2} \left( \frac{b^2 - 1}{4} + \frac{ab}{4} - \frac{(a + d)(abd - c)}{12} \right) \right) Q^{1/24} q^{1/12} f(x).$$

6. Approximation of the integrand. Let

$$z = k(N^{-2} - i\theta).$$

Then

$$I_1 = \sum_{1 \leq h \leq N} \sum_{1 \leq k \leq \frac{N}{2}} \int_{k^{-1}(N+1)^{-1}}^{k^{-1}(N+1)^{-1}} g(\theta) f(e^{2\pi ih - 2\pi x/z}) e^{-2\pi ihn/k} \frac{q(\nu)}{k^2} d\theta$$

$$= \sum_{1 \leq h \leq N} \sum_{1 \leq k \leq \frac{N}{2}} \int_{k^{-1}(N+1)^{-1}}^{k^{-1}(N+1)^{-1}} g(\theta) \omega_{h,k} e^{(\pi/12k)(-1/z)} f(x') e^{-2\pi ihn/k} \frac{q(\nu)}{k^2} d\theta$$

$$= \sum_{1 \leq h \leq N} \sum_{1 \leq k \leq \frac{N}{2}} \int_{k^{-1}(N+1)^{-1}}^{k^{-1}(N+1)^{-1}} g(\theta) \omega_{h,k} e^{(\pi/12k)(-1/z)} e^{-2\pi ihn/k} \frac{q(\nu)}{k^2} d\theta$$

$$= \sum_{1 \leq h \leq N} \sum_{1 \leq k \leq \frac{N}{2}} \int_{k^{-1}(N+1)^{-1}}^{k^{-1}(N+1)^{-1}} \sum_{\nu = 0}^{\infty} q(\nu) e^{-2\pi i/k} (x + 1/24) + (2\pi i/k) (n+1/24)$$

$$\sum_{r=1}^{k} b_r e^{2\pi i k h'/k} \omega_{h,k} e^{-2\pi ihn/k} h' e^i k d\theta.$$
Since \((1/k)\Re(1/z) \geq \frac{1}{2}\), we have

\[
|I_1| \leq \sum_{1 \leq k \leq N, 2|k} \int_{-k^{-1}(N+1)^{-1}}^{k^{-1}(N+1)^{-1}} \sum_{\nu=0}^{\infty} q(\nu) \cdot \exp \left\{ -\frac{2\pi}{k} \left( \nu + \frac{1}{24} \right) \Re \frac{1}{z} + \frac{2\pi}{k} \left( \nu + \frac{1}{24} \right) \Re z \right\} \sum_{(h,k)=1}^{k} \omega_{h,k} e^{-2\pi i h n/k + 2h'(r-v) \pi i/k} \right| d\theta
\]

\[
= O \left( \sum_{k=1}^{N} \int_{-k^{-1}(N+1)^{-1}}^{k^{-1}(N+1)^{-1}} \sum_{\nu=0}^{\infty} q(\nu) e^{-\pi (r+1/24)} \sum_{r=1}^{k} b_r \left| k \cdot k^{2/3} \right| d\theta \right)
\]

\[
= O \left( \sum_{k=1}^{N} \log k \cdot k^{2/3} \frac{1}{kN} \right) = O \left( \frac{1}{N} \sum_{k=1}^{N} k^{-1/3+\epsilon} \right)
\]

\[
= O(N^{-1/3+\epsilon}).
\]

Let

\[
J = \frac{1}{2^{1/2}} \sum_{k=1, k \text{ odd}}^{N} \sum_{(h,k)=1, 0 < h \leq k} \int_{-k^{-1}(N+1)^{-1}}^{k^{-1}(N+1)^{-1}} g(\theta) \omega_{h,k} \cdot e^{2\pi i (n/24)+\pi/24 k^2 \theta} d\theta.
\]

The same method will give us that \(|I_2 - J| = O(N^{-1/3+\epsilon})\).

7. A contour integration. Let \(w = N^{-2} - i\theta\). Then

\[
J = \frac{-i}{2^{1/2}} \sum_{1 \leq k \leq N, k \text{ odd}} \sum_{(h,k)=1, 0 < h \leq k} \omega_{h,k} e^{-2\pi i h n/k} \int_{N^{-2} - i\theta_1}^{N^{-2} + i\theta_1} e^{2\pi i w(n/24)+\pi/24 k^2 \theta} d\theta
\]

\[
= \frac{i}{2^{1/2}} \sum_{1 \leq k \leq N, k \text{ odd}} \sum_{(h,k)=1, 0 < h \leq k} \omega_{h,k} e^{-2\pi i h n/k} \left( \int_{N^{-2} + i\theta_1}^{N^{-2} + ik^{-1}(N+1)^{-1}} + \int_{N^{-2} - ik^{-1}(N+1)^{-1}}^{N^{-2} - i\theta_1} \right)
\]

\[
+ \int_{N^{-2} - ik^{-1}(N+1)}^{N^{-2} - i\theta_1} - 2\pi i \text{ Residue at } 0 \right)
\]

\[
= K_1 + K_2 + K_3 + K_4 + K_5 + L \text{ (say)}.
\]

We have

\[
K_1 = \frac{i}{2^{1/2}} \sum_{1 \leq k \leq N, k \text{ odd}} \sum_{(h,k)=1, 0 < h \leq k} \omega_{h,k} e^{-2\pi i h n/k} \int_{N^{-2} + ik^{-1}(N+1)^{-1}}^{N^{-2} + i\theta_1} g(\theta) e^{2\pi i w(n/24)+\pi/24 k^2 \theta} d\theta.
\]
By Lemma 3.1, we have

\[
K_1 = O\left(\sum_{1 \leq k \leq N, k \text{ odd}} k^{2/3 + \epsilon} \int_{k^{-1}(N+k)^{-1}}^{k^{-1}(N+1)^{-1}} \exp \left\{ 2\pi \left( n + \frac{1}{24} \right) R_w + \frac{\pi}{24k^2} R \frac{1}{w} \right\} dw \right).
\]

\[
= O\left( \sum_{k=1}^{N} k^{2/3 + \epsilon} e^{-2\pi n N^{-1}} \int_{k^{-1}(N+1)^{-1}}^{k^{-1}(N+k)^{-1}} d\theta \right).
\]

\[
= O(N^{-1/3 + \epsilon}).
\]

Similar result holds for \( K_6. \)

We have

\[
R \frac{1}{k^2 w} = \frac{N^{-2}}{k^2 N^{-2} + N^2}, \quad K_2 = O\left( \sum_{k=1}^{N} N^{-2} k^{2/3 + \epsilon} \right) = O(N^{-1/3 + \epsilon}).
\]

Similar result holds for \( K_4. \)

Applying again Kloosterman's argument to \( K_4, \) we have also \( K_3 = O(N^{-1/3}). \)

Finally we find the residue of \( \exp \left( 2\pi w (n + 1/24) + \pi/24 k^2 w \right) \) at \( w = 0. \) We have the expansion

\[
e^{2\pi w (n+1/24)} = \sum_{\nu=1}^{\infty} \frac{2\pi w(n+1/24)^\nu}{\nu!},
\]

\[
e^{\pi/24 k^2 w} = \sum_{\mu=1}^{\infty} \frac{1}{\mu!} \left( \frac{\pi}{24k^2 w} \right)^\mu.
\]

The residue is, therefore,

\[
\sum_{\mu=1}^{\infty} \frac{1}{\mu!} \left( \frac{\pi}{24 k^2} \right)^\mu \frac{1}{(\mu - 1)!} (2\pi(n + \frac{1}{24}))^{\mu-1}
\]

\[
= \frac{1}{2\pi} \frac{d}{dn} \sum_{\mu=1}^{\infty} \frac{1}{(\mu!)^2} (2\pi(n + \frac{1}{24}))^\mu
\]

\[
= \frac{1}{2\pi} \frac{d}{dn} \sum_{\mu=1}^{\infty} \frac{1}{2^{\mu} (\mu!)^2} \left( \frac{\pi}{k} \{ \frac{1}{2}(n + \frac{1}{24}) \}^{1/2} \right)^{2\mu}
\]

\[
= \frac{1}{2\pi} \frac{d}{dn} J_0 \left( \frac{i\pi}{k} \{ \frac{1}{2}(n + \frac{1}{24}) \}^{1/2} \right).
\]

Therefore

\[
g(n) = \frac{1}{2^{1/2}} \sum_{k=1, k \text{ odd}}^{N} \omega_{k, \alpha} e^{-2\pi i n k / \alpha} \frac{d}{dn} J_0 \left( \frac{i\pi}{k} \{ \frac{1}{2}(n + \frac{1}{24}) \}^{1/2} \right)
\]

\[
+ O(N^{-1/3 + \epsilon}).
\]

Let \( N \to \infty; \) we obtain the theorem.

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