ERGODIC THEOREMS FOR ABELIAN SEMI-GROUPS

BY

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This paper adapts the method of F. Riesz to the proof of certain general ergodic theorems for Abelian semi-groups of operators on a Banach space to itself. The main features of the method are that no measurability conditions are imposed on the semi-group under consideration and that consistent use of the second conjugate space and its compactness properties make it possible to replace the compactness conditions often imposed by a more natural restriction on the transforms of points. Theorem 1 and the various supplementary results include as special cases theorems of Lorch [10], Dunford [7], Yosida [15], F. Riesz [12], and Cohen [6]. It overlaps the work of Alaoglu and Birkhoff [3] at those points where they consider Abelian cases; for example, Corollary 8 is a great generalization of their Theorem 5.

Section 1 contains some introductory material on conjugate spaces and adjoint operations. Section 2 introduces bounded Abelian semi-groups of operators and near invariance of a system of set functions on such a semi-group; this section also contains the principal theorem (Theorem 1) of the paper. The form of this theorem raises three questions ((A) to (C) at the beginning of §3). The answer to (A) shows, among other things, that every Abelian semi-group has a property much like "ergodicity" in the sense of Alaoglu and Birkhoff; Theorem 3 is the main result here. The answer to (B) again indicates the importance of reflexivity in theorems of this type; Corollary 8 is one example. Two special cases of (C) give a generalization of Dunford's theorem (Theorem 5) and a theorem on bounded Abelian semi-groups of projections (Theorem 6) which has not, so far as I know, been considered before.

1. Some properties of Banach spaces. If \( B \) is a Banach space\(^{(2)} \), let \( B^* \) be the set of all linear—that is, additive and continuous—real-valued functions on \( B \). If, for \( \beta \) in \( B^* \), \( \| \beta \| = \sup_{\| b \| \leq 1} |\beta(b)| \), then \( B^* \) is also a Banach space. As is usual, the weak neighborhood topology in \( B \) is defined as follows: For each \( \varepsilon_0 \) in \( B \) the weak neighborhoods of \( b_0 \) are the sets of the form\(^{(3)} \) \( \{ b \mid |\beta_i(b) - \beta_i(b_0)| < \varepsilon \text{ for } i = 1, \cdots, k \} \) for every choice of \( \varepsilon > 0, k \) a positive integer, and \( \beta_1, \cdots, \beta_k \) in \( B^* \). With this topology \( B \) is a linear topological

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\(^{(1)} \) Part of the work on this paper was done at the Institute for Advanced Study while the author held the Corinna Borden Keen Research Fellowship of Brown University.

\(^{(2)} \) See Banach [4]. A Banach space is a complete norm vector space.

\(^{(3)} \) \( \{ b \mid \cdots \} \) is the set of all points \( b \) satisfying the condition after the vertical bar.
space (see Wehausen [12]) and hence is a regular Hausdorff space in which addition of elements and multiplication by real numbers are continuous operations.

Since $B^*$ is a Banach space, it has a weak topology defined just as in $B$; however, there is another topology in a conjugate space which cannot be defined in every space. This is the weak* topology in which the neighborhoods of a point $\beta_0$ in $B^*$ are the sets of the form \( \{ \beta \mid |\beta(b_i) - \beta_0(b_i)| < \epsilon \text{ for } i = 1, \ldots, k \} \) for all choices of $\epsilon > 0$, $k$ a positive integer, and $b_1, \ldots, b_k$ in $B$. Each weak* neighborhood of $\beta_0$ is a weak neighborhood of $\beta_0$ but the converse is not true unless $B$ is reflexive. The importance of the weak* topology in this paper arises from

**Lemma 1.** The unit sphere in $B^*$ is always compact\(^\dagger\) in the weak* neighborhood topology.

This has been proved by Alaoglu [2], Kakutani [9], and Šmulian [13].

If $B$ is a Banach space, let $B^{**}$ be the space $(B^*)*$. Then there is a natural imbedding of $B$ in $B^{**}$ which associates to each $b$ in $B$ the point $b_b$ in $B^{**}$ such that $b_b(\beta) = \beta(b)$ for every $\beta$ in $B^*$. $B$ is reflexive if $B$ fills up $B^{**}$ under this imbedding. For the rest of this paper $B$ will be considered to be imbedded in this way in $B^{**}$ whenever it seems convenient.

If $T$ is a linear operator defined on $B$ with values in $B$, let $T^*$, the adjoint of $T$, be the operator on $B^*$ to $B^*$ such that $\beta(Tb) = T^*\beta(b)$ for every $\beta$ in $B^*$ and $b$ in $B$. Then:

1. $||T^*|| = ||T||$.
2. $(T_1T_2)^* = T_2^*T_1^*$ so $T_1^*$ and $T_2^*$ commute if $T_1$ and $T_2$ do.
3. If $T^{**} = (T^*)^*$, then $T^{**}b_b = b_{T_b}$, since $T^{**}b_b(\beta) = b_b(T^*\beta) = T^*\beta(b) = \beta(Tb) = b_{T_b}(\beta)$ for every $\beta$ in $B^*$.

For brevity $T$ will sometimes be used for $T^{**}$.

If $Y$ is any set of elements $y$, $M_Y$ is the Banach space of all real-valued bounded functions $\phi$ on $Y$ with\(^\ddagger\) $||\phi||_{M_Y} = \sup_{y \in Y} |\phi(y)|$. If $B$ is any Banach space, $M_Y(B)$ is the Banach space of all bounded functions $f$ on $Y$ with values in $B$ where $||f||_{M_Y(B)} = \sup_{y \in Y} ||f(y)||_B$. If $T$ is any element of $M_Y^*$, it is possible to define $U$ on $M_Y(B)$ to $B^{**}$ by letting $U(f)$ be that point $b$ of $B^{**}$ such that $b(\beta) = T(\beta f)$ for every $\beta$ in $B^*$, where $\beta f$ is the element of $M_Y$ defined by $\beta f(y) = \beta(f(y))$. For each $T$ in $M_Y^*$ there is defined a unique, bounded, additive\(^\ddagger\) set function $\Psi$ by the relation $\Psi(E) = T(\phi_E)$, where $\phi_E$ is the charac-

\(^\ddagger\) $T$ is compact (bicom pact in the sense of Alexandroff and Hopf, Topologie, I) if every covering of $E$ by open sets contains a finite subcovering; that is, if $E \subseteq \bigcup_{a} O_a$, where the $O_a$ are open, there exist $a_1, \ldots, a_k$ such that $E \subseteq \bigcup_{i=1}^{k} O_{a_i}$; this is equivalent to the following condition on closed sets: If the closed sets $C_a \subseteq E$ are such that every finite set of the $C_a$ have a point in common, then $\prod_a C_a$ is not empty.

\(^\ddagger\) The subscript on the norm symbol indicates the space in question; it will be omitted when there is no danger of confusion.

\(^\ddagger\) $\Psi$ is bounded if $\Psi(E) \leq K$ if $E \subseteq Y$, $\Psi$ is additive if $\Psi(E_1 + E_2) = \Psi(E_1) + \Psi(E_2)$ whenever
teristic function of $E$. Conversely, for each bounded additive $\Psi$, setting $T(\phi) = \int \phi d\Psi$ defines an $T$ in $M_{\mathbb{S}}$, where the integral is, say, that of Radon-Stieltjes (1); moreover $\|T\| = V\Psi(Y)$. Because of this relation between $\Psi$ and $T$ it is possible to define $\int \phi d\Psi$ for $f$ in $M_{\mathbb{S}}(B)$ to be the element $b$ of $B^*$ for which $b(\beta) = \int f d\Psi$ for every $\beta$ in $B^*$. All integrals used hereafter will be of this nature (8).

A set $X$ is directed if there is a relation $>$ ("follows") among some pairs of its points such that $x > x'$ and $x' > x''$ implies $x > x''$ and such that each pair, $x'$ and $x''$, of points in $X$ has a common successor, $x$ in $X$; that is, $x > x'$ and $x > x''$. If for each $x$ in $X$, $s_x$ is a point of the topological space $S$, then $s = \lim_x s_x$ if and only if for each neighborhood $N$ of $s$ there is an $x_N$ in $X$ such that $s_N \subseteq N$ if $x > x_N$.

**Lemma 2.** If $X$ is a directed set, if for each $x$ in $X$ $b_x$ is a point in $B^*$ ($B$ any Banach space), and if $\|b_x\|$ is ultimately bounded—that is, if there exist $K > 0$ and $x_0 \in X$ such that $\|b_x\| \leq K$ if $x > x_0$—then there is a $b_0$ in $B^*$ such that there exists the weak* closure of $\{b_x : x' > x\}$ for every $x$ in $X$; that is, for every $\epsilon > 0, \beta_1, \ldots, \beta_k$ in $B^*$ and $x$ in $X$ there is an $x'$ in $X$ such that $x' > x$ and $\|b_0(\beta_i) - b_x'(\beta_i)\| < \epsilon$ for $i = 1, \ldots, k$.

For each $x$ in $X$ let $E_x = \{b_x' : x' > x\}$; since $X$ is directed, any finite number of the $E_x$ have a point in common, so the sets $F_x$ which are obtained by taking the weak* closure of $E_x$ are weak* closed sets with non-empty finite intersections. Since the sphere $\|b_x\| \leq K$ is weak* compact, by the condition for this in terms of closed sets some $b_0$ exists in all these $F_x$; the last clause in the lemma is merely a full statement of the fact that $b_0$ is in the weak* closure of every $E_x$.

2. The principal theorem. The terms next defined are the ones used in the statement of the theorem and not merely in its proof.

A set $Y$ is called an Abelian semi-group if there is defined for each pair of elements $y, y'$ in $Y$ a sum $y + y'$ in $Y$ such that $y + y' = y' + y$, and such that $y + (y' + y'') = (y + y') + y''$; that is, addition is commutative and associative. If $E$ is a subset of $Y$, then $E \pm y$ is the set $\{y' \mid y' + y \in E\}$. It is clear that if $E$ is any partition of $Y$ into any number of disjoint sets, then $E \pm y$ is also such a partition.

Let $X$ be a directed set and $Y$ an Abelian semi-group; for each $x$ in $X$ let $\Psi_x$ be a bounded, additive set function over $Y$. For each $y$ in $Y$ and $E_1$ and $E_2$ are disjoint subsets of $Y$. If $\Psi$ is bounded $V\Psi(Y) = \sup \sum_{i \leq k} |\Psi(E_i)| \leq 2K$, where the "sup" is taken over all partitions of $Y$ into a finite number of disjoint subsets $E_i$.

(1) The pertinent properties are these: (1) If $\phi = \sum_{i \leq k} \alpha_i \phi_{E_i}$, $\int \phi d\Psi = \sum_{i \leq k} \alpha_i \Psi(E_i)$. (2) If $\|\phi_n - \phi\|_{M_{\mathbb{S}}} \to 0$, then $\int \phi_n d\Psi \to \int \phi d\Psi$. See Fichtenholz and Kantorovich [7].

$x$ in $X$ let $\Psi_{sy}$ be the bounded additive set function over $Y$ defined by $\Psi_{sy}(E) = \Psi_s(E - y) - \Psi_s(E)$ for every $E \subseteq Y$; the system of set functions $\{\Psi_s\}$ is called nearly invariant over $Y$ if $\lim_s V\Psi_{sy}(Y) = 0$ for each $y$ in $Y$, if $V\Psi_s(Y)$ is bounded, and if $\lim_s \Psi_s(Y) = 1$. $\Psi$ is called invariant if $\Psi(E - y) = \Psi(E)$ for every $E \subseteq Y$ and $\Psi(Y) = 1$.

If $Y$ is an Abelian semi-group and $B$ is a Banach space and if for each $y$ in $Y$, $T^y$ is an element of $\mathcal{Y}$, the space of linear operators on $B$ to $B$, the semi-group $\{T^y\}$ is called a bounded representation of $Y$ if $T^yT^{y'} = T^{y+y'}$ for every $y$ and $y'$ in $Y$, and if there is a $K \geq 0$ such that $\|T^y\| \leq K$ for every $y$. If $\{T^y\}$ is any bounded representation of $Y$, let $B'$ and $B''$ be the subsets of $B$ defined by $B'' = \{b \mid Tb = b$ for every $y$ in $Y\}$ and $B'$ is the smallest closed linear subset of $B$ containing all the points $b - T^yb$ for every $b$ in $B$ and $y$ in $Y$; $M$ is the smallest linear subset of $B$ containing $B'$ and $B''$. Clearly, $b \in M$ if and only if one (or all) $T^yb \in M$, since $b = T^yb + (b - T^yb)$ and $b - T^yb$ is in $B'$.

**Theorem 1.** Let $Y$ be any Abelian semi-group, $B$ a Banach space and $\{T^y\}$ a bounded representation of $Y$ in $\mathcal{Y}$. Let $X$ be a directed set and $\{\Psi_s\}$ a nearly invariant system of set functions over $Y$, and for each $b$ in $B$ and $x$ in $X$ let $T_x b = \int f^b d\Psi_s$ where $f^b$ in $MY(B)$ is defined by $f^b(y) = T^yb$ for each $y$ in $Y$; for each $b$ in $B$ let $Tb$ be one of the points which Lemma 1 associates with the points $T_x b$. Then:

1. $Tb$ can be taken in $B$ instead of merely in $B''$ if and only if $b$ is in $M$.
2. $M$ is a closed linear subset of $B$ and is the direct sum of $B'$ and $B''$, that is, each $b$ in $M$ is the sum of a $b'$ in $B'$ and a $b''$ in $B''$, where $b'$ and $b''$ are uniquely determined; in fact $b'' = Tb$.
3. $Tb$ is uniquely determined if $b \in M$ and, in $M$, $T$ is a linear operator with values in $B''$.
4. If $b \in M$, $Tb = Tb$ and $T Tb = T Tb = T Tb$ for every $y$ in $Y$.
5. $Tb = b$ if and only if $b \in B''$; $Tb = 0$, the zero element in $B$, if and only if $b \in B'$.
6. If $b \in M$, $\|T \phi s b - Tb\| \to 0$.

The main body of the proof will be divided into a number of simple steps.

(a) $T^y T_x b = T_x T^y b$ for every $x$ in $X$, $y$ in $Y$ and $b$ in $B$.

For each $b$ in $B^*$,

$$T^y T_x b(\beta) = T_x b(T^y \beta) = \int T^y \beta f^b d\Psi_s = \int \beta f^y T^b d\Psi_s = \tau_s T^y b(\beta)$$

since

$$T^y \beta f^b(t) = T^y \beta(T' b) = \beta(T^y T' b) = \beta(T' T^y b) = \beta f^y T^b(t)$$

for each $t$ in $Y$.

(b) $\|T^y T_x b - \tau_s b\| \to 0$ for each $b$ in $B$ and $y$ in $Y$.

(*) It should be noted that the condition that $V\Psi_{sy}(Y) \to 0$ for each $y$ which is used in the
\[ \| T^\tau x b - \tau_x b \| = \| \tau_x T^\tau b - \tau_x b \| = \sup_{|\beta| \leq 1} \| \tau_x T^\tau b(\beta) - \tau_x b(\beta) \| = \sup_{|\beta| \leq 1} \left| \int \beta f T^\tau b d\Psi_x - \int \beta f b d\Psi_x \right| = \sup_{|\beta| \leq 1} \left| \int \beta f b d\Psi_x \right| \leq \| f \| \| V \Psi_x(Y) \| \]

which tends to zero for each \( y \). The only difficulty is in justifying the last equality which can be done as follows: since \( \beta f \in M_Y \) for each \( \beta \) in \( B^* \) and \( b \) in \( B \), it suffices to prove that \( \int \phi d\Psi_x y = \int \phi d\Psi_x z - \int \phi d\Psi_x \) where \( \phi_y \) is defined by \( \phi_y(y') = \phi(y + y') \) and \( \phi \) is any function in \( M_Y \). If \( \phi \) is a simple function; that is, \( \phi = \sum i \leq k \alpha_i \phi_E \) where the \( E_i \) are disjoint subsets of \( Y \) and the \( \alpha_i \) are real numbers; then \( \int \phi d\Psi_x y = \int \sum i \leq k \alpha_i \Psi_x(E_i) = \sum i \leq k \alpha_i \Psi_x(E_i - y) - \sum i \leq k \alpha_i \Psi_x(E_i) = \int \phi d\Psi_x z - \int \phi d\Psi_x \) since \( \phi_y = \sum i \leq k \alpha_i \phi_E \). Since the simple functions are dense in \( M_Y \) the same is true for any \( \phi \) in \( M_Y \).

(c) \( T^\tau rb = rb \) for any \( y \) in \( Y \) and \( b \) in \( B \).

It suffices to show for any \( \epsilon > 0 \) and \( \beta \) in \( B^* \) that \( \left| T^\tau rb(\beta) - rb(\beta) \right| < \epsilon. \) But

\[ \left| T^\tau rb(\beta) - rb(\beta) \right| \leq \left| T^\tau rb(\beta) - T^\tau \tau_x b(\beta) \right| + \left| T^\tau \tau_x b(\beta) - \tau_x b(\beta) \right| + \left| \tau_x b(\beta) - rb(\beta) \right| ; \]

by (b) the middle term is less than \( \epsilon/3 \) if \( x > x_0 \). The first term is equal to \( \left| T^\tau b(T^\tau b(\beta) - \tau_x b(T^\tau b(\beta)) \right| \); by the definition of \( rb \) and Lemma 1, \( rb \) is in the weak* closure of \( \{ \tau_x b \mid x > x_0 \} \) so there is an \( x > x_0 \) such that \( \left| \tau_x b(\beta) - \tau_x(\beta) \right| < \epsilon/3 \) and \( \left| \tau_x b(T^\tau b(\beta) - rb(T^\tau b(\beta)) \right| < \epsilon/3 \) for this \( x \) all three terms are less than \( \epsilon/3 \) and (c) follows.

(d) If \( d \in B' \), then \( \| \tau_x d \| \to 0 \).

If \( d = b - T^\tau b \) for some \( b \) in \( B \) and \( y \) in \( Y \) then \( \| \tau_x d \| = \| T^\tau x b - \tau_x b \| \to 0 \) by (b). Since all the operations in question are additive and homogeneous, \( \| \tau_x d \| \to 0 \) if \( d = \sum i \leq k \alpha_i (b_i - T^\tau b_i) \) for any choice of \( y_i \) in \( Y \), \( b_i \) in \( B \), and \( \alpha_i \) real. If \( d \in B' \) then there is a \( d_e \) of this last form such that \( \| d - d_e \| < \epsilon \); then \( \| \tau_x d \| < \| \tau_x (d - d_e) \| + \| \tau_x d_e \| < (K + 1) \epsilon \), where \( K \) is the upper bound of \( V \Psi_x(Y) \), if \( x \) is large enough.

(e) \( \| \tau_x rb - rb \| \to 0 \) for each \( b \) in \( M' \), where \( M' \) is the set of those \( b \) in \( B \) such that \( rb \) can be chosen in \( B \), not only in \( B^* \).
\[
\tau_x \tau_b(\beta) = \int \beta f^\beta d\Psi_x, \quad \text{but by (c) } f^\beta(y) = T^\nu \tau_b = \tau_b \text{ for each } y \text{ so } \tau_x \tau_b(\beta) = \int \beta(\tau_b) d\Psi_x = \tau_x(\beta(\tau_b)) = \beta(\tau_x(\tau_b)) \text{ for each } \beta, \quad \|\tau_x \tau_b - \tau_b\| = \|\tau_b\| \|\Psi_x(Y) - 1\|,
\]
which tends to 0 by the hypotheses on \{\Psi_x\}.

(f) \( b - \tau_b \in B' \) for each \( b \) in \( M' \) so \( \|\tau_x b - \tau_x \tau_b\| \to 0 \) for each \( b \) in \( M' \).

If \( b_1 = b - \tau b \) is in \( B - B' \), there exists a \( \beta_0 \) in \( B^* \) for which \( \beta_0(d) = 0 \) if \( d \in B' \) while \( \beta_0(b_1) = 1 \). Then \( \tau_x b(\beta_0) = \int \Psi_x d\Psi_x, \) but \( \beta_0 f^\beta(y) = \beta_0(T^\nu \tau_b) = \beta_0(b) \) for each \( y \) so \( \tau_x b(\beta_0) = \Psi_x(Y) \beta_0(b) \) and \( \lim_x \tau_x b(\beta_0) = \lim_x \Psi_x(Y) \beta_0(b) = \beta_0(b) \). On the other hand, for each \( x \) in \( X \) and \( \epsilon > 0 \) there is an \( x' \) in \( X \) such that \( x' > x \) and \( |\tau_x b(\beta_0) - \beta_0(\tau b)| < \epsilon \), so \( \beta_0(\tau b) = \beta_0(b) \) and \( \beta_0(b_1) = 0 \) contrary to the choice of \( \beta_0 \).

The conclusions mentioned can now be drawn. (1) is true if \( M' = B \). (B) \( \tau \in \tau \) and \( \tau \in \tau \) are clearly additive on \( M \) and for each \( b \) in \( M \) \( \|\tau b\| = \sup_x \|\tau_x b\| = \sup_x \|\int f^\beta d\Psi_x\| \leq \|f^\beta\| \sup_x \Psi_x(Y) \leq \|\tau\| \sup_y \|T^\nu\| \sup_x \Psi_x(Y) \), so \( \tau \) is linear on \( M \). All of (2) is proved except that \( M \) is closed; let \( N = \{b \mid \tau_0 b = \lim_x \tau_x b \} \), exists in the norm topology in \( B^* \} \). Then \( N \), as the set of points of convergence of \( \{\tau_x\} \), is a closed linear manifold and \( M \) is the inverse image by \( \tau_0 \) of \( B \) (considered as imbedded in \( B^* \)). Hence \( M \) is closed in \( N \) and therefore in \( B \). For (5) the remarks above showed that \( \tau b = b \) if \( b \in B' \); \( \tau b = b \), \( T^\nu \tau b = T^\nu \tau b = b \) by (c), so \( b \in B' \). (d) shows that \( \tau b = 0 \) if \( b \in B' \); \( \tau b = 0 \), \( b = b - \tau b \in B' \) by (i) since \( b \in M \) in this case. For (4) \( \tau b \in B' \) if \( b \in M \) so \( \tau b = \tau b = \tau b = b \) by (c); if \( b \in M \), \( T^\nu b = T^\nu (b) \) too, since \( T^\nu b - b \in B' \) \( \in M \); then \( \tau T^\nu b = \lim_x \tau_x T^\nu b = \lim_x T^\nu \tau_x b = T^\nu (\lim_x \tau_x b) = T^\nu \tau b = \tau b \).

3. Related and supplementary theorems. One point stands out strongly in this set of conclusions: most of them do not directly involve \( X \) or \( \{\Psi_x\} \) except in so far as the existence of the nearly invariant system \( \{\Psi_x\} \) was required to prove existence here. \( M \) is already defined in terms of the bounded Abelian semi-group \( \{T^\nu\} \) and (5) defines \( \tau \) in \( M \) in terms of \( \{T^\nu\} \) alone, so that \( M \) and \( \tau \) are the same no matter what \( X \) and \( \{\Psi_x\} \) are used so long as the system \( \{\Psi_x\} \) is nearly invariant over \( Y \). This raises three questions:

(A) If \( Y \) is a given Abelian semi-group, is there a nearly invariant system \( \{\Psi_x\} \) for some \( X \)?

(B) Under what conditions on \( Y \) and \( \{T^\nu\} \) does \( M = B \)?

(C) If some natural choice of \( X \) and \( \{\Psi_x\} \) is suggested by the nature of \( Y \), is this system nearly invariant?

(A) can be completely answered (yes, for any \( X \)); (B) partially; (C) depends on the case in question and obviously has no general answer. This section contains the discussion of (A) and (B).

Theorem 2. If a nearly invariant system \( \{\Psi_x\} \) of set functions over \( Y \) exists

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then there is an invariant function \( \Psi \); that is, a \( \Psi \) such that \( \Psi(E \ominus y) = \Psi(E) \) for every \( E \subset Y \) and \( y \in Y \) and \( \Psi(Y) = 1 \).

**Proof.** If \( \{ \Psi_x \} \) is the nearly invariant system, let \( T_x \) be the element of \( M_{\Psi}^* \) defined by \( T_x(\phi) = \int \phi d\Psi_x \) for every \( \phi \in M_Y \). Applying Lemma 1 with \( B^* = M_Y, b_x = T_x \) gives an \( T \) in \( M_{\Psi}^* \) such that \( T \) is in the weak* closure of every set \( \{ T_x \mid x > x_0 \} \) for every \( x_0 \) in \( X \). Let \( \Psi(E) = T(\phi_F) \) for each \( E \subset Y \), then \( \Psi \) is additive and bounded, of course, since \( \sup \Psi(E) \leq \lim \sup \forall \Psi(E) \leq 1 \) and, since \( \lim \Psi_x(Y) = 1, \Psi(Y) = 1 \).

To show \( \Psi \) invariant it suffices to show that for each \( \epsilon > 0 \) \( | \Psi(E \ominus y) - \Psi(E) | < \epsilon \). Now

\[
| \Psi(E \ominus y) - \Psi(E) | \leq | \Psi(E \ominus y) - \Psi_x(E \ominus y) | + | \Psi_x(E \ominus y) - \Psi_x(E) | + | \Psi_x(E) - \Psi(E) |
\]

By the near invariance of \( \{ \Psi_x \} \), there is an \( x_0 \) such that the middle term is less than \( \epsilon/3 \) whenever \( x > x_0 \); then by the fact that \( T \) is in the weak* closure of \( \{ T_x \mid x > x_0 \} \) the other two terms can be made less than \( \epsilon/3 \) by proper choice of \( x > x_0 \), so \( \Psi(E \ominus y) = \Psi(E) \) for all \( E \subset Y \) and \( y \in Y \).

Note that the full strength of near invariance is not used in this proof but only that \( \Psi_x(E) \rightarrow 0 \) for every \( E \subset Y \). Naturally if an invariant function \( \Psi \) exists, the system \( \{ \Psi_x \} \) such that every \( \Psi_x = \Psi \) for each \( x \in X \), is nearly invariant over \( Y \) no matter what \( X \) is.

**Corollary 1.** If \( Y \) is a family of subsets \( y \) of a given set \( A \), with addition in \( Y \) ordinary addition as subsets of \( A \), then there exists an invariant function \( \Psi \) over \( Y \).

This can be proved directly by proper application of the Hahn-Banach theorem. To derive it from Theorem 3, let \( X = Y \), order \( Y \) by letting \( y > y' \) mean \( y \supset y' \), and let \( \Psi_y \) for each \( y \) in \( Y \) be defined on the subsets \( E \) of \( Y \) by \( \Psi_y(E) = 1 \) if \( y \in E \), \( \Psi_y(E) = 0 \) if not. To show that the system \( \{ \Psi_y \} \) is nearly invariant, for any \( y_0 \) in \( Y \) let \( y \supset y_0 \); then \( \Psi_y(E \ominus y_0) = 1 \) if and only if \( y \) is in \( E \ominus y_0 \), that is, if and only if \( y + y_0 \) is in \( E \), that is, if and only if \( y + y_0 \) is in \( E \), but \( y + y_0 = y \) if \( y \supset y_0 \), so \( \Psi_y(E \ominus y_0) = 1 \) if and only if \( y \in E \), i.e., if and only if \( \Psi_y(E) = 1 \). Therefore \( \Psi_y(E) = \Psi_y(E \ominus y_0) \) for every \( E \) or \( \forall \Psi_y(y_0) = 0 \) if \( y > y_0 \).

This particular nearly invariant system will be used later.

**Corollary 2.** To the conclusions of Theorem 1 can be added:

(7) There exists an invariant function \( \Psi \) defined over \( Y \) such that \( \tau b = \int b \Psi \) if \( b \in M \), while \( \int b \Psi \in B^{**} \) if \( b \in M \). Moreover \( \int b \Psi \) is in the weak* closure of \( \{ \tau_x b \mid x > x_0 \} \) for every \( x_0 \) in \( X \).

Letting \( \Psi' = \Psi \) for any \( X \) gives a nearly invariant system \( \tau' b = \tau_x b = \int b \Psi \).
shows that \( r'b \in B \) if and only if \( b \in M \) (by Theorem 1) and in \( M r'b = r'b \). The definition of \( \mathcal{T} \), and hence of \( \Psi \), in Theorem 2 shows that \( \int f^b d\Psi \) lies in the given sets.

The important result is

**Theorem 3.** If \( Y \) is an Abelian semi-group, there is an invariant set function \( \Psi \) over \( Y \) such that \( \Psi(Y) = V \Psi(Y) = 1 \) and \( \Psi(E) \geq 0 \) if \( E \subset Y \).

The proof is essentially that of a result of Morse and Agnew [1, Lemma 2.01]. Let \( T_0 \) be defined on the multiples of \( \phi_Y \) by \( T_0(\alpha \phi_Y) = \alpha \); then by the Hahn-Banach theorem there is an \( T_1 \) in \( M_Y^* \) such that \( \|T_1\| = 1 \) and \( T_1(\alpha \phi_Y) = T_0(\alpha \phi_Y) \). For each \( y \) in \( Y \) let \( T^y \) be the function on \( M_Y \) to \( M_Y \) defined by \( T^y \phi = \phi(y + y') \) for each \( \phi \) in \( M_Y \) and \( y' \) in \( Y \). \( \|T^y\| = 1 \) for each \( y \) since \( \sup_{\phi' \in Y} |\phi(y + y')| \leq \sup_{\phi' \in Y} |\phi(y')| \) for every \( y \) in \( Y \) and \( T^y \phi = \phi_Y \); \( T_0 \) is invariant under all \( T^y \). Following [1] let

\[ p(\phi) = \inf \left( \sup_{\phi' \in Y} \frac{1}{n} \sum_{i=1}^{n} T_1(T^{yo} T^{vi} \phi) \right) \]

where the "\( \inf \)" is taken over all choices of the integer \( n \) and the points \( y_1, \ldots, y_n \) in \( Y \). As in [1] it can be shown that \( p(\alpha \phi) = \alpha p(\phi) \) if \( \alpha \geq 0 \), that \( p(\phi + \phi') \leq p(\phi) + p(\phi') \), and that \( p(\phi) \leq \|\phi\| \) for every \( \phi \) in \( M_Y \). Since \( T_0(\alpha \phi_Y) \leq p(\alpha \phi_Y) \), applying the Hahn-Banach theorem again gives an \( T \) in \( M_Y^* \) such that \( T(\alpha \phi_Y) = \alpha \) and \( T(\phi) \leq p(\phi) \leq \|\phi\| \) for every \( \phi \); the proof that \( T \) is invariant under the \( T^y \) is as in [1].

Let \( \Psi(E) = T(\phi_E) \) if \( E \subset Y \). Then \( \Psi(E - y) = T(\phi_{E - y}) = T(T^y \phi_E) = T(\phi_E) = \Psi(E) \), so \( \Psi \) is invariant; \( \|T\| = T(\phi_Y) = 1 \) so \( V \Psi(Y) = \Psi(Y) = 1 \). \( \Psi \) is non-negative, for if \( E \) exists with \( \Psi(E) < 0 \), then \( V \Psi(Y) \leq \|\Psi(E)\| + \|\Psi(Y - E)\| = -\Psi(E) + 1 - \Psi(E) > 1 \).

**Corollary 3.** \( \|\tau b\| \leq \|f^b\|_{M_Y(B)} = \sup_{\phi \in Y} \|T^y b\| \leq \|b\| \sup_{\phi \in Y} \|T^y\| \) if \( b \in M \).

If \( b \in M \), \( \tau b = 1 \int f^b d\Psi \) where \( \Psi \) is defined by Theorem 3; hence \( \|\tau b\| = \|\int f^b d\Psi\| \leq \|f^b\| \Psi(Y) = \|f^b\| \).

**Corollary 4.** If \( b \in M \), then \( \tau b \) is in the closed convex hull \( K(b) \) of the set \( \{T^y b \mid y \in Y\} \).

For each \( \beta \in B^* \), \( \beta(\tau b) = \int \beta f^b d\Psi \leq V \Psi(Y) \sup_{\phi \in Y} \beta f^b(\phi) = \sup_{\phi \in Y} \beta(T^y b) \). If \( b_0 \) is not in \( K(b) \) there is a \( \beta_0 \) such that \( \beta_0(b_0) > \sup_\beta \beta_0(T^y b) \) (by a theorem of Mazur [9]) so \( \tau b \) is in \( K(b) \).

 Alaoglu and Birkhoff call a point \( b \) in \( B \) ergodic relative to the bounded semi-group \( \{T^y\} \) if there is just one point of \( B'' \) in \( K(b) \) and all \( K(T^y b) \). (10)

(10) This is not their definition but one of the properties shown to be necessary and sufficient.
Corollary 5. \( b \) is ergodic if and only if \( b \in M \).

If \( b \in M \) and \( T^\#b \) and \( T^\#b = T^\#b \), and \( T^\#b = \tau b \), so \( b = \tau K(T^\#b) \) also and \( b \) is ergodic if these sets contain no other points in \( B'' \); this is asserted by Lemmas 1 and 2 of [2]. If \( b \) is ergodic and \( b_0 \) is the unique point in \( B'' \cdot K(b) \), then \( \theta \) is the unique point in \( B'' \cdot K(b-b_0) \); let \( b_1 = b - b_0 \). Since \( \theta \) is in the closed convex hull of the set \( \{ T^\#b_1 | y \in Y \} \), for each \( \epsilon > 0 \) there is a point \( b_2 = \sum \alpha_i T^\#b_1 \) with \( \| b_2 \| < \epsilon / K \), where \( K = \sup \| T^\#b \| \). If \( \tau b' = \int \beta d\Psi \), with \( \Psi \) as in Theorem 3, \( \| \tau b'_1 = \sum \alpha_i \tau T^\#b_1 = \tau b_1 \) and \( \| \tau b_1 \| = \| \tau b_2 \| < \epsilon \). Hence \( \| \tau b_1 \| = 0 \) or \( b_1 \in B'' \); so \( b = b_1 + b_0 \in M \).

To complete the relation of these results to those of Alaoglu and Birkhoff requires some study of Problem (B).

Lemma 3. If \( f \in M_Y(B) \) and \( \Psi \) is a bounded additive set function over \( Y \), \( \int f d\Psi \) is in \( Y \) if \( P \), the smallest closed linear subspace of \( B \) containing all the points \( \{ f(y) | y \in Y \} \), is reflexive.

Each \( \beta \) in \( B^* \) defines a \( \pi_\beta \) in \( P^* \) by \( \pi_\beta(\rho) = \beta(\rho) \) for each \( \rho \) in \( P \); then \( \int f d\Psi(\beta) = \int f d\Psi = \int \pi_\beta d\Psi \). Since the \( \pi_\beta \) cover \( P^* \) this defines a unique \( \rho \) in \( P^{**} \) such that \( \rho(\pi) = \int f d\Psi \) if \( \pi \in P^* \). Since \( P \) is reflexive there is a \( \rho \) in \( P \) such that \( \pi(\rho) = \int f d\Psi \) if \( \pi \in P^* \). Hence \( \beta(\rho) = \pi_\beta(\rho) = \int \pi_\beta d\Psi = \int \beta d\Psi \); so \( \int f d\Psi = \rho \in P^* \subset B \).

Corollary 6. If the set \( \{ T^\#b | y \in Y \} \) lies in some reflexive subspace of \( B \), then \( b \in M \).

Corollary 7. If \( B \) is reflexive, \( M = B \).

Corollary 8. If \( B \) is reflexive and \( \{ T^\# \} \) is any bounded Abelian semi-group of operators on \( B \) to \( B \), then each \( b \) in \( B \) is ergodic; that is, for each \( b \) in \( B \) \( K(b) \) contains just one fixed point \( \tau b \) of all the \( T^\# \).

It is to be noticed that this is a great strengthening of Theorem 5 of [2] since, as is known, every uniformly convex space is reflexive, and since \( \| T^\# \| \leq 1 \) can be replaced by \( \| T^\# \| \leq K \). On the other hand, that result can be proved with far less machinery.

Corollaries 3 and 7 together imply that if \( B \) is reflexive the set \( B'' \) of common fixed points of any bounded Abelian semi-group \( \{ T \} \) of operators on \( B \) to \( B \) is the range of a projection operator \( \tau \) defined on all of \( B \) and \( \| \tau \| \leq \sup \| T \| \).

A trivial result is this:

If \( \{ T^\# \} \) is any bounded Abelian semi-group of operators on \( B \) to \( B \) and if there is a \( y_0 \) such that \( \| T^\#y_0 \| < 1 \), then \( B' = B \); in fact \( \lim_y \| T^\#y \| = 0 \) if \( y \) is directed by letting \( y > y' \) if there is a \( y'' \) such that \( y = y'' + y' \).

For each \( \epsilon > 0 \), there is an \( n \) such that \( \| (T^\#y_0)^n \| < \epsilon / K \), where \( K \) is the bound of the norms of all \( T^\# \), then \( \| T^\#(T^\#y_0)^n \| < \epsilon \) for every \( y \) or \( \| T^\# \| < \epsilon \) if \( y > ny_0 \).
In some cases a certain $X$ and a nearly invariant system $\{\Psi_x\}$ over $Y$ arise naturally. Under certain conditions on $X$ the reflexivity condition of Corollary 7 can be weakened (at least formally). A directed set $X$ has a countable cofinal subset if there is a countable subset $X'$ of $X$ such that each $x$ in $X$ is followed by some $x'$ in $X'$.

**Theorem 4.** If $\{T^y\}$ is a bounded representation of the Abelian semi-group $Y$, if $X$ is a directed set with a countable cofinal subset, and if the system $\{\Psi_x\}$ is nearly invariant over $Y$, then $b \in M$ if and only if there is a countable cofinal subset $\{x_n\}$ of $X$ and a $b_0$ in $B$ such that $\lim_n \tau_{x_n} b(\beta) = \beta(b_0)$ for every $\beta$ in $B^*$. In this case $b_0 = \tau b$.

For such an $X$ this sequential compactness condition assures that $\tau b$ is in $B$; that is, that $b$ is in $M$. Since norm convergence implies weak convergence this condition is satisfied if $b \in M$.

**Corollary 9.** If $B$ is a Banach space with sequentially weakly compact unit sphere, if $\{T^y\}$ is a bounded Abelian semi-group of operators on $B$ to $B$, if $X$ has a countable cofinal subset and if $\{\Psi_x\}$ is a nearly invariant system such that $\{f^b \Psi_x \} \subseteq B$ for each $b$ in $B$, then $M = B$.

Since every reflexive space has a sequentially weakly compact unit sphere, this result is related to Corollary 7; since it is not known whether or not sequential weak compactness implies reflexivity, it is not known whether the hypotheses on $X$ and $\{\Psi_x\}$ are needed.

4. Special semi-groups and systems of set functions. A theorem of Dunford [6] uses $E_n$, euclidean $n$-space with coordinates $y_1, \ldots, y_n$, for $Y$ and the class of $n$-dimensional cubes $x = \{ y \mid \alpha_j < y_j < \alpha_j + r, j = 1, \ldots, n \}$, where $r > 0$ and the $\alpha_j$ are arbitrary real numbers, for $X$, defining $\Psi_x(E) = m(Ex)/m(x)$, where $m$ is Lebesgue measure, for every Lebesgue measurable set $E \subseteq Y$. (He has then to assume measurability for each $f^b$ in order to integrate.) $\tau_x b$, then, is the arithmetic mean of $f^b$ over the cube $x$; that is, $\tau_x b = \tau_n f_x f^b dm$, where $\tau$ is the length of edge of the cube $x$. $X$ is ordered by the size of the cubes, that is, $x > x'$ if the edges of $x$ are longer than those of $x'$.

A more general result follows from a simple property of convex bodies with interior points in $E_n$. In what follows let $S_\alpha(y)$ be the closed sphere about $y$ of radius $\alpha$: as in any linear space if $E, E' \subseteq E_n$ let $E + E' = \{ e + e' \mid e \in E$ and $e' \in E' \}$ and for any real $\alpha$ and $E \subseteq E_n$ let $\alpha E = \{ \alpha e \mid e \in E \}$.

**Lemma 4.** If $E$ is a convex subset of $E_n$ and if $E$ contains a sphere $S_\alpha = S_\alpha(0)$, and if $S_\alpha = S_\alpha(0)$, then $S_\alpha + E \subseteq [(r + \alpha)/r]E$ (12).

---

(11) A set $B_0 \subseteq B$ is sequentially weakly compact if for each sequence $\{b_n\} \subseteq B_0$ there is a subsequence $\{n_i\}$ and a $b_0$ in $B_0$ such that $\lim_i \beta(b_{n_i}) = \beta(b_0)$ for every $\beta$ in $B^*$.

(12) This proof is due to the referee who notes that it holds in any normed vector space.
\[
y \in S_\alpha \text{ if and only if } y = (\alpha/r)y' \text{ for some } y' \text{ in } S_r. \text{ For each } y \text{ in } S_\alpha \text{ and } y'' \text{ in } E
\]
\[
y + y'' = \frac{\alpha}{r} y' + y'' = \frac{r + \alpha}{r} \left[ \frac{\alpha}{r + \alpha} y' + \frac{r}{r + \alpha} y'' \right] = \frac{r + \alpha}{r} y_0
\]
where \( y_0 \in E \) since \( y', \ y'' \in E, \ \alpha/(r+\alpha)+r/(r+\alpha)=1, \) and \( \alpha/(r+\alpha)>0, \ r/(r+\alpha)>0. \) Hence \( E + S_\alpha \subset [(r+\alpha)/r]E. \)

For each bounded convex set \( E \) with interior points contained in \( E_n \) let \( r(E) \) be the least upper bound of the radii of the spheres contained in \( E. \) Then there will be at least one sphere of radius \( r(E) \) contained in \( E, \) the closure of \( E, \) since any bounded closed set in \( E_n \) is compact.

**Lemma 5.** If \( X \) is the set of all bounded convex sets with interior points in \( E_n \) and if \( X \) is directed by the relation \( x > x' \) if and only if \( r(x) \geq r(x'), \) then for each \( y \) in \( E_n, \lim_{x \to y} m(x)/m(x)=1. \)

Since \( x - y = x(x - y) + [(x - y) - x], \) it suffices to show that \( m[(x - y) - x]/m(x) \to 0. \) If \( \alpha \) is the distance from \( y \) to 0, then \( x - y \subset x + S_\alpha; \) by Lemma 4, \( x + S_\alpha \) is contained in a dilation of \( x \) in the ratio \( (r(x) + \alpha)/r(x) \) about the center of any sphere of radius \( r(x) \) contained in \( x. \) The ratio of the measures of the dilated set and \( x \) is precisely \( [(r(x) + \alpha)/r(x)]^n \) so \( m[(x - y) - x]/m(x) \leq m[(x + S_\alpha) - x]/m(x) \leq [(r(x) + \alpha)/r(x)]^n - 1 \) which tends to 0 as \( r(x) \to \infty. \)

**Lemma 6.** There is a non-negative additive set function \( \mu \) defined on all subsets of \( E_n \) of finite outer measure, such that \( \mu(\alpha A - y) = |\alpha| \mu(A) \) and such that \( \mu(A) = m(A) \) if \( A \) is Lebesgue measurable and of finite measure.

This follows from the work of Morse and Agnew [1]; they gave the construction for the case \( n=1. \)

**Theorem 5.** Let \( A \) be any convex set with interior points in \( E_n \) and let \( Y = \{ \alpha y \mid y \text{ in } A \text{ and } \alpha \geq 1 \}; \) then \( Y \) is an Abelian semi-group under vector addition in \( E_n; \) for any Banach space \( B \) let \( \{ TV \} \) be a bounded representation of \( Y. \) Let \( X \) be the set of all convex sets of finite nonzero measure contained in \( Y \) and for each \( x \) in \( X \) define \( \Psi_x \) by \( \Psi_x(E) = \mu(Ex)/\mu(x). \) Then the system \( \{ \Psi_x \} \) is nearly invariant and the conclusions of Theorem 1 hold.

Near invariance of \( \{ \Psi_x \} \) is all that needs be verified. \( VV_x(Y) = \Psi_x(Y) = \mu(x)/\mu(x)=1; \)
\[
\Psi_{x(E)} = \Psi_x(E - y) = \Psi_x(E) = \left[ \mu(x) \right]^{-1} \left\{ \mu(E - y) x - \mu(Ex) \right\}
\]
\[
= \left[ \mu(x) \right]^{-1} \left\{ \mu(E - y)(x - y) - \mu(E - y)((x - y) - x) \right\}
= \mu \left[ (E - y)(x - (x - y)) \right] - \mu(Ex)
\]
\[
= \left[ \mu(x) \right]^{-1} \left\{ \mu(Ex - y) - \mu(Ex) - \mu(E - y)((x - y) - x) \right\}
= \left[ \mu(x) \right]^{-1} \left\{ - \mu(E - y)((x - y) - x) + \mu(E - y)(x - (x - y)) \right\}.
\]
If \( Y = \sum_{i \leq k} E_i \) with the \( E_i \) disjoint, then the sets \( E_i \cap y \) are disjoint and have sum \( Y \). Hence

\[
\sum_{i \leq k} |\Psi_{x_k}(E_i)| \leq [\mu(x)]^{-1} \sum_{i \leq k} \{ \mu[(E_i \cap y)((x \cap y) - x)]
+ [\mu((E_i \cap y)(x - (x \cap y)))] 
\leq [\mu(x)]^{-1} \left\{ \mu \left[ \sum_{i \leq k} (E_i \cap y)((x \cap y) - x) \right] 
+ [\mu \left( \sum_{i \leq k} (E_i \cap y)(x - (x \cap y)) \right)] \right\}
\]

\[
\leq [\mu(x)]^{-1} \left\{ \mu[(x \cap y) - x] + \mu[x - (x \cap y)] \right\}.
\]

By Lemma 3, this is small for \( r(x) \) large, independent of the decomposition of \( Y \) into the sets \( E_i \), so \( \forall \Psi_{x_k}(Y) \rightarrow 0 \) and this system \( \{\Psi_x\} \) is nearly invariant.

The set \( X' \) of cubes used by Dunford if ordered by edge length has the same ordering as if ordered by the radius of the largest sphere inside; so \( X' \) is a cofinal subset of this family \( X \) of convex sets of finite, nonzero measure; so \( \lim_x \tau_x b \) exists for every \( b \) in \( M \) since \( \lim_x \tau_x b \) exists for such \( b \); moreover if the functions \( f^b \) are all measurable, the \( \tau_x \) reduce to Dunford's transformations and Theorem 5 offers a simple proof, without differentiation theorems, of Dunford's result.

Note that this \( X \) has a countable cofinal subset, in fact any sequence \( \{x_n\} \subset X \) such that \( r(x_n) \rightarrow \infty \) will do. Hence if the assumption is made that each \( T^b \) is a measurable function, each \( \tau_x b \in B \) and Corollary 9 can be applied with proper choice of \( B \).

For a second application (not considered anywhere in the literature so far as I know) take \( Y \) to be the stack \( \Delta \) whose elements are the finite subsets of some given set \( D \) of elements \( d \) where addition is, as in Corollary 1, ordinary point set addition. If \( B \) is a Banach space and \( \{T^\delta \mid \delta \in \Delta \} \) is a bounded representation of \( \Delta \) in the space of linear operators on \( B \), \( T^{\delta} T^{\delta'} = T^{\delta + \delta'} = T^{\delta} \) so every \( T^\delta \) must be a projection; moreover \( T^\delta = \prod_{d \in \delta} T^d \) for each \( \delta \in \Delta \). \( \Delta \) is also a directed set if \( \delta > \delta' \) means \( \delta \supset \delta' \); for each \( \delta \in \Delta \) let \( \Psi_\delta \) be defined over the subsets of \( \Delta \) by \( \Psi_\delta(E) = 1 \) if \( \delta \in E \), \( \Psi_\delta(E) = 0 \) if \( \delta \notin E \). Then, as in Corollary 1, the system \( \{\Psi_\delta\} \) is nearly invariant over \( \Delta \).

**Theorem 6.** If \( B \) is a Banach space and the \( T^d \) are commuting projections on \( B \) to \( B \) such that \( \|T^\delta\| \leq K \) for all \( \delta \), where \( T^\delta = \prod_{d \in \delta} T^d \); then \( \lim_\delta T^\delta b \) exists (in the norm topology) if and only if \( b \in M \) (where \( M \) is defined as in Theorem 1) and for such \( b \), \( \lim_\delta \|T^\delta b - \tau b\| = 0 \).

By Theorem 1, \( \|\tau_\delta b - \tau b\| \rightarrow 0 \) with \( \tau b \) in \( B \) if and only if \( b \in M \). But \( \tau_\delta b = \int f^b d\Psi_\delta = T^\delta b \), and if the \( T^\delta b \) converge at all their limit must be in \( B \).
It is to be noted that this theorem has a much stronger conclusion than can be obtained in general; this is true because the bounded representations of a stack are so greatly restricted.

**Corollary 10.** If $B$ is reflexive and if $T^d$ are commuting projections with $\|T^d\|$ uniformly bounded, then $\lim_b T^b = \tau b$ for every $b$ in $B$, where $\tau$ is a projection defined on all of $B$ which has all the properties ascribed to $\tau$ in Theorem 1.

One more consequence of Theorem 1 is this: If the collection $\{T^y\}$ of operators on $B$ to $B$ is a bounded representation of the semi-group $Y$, then the collection $\{T^{y*}\}$ is such a representation in the set of operators on $B^*$ to $B^*$. In general, defining $B^{(n*)}$ by induction from $B^{(n*)} = (B^{((n-1)*)})^*$, and $T^{(n*)} = (T^{((n-1)*)})^*$, the same is true of the collection $\{T^{y(n*)}\}$. Hence Theorem 1 defines in $B^{(n*)}$ a set $M^{(n*)}$ consisting of the direct sum of $B^{((n+1)*)}$, the set of fixed points of the $T^{y(n*)}$, and $B^{((n+1)*)}$, the smallest closed linear set in $B^{(n*)}$ containing all $T^{y(n*)}b^{(n*)} - b^{(n*)}$ for all choices of $y$ in $Y$ and $b^{(n*)}$ in $B^{(n*)}$. Since $B^{((n+2)*)} \supseteq B^{(n*)}$ for every $n$, and the $T^{y(n+2)*)}$ agree with the $T^{y(n*)}$ in $B^{(n*)}$, $B^{((n+2)*)} = B^{((n+1)*)}$ and similar relations hold for $B^{((n+1)*)}$ and $M^{(n*)}$.

(c) of the proof of Theorem 1 shows that for any $b$ in $B$ and any possible choice of $\tau b$, $\tau b$ is in $B^{(n*)}$.

An example of the results obtained from this point of view is

**Theorem 7.** If $B$ is reflexive and if $\{T^y \mid y \in Y\}$ is a bounded Abelian semi-group of operators on $B$ to $B$, then there exists a projection $\tau$ defined over all $B$ such that for every nearly invariant system of set functions $\{\Psi_x\}$ over $Y$, $\lim_x \|\tau_x b - \tau b\| = 0$ for each $b$ in $B$ and $\lim_x \|\tau_x^* \beta - \tau^* \beta\| = 0$ for each $\beta$ in $B^*$. $\tau$ and $\tau^*$ have in their respective spaces the properties specified by Theorem 1.

All that need be verified is that $\tau_0^*$, the projection in $B^*$ that exists by direct application of Theorem 1 to that space, is equal to $\tau^*$, the adjoint of $\tau$. Since $\|\tau_x^* \beta - \tau_0^* \beta\| \to 0$ for each $\beta$ in $B^*$, $\tau_x^* \beta(b) - \tau_0^* \beta(b) = \beta(\tau_x b) - \tau_0^* \beta(b) \to 0$ for each $b$ in $B$ but $\beta(\tau_x b) \to \beta(\tau b)$ so $\tau_x^* \beta(b) = \beta(\tau b) = \tau_0^* \beta(b)$ for each $b$ in $B$ and $\beta$ in $B^*$, or $\tau^* = \tau_0^*$.

I see no way of proving anything quite of this nature if $M \neq B$.

It may be noted that neither the methods nor the results of this paper carry over to noncommutative semi-groups; in fact, an example [2, §13, Example 1] shows that for non-abelian semi-groups the set $M$ and the set of ergodic points of $B$ need not be the same.

**Bibliography**


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